

Title: Lecture 2: Bootstrapping global symmetries. Cutting surface algorithm

Speakers: Ning Su

Collection: Mini-Course of Numerical Conformal Bootstrap

Date: April 25, 2023 - 10:00 AM

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Plan for today

- Derivation of $O(N)$ bootstrap equations for all four-point functions involving $\{v = \phi_i, s = \phi^2, t = \phi_i \phi_j\}$
- Cutting surface algorithm
- Tutorial A : Learn how to use autoboot, how to use projectors, run cutting surface algorithm
- Tutorial B : Using Hyperion to bootstrap 3d Ising

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Group theory: CG coeff, 3j symbol

[Reference : autoboot arXiv:1903.10522, <<Birdtracks, Lie's, and Exceptional Groups>> Predrag Cvitanovic]

Consider a group with real representations generic reps s, t, r, \dots

Tensor product decomposition : $s \otimes t = r_1^{n_1} \oplus r_2^{n_2} \oplus r_3^{n_3} \oplus \dots$, (n_i : multiplicity)

Clebsch Gordan coefficient : decomposition $r \otimes s \rightarrow t^n$ is given explicitly by $\left\{ \begin{matrix} c \\ t \end{matrix} \middle| \begin{matrix} a, b \\ r, s \end{matrix} \right\}_{i=1 \dots n}$, (a : indices of rep r , etc.) i.e. $\phi_c^{(t)} = \left\{ \begin{matrix} c \\ t \end{matrix} \middle| \begin{matrix} a, b \\ r, s \end{matrix} \right\} \phi_a^{(r)} \phi_b^{(s)}$

Normalization of CG coeff : $\sum_{a,b} \left\{ \begin{matrix} c \\ t \end{matrix} \middle| \begin{matrix} a, b \\ r, s \end{matrix} \right\}_n \left\{ \begin{matrix} c' \\ t' \end{matrix} \middle| \begin{matrix} a, b \\ r, s \end{matrix} \right\}_{n'} = \delta_{tt'} \delta_{nn'} \delta_{cc'}$

Define the 3j symbol : $\left\langle \begin{matrix} a, b, c \\ r, s, t \end{matrix} \right\rangle_n = \frac{1}{\sqrt{\dim(t)}} \left\{ \begin{matrix} c \\ t \end{matrix} \middle| \begin{matrix} a, b \\ r, s \end{matrix} \right\}_n$, normalization: $\sum_{a,b,c} \left\langle \begin{matrix} a, b, c \\ r, s, t \end{matrix} \right\rangle_n \left\langle \begin{matrix} a, b, c \\ r, s, t \end{matrix} \right\rangle_n = 1$

Example : $O(n)$ group with S (singlet), V (vector) reps:

CG coefficient: $\left\{ \begin{matrix} i & j & \square \\ V & V & S \end{matrix} \right\} = \frac{1}{\sqrt{n}} \delta_{ij}$, 3j symbol : $\left\langle \begin{matrix} i & j & \square \\ V & V & V \end{matrix} \right\rangle = \frac{1}{\sqrt{n}} \delta_{ij}$, i, j : $O(n)$ vector indices

Exercise : check $\left\langle \begin{matrix} i & j & \square \\ V & V & V \end{matrix} \right\rangle$ is symmetric (In general, 3j symbol is symmetric / antisymmetric)

Group theory: CG coeff, 3j symbol

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Group theory: generalized delta

Let i, j, k, \dots be the $O(n)$ vector indices. We can use the vector indices to label an arbitrary rep R in $O(n)$

“generalized delta function” $\delta_{i,j}^{(R)} : \delta_{i,j}^{(R)} \phi_j^{(R)} = \phi_i^{(R)}$, $\mathbf{i} = \{i_1, \dots, i_n\}$ and $\sum_j = \sum_{j_1} \sum_{j_2} \dots$

Properties: $\delta_{i,j}^{(R)} \delta_{j,k}^{(R)} = \delta_{i,k}^{(R)}$ and $\delta_{j,j}^{(R)} = \dim(R)$

$\delta_{i,j}^{(R)}$: projection $V \otimes V \otimes V \dots \rightarrow R$

Example: traceless symmetric rank-2 tensor $T : \delta_{(ij),(kl)}^{(T)} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{1}{n} \delta_{ij} \delta_{kl}$

Example: $\delta_{(ij),(kl)}^{(T)} \phi^i \phi^j = \phi_k \phi_l - \frac{1}{n} \phi^i \phi_i \delta_{kl}$ is a traceless symmetric rank-2 tensor

Exercise: $\left\langle \frac{i}{R} \frac{j}{R} \middle| \frac{\square}{S} \right\rangle = \frac{1}{\sqrt{\dim(R)}} \delta_{ij}^{(R)}$, $\left\langle \frac{i}{R} \frac{j}{R} \frac{\square}{S} \right\rangle = \frac{1}{\sqrt{\dim(R)}} \delta_{ij}^{(R)}$ for any rep R



CFT : 2-point, 3-point function

We define 3pt coefficient λ_{ijk} as

$$\langle \phi_{1,r|a|}(x) \phi_{2,s|b|}(y) \phi_{3,t|c|}^{\mu_1 \dots \mu_\ell}(z) \rangle = \lambda_{123} \left\{ \frac{c}{t} \middle| \frac{a,b}{r,s} \right\} \frac{Z^{\mu_1 \dots \mu_\ell} \text{-trace}}{|x-y|^{\Delta_i + \Delta_j - (\Delta_k - \ell)} |y-z|^{\Delta_j + (\Delta_k - \ell) - \Delta_i} |z-x|^{\Delta_k - \ell + \Delta_i - \Delta_j}}, \quad Z^\mu = \frac{(x-z)^\mu}{(x-z)^2} - \frac{(y-z)^\mu}{(y-z)^2}$$

$\phi_{1,r|a|}(x)$: ϕ is in representation r with indices a .

We define 3pt coefficient α_{ijk} as $\lambda_{ijk} \left\{ \frac{c}{t} \middle| \frac{a,b}{r,s} \right\} = \alpha_{ijk} \left\langle \frac{a,b,c}{r,s,t} \right\rangle$, i.e. $\alpha_{ijk} = \lambda_{ijk} \sqrt{\dim(\phi_k)}$

2pt function :

$$\langle \phi_{A|a|}(x) \phi_{B|b|}(y) \rangle = \langle \phi_{A|a|}(x) \phi_{B|b|}(y) \mathbb{1} \rangle = \lambda_{\phi\phi\mathbb{1}} \left\{ \frac{\mathbb{1}}{S} \middle| \frac{a,b}{A,B} \right\} \frac{\delta_{AB}}{|x-y|^{2\Delta}} = \frac{\alpha_{\phi\phi\mathbb{1}}}{\sqrt{\dim(\phi)}} \frac{\delta_{AB} \delta_{a,b}^{(A)}}{|x-y|^{2\Delta}}$$

(difference choices of $\alpha_{\phi\phi\mathbb{1}}$ lead to different conventions.)

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CFT : OPE

Define $C^{\nu_1 \dots \nu_\ell}(x, -\partial_z)$ by

$$\frac{Z^{\mu_1 \dots \mu_\ell} \text{-trace}}{|z|^{\Delta_i + \Delta_j - (\Delta_k - \ell)} |z|^{\Delta_j + (\Delta_k - \ell) - \Delta_i} |z - x|^{\Delta_k - \ell + \Delta_i - \Delta_j}} = C^{\nu_1 \dots \nu_\ell}(x, -\partial_z) \frac{I_1^{\mu_1} \dots I_{\nu_\ell}^{\mu_\ell} \text{-trace}}{|z|^{2 \Delta_k}}, \quad I_\nu^\mu = \delta_\nu^\mu - 2 \frac{(y-z)^\mu (y-z)_\nu}{(y-z)^2}$$

$$\text{OPE : } \phi_{1, r_1 | a_1}(x_1) \times \phi_{2, r_2 | a_2}(x_2) = \sum_r \sum_{O:r} \sum_n \frac{\sqrt{\dim(O)}}{\alpha_{OO1}} \alpha_{\phi_1 \phi_2 O}^{(n)} \sum_a \left\langle \frac{a, a_1, a_2}{r, r_1, r_2} \right\rangle_n C_{\phi_1 \phi_2 O}^{\mu_1 \dots \mu_k}(x_1 - x_2, \partial_{y_1}) O_{r|a}^{\mu_1 \dots \mu_k}(y_1)$$

(group decomposition $r_1 \otimes r_2 \rightarrow r \oplus \dots$)

Exercise : show $\alpha_{\phi_1 \phi_2 O}$ in the OPE is the same $\alpha_{\phi_1 \phi_2 O}$ in the definition of 3pt.

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Group theory : 4pt structure

For simplicity, let's assume there is no multiplicity.

Define 4pt structure as

$$P_{r_1[a_1]r_2[a_2]r_3[a_3]r_4[a_4]}^s := \sum_b \left\{ \begin{matrix} b \\ s \end{matrix} \middle| \begin{matrix} a_1, a_2 \\ r_1, r_2 \end{matrix} \right\} \left\{ \begin{matrix} b \\ s \end{matrix} \middle| \begin{matrix} a_3, a_4 \\ r_3, r_4 \end{matrix} \right\} \frac{1}{\sqrt{\dim(s)}} \frac{1}{\alpha_{s s 1}}$$

Taking $\alpha_{s s 1} = \frac{1}{\sqrt{\dim(s)}}$ (in this case P is called 4pt projector), P has the following properties:

- 1, (completeness), $(P_{r_1 r_2 r_1 r_2}^{s_1} + P_{r_1 r_2 r_1 r_2}^{s_2} + \dots + P_{r_1 r_2 r_1 r_2}^{s_N})_{a_1 a_2 a_3 a_4} = \delta_{a_1 a_3}^{(r_1)} \delta_{a_2 a_4}^{(r_2)}$
- 2, (projection), $\sum_{a_3 a_4} (P_{r_1 r_2 r_3 r_4}^r)_{a_1 a_2 a_3 a_4} (P_{r_3 r_4 r_5 r_6}^s)_{a_3 a_4 a_5 a_6} = \delta_{r s} (P_{r_1 r_2 r_5 r_6}^s)_{a_1 a_2 a_5 a_6}$
- 3, (orthogonality), $\sum_{a_1 a_2 a_3 a_4} (P_{r_1 r_2 r_3 r_4}^r)_{a_1 a_2 a_3 a_4} (P_{r_1 r_2 r_3 r_4}^s)_{a_1 a_2 a_3 a_4} = \delta_{r s} \dim(r)$

Exercise : proof above properties using the properties of $\left\{ \begin{matrix} b \\ s \end{matrix} \middle| \begin{matrix} a_1, a_2 \\ r_1, r_2 \end{matrix} \right\}$



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CFT : 4pt function

4pt function:

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle =$$

$$\frac{1}{|x_{12}|^{\Delta_1+\Delta_2} |x_{34}|^{\Delta_3+\Delta_4}} \left(\frac{|x_{24}|}{|x_{14}|} \right)^{\Delta_{12}} \left(\frac{|x_{14}|}{|x_{13}|} \right)^{\Delta_{34}} \sum_s \sum_{O:s} \sum_n \sum_m \alpha_{\phi_1 \phi_2 O}^n \alpha_{\phi_3 \phi_4 O}^m \sum_b \left\langle \frac{b, a_1, a_2}{s, r_1, r_2} \right\rangle \left\langle \frac{b, a_3, a_4}{s, r_3, r_4} \right\rangle \frac{\sqrt{\dim(O)}}{\alpha_{OO1}} g_O^{\Delta_{12}, \Delta_{34}}(u, v)$$

where

$$g_O^{\Delta_{12}, \Delta_{34}}(u, v) = \left| \frac{x_{24}}{x_{14}} \right|^{-\Delta_{12}} \left| \frac{x_{14}}{x_{13}} \right|^{-\Delta_{34}} C_{\phi_1 \phi_2 O, k}(x_{12}, \partial_{y_1}) C_{\phi_3 \phi_4 O, j}(x_{34}, \partial_{y_2}) \frac{I^{ij}(y_{12})}{|y_{12}|^{2\Delta}}, \quad x_{ij} = x_i - x_j$$

(Convention : Poland, Rychkov, Vichi (2018) Row 1 of Table I)

$\mathcal{N}_{d,\ell}$	Reference
$\frac{v}{(v-2)^\ell (v/2-1)^\ell}$	Dolan and Osborn (2001b, 2004), Battazzi <i>et al.</i> (2008), Penedones <i>et al.</i> (2016), this review ★ Dolan and Osborn (2011), Hogervorst and Rychkov (2015), El-Showk <i>et al.</i> (2012, 2014b), Costa <i>et al.</i> (2016b), JuliBoots (Paulos, 2014f), eboot (Ohtsuki, 2016)
$\frac{v-1}{2^{\ell+1} (v/2-1)^\ell}$	Kos <i>et al.</i> (2014, 2015b, 2016), Li <i>et al.</i> (2017b) PyCFTBoot (Behan, 2017a)
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$\frac{v}{2^{\ell+1} (v/2-1)^\ell}$	Kos <i>et al.</i> (2014b) Mathematica notebook (Simmons-Duffin, 2015b)
$\frac{v-1}{2^{\ell+1} (v/2-1)^\ell}$	Simmons-Duffin (2017c)

TABLE 1 Summary of various conformal block normalizations $\mathcal{N}_{d,\ell}$, Eqs. (52, 62), used in the literature.

Exercise : plug in OPE and derive above expansion.

CFT : 4pt function

4pt function:

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle = \text{I} \frac{1}{|x_{12}|^{\Delta_1+\Delta_2} |x_{34}|^{\Delta_3+\Delta_4}} \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{12}} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{34}} \sum_s \sum_{O:s} \sum_n \sum_m \alpha_{\phi_1 \phi_2 O}^n \alpha_{\phi_3 \phi_4 O}^m \sum_b \left\langle \frac{b, a_1, a_2}{s, r_1, r_2} \right\rangle_n \left\langle \frac{b, a_3, a_4}{s, r_3, r_4} \right\rangle_m \frac{\sqrt{\dim(O)}}{\alpha_{OO1}} g_O^{\Delta_{12}, \Delta_{34}}(u, v)$$

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$$\frac{1}{|x_{12}|^{\Delta_1+\Delta_2} |x_{34}|^{\Delta_3+\Delta_4}} \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{12}} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{34}} \sum_s \sum_{O:s} \sum_n \sum_m \alpha_{\phi_1 \phi_2 O}^n \alpha_{\phi_3 \phi_4 O}^m \sum_b \left\langle \frac{b, a_1, a_2}{s, r_1, r_2} \right\rangle_n \left\langle \frac{b, a_3, a_4}{s, r_3, r_4} \right\rangle_m \frac{\sqrt{\dim(O)}}{\alpha_{OO1}} g_O^{\Delta_{12}, \Delta_{34}}(u, v)$$

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CFT : OPE

Define $C^{\nu_1 \dots \nu_\ell}(x, -\partial_z)$ by

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$$\text{OPE : } \phi_{1, r_1[a_1]}(x_1) \times \phi_{2, r_2[a_2]}(x_2) = \sum_r \sum_{O:r} \sum_n \frac{\sqrt{\dim(O)}}{\alpha_{OO1}} \alpha_{\phi_1 \phi_2 O}^{(n)} \sum_a \left\langle \frac{a, a_1, a_2}{r, r_1, r_2} \right\rangle_n C_{\phi_1 \phi_2 O}^{\mu_1 \dots \mu_k}(x_1 - x_2, \partial_{y_1}) O_{r[a]}^{\mu_1 \dots \mu_k}(y_1)$$

(group decomposition $r_1 \otimes r_2 \rightarrow r \oplus \dots$)

Exercise : show $\alpha_{\phi_1 \phi_2 O}$ in the OPE is the same $\alpha_{\phi_1 \phi_2 O}$ in the definition of 3pt.

Bootstrap equations

$$r_1 \otimes r_2 \rightarrow R \oplus \dots$$

Define crossing matrix “ M ” by $P_{r_3 r_2 r_1 r_4}^S = \sum_R M_{RS} P_{r_1 r_2 r_3 r_4}^S$.

Define convolved block : $F_{\pm, O}^{12;34}(u, v) = v^{\frac{\Delta_2 + \Delta_3}{2}} g_{O}^{\Delta_{12}, \Delta_{34}}(u, v) \pm u^{\frac{\Delta_1 + \Delta_2}{2}} g_{O}^{\Delta_{32}, \Delta_{14}}(v, u)$

The bootstrap equation is

$$\pm \sum_R \sum_{O:R} \alpha_{\phi_3 \phi_2 O} \alpha_{\phi_1 \phi_4 O} (M^T)_{S,R} F_{\pm, O}^{32;14}(u, v) - \sum_{O:S} \alpha_{\phi_1 \phi_2 O} \alpha_{\phi_3 \phi_4 O} F_{\pm, O}^{12;34}(u, v) = 0$$

Special case $\phi_1 = \phi_3$: $\sum_R \sum_{O:R} \alpha_{\phi_1 \phi_2 O} \alpha_{\phi_1 \phi_4 O} (\pm M^T - \mathbb{1})_{S,R} F_{\pm, O}^{12;14}(u, v) = 0$

Note : M_{RS} depends on the choice of $\alpha_{\phi\phi_1}$

Exercise : derive above equations



OPE coefficient conventions

Let's now fix $\alpha_{\phi\phi 1}$. Different choices of $\alpha_{\phi\phi 1}$ correspond to different OPE coefficient conventions.

$$\text{For } \alpha_{s s 1} = 1, \quad P_{i j k l}^{s,(\text{sim})} = \frac{1}{\sqrt{\dim(s)}} \sum_b \left\{ \begin{matrix} b \\ s \end{matrix} \middle| \begin{matrix} i, j \\ r_1, r_2 \end{matrix} \right\} \left\{ \begin{matrix} b \\ s \end{matrix} \middle| \begin{matrix} k, l \\ r_3, r_4 \end{matrix} \right\} = \sqrt{\dim(s)} \sum_b \left\langle \begin{matrix} b, i, j \\ s, r_1, r_2 \end{matrix} \right\rangle \left\langle \begin{matrix} b, k, l \\ s, r_3, r_4 \end{matrix} \right\rangle$$

$$\text{For } \alpha_{s s 1} = \frac{1}{\sqrt{\dim(s)}}, \quad P_{i j k l}^{s,(\text{prj})} = \sum_b \left\{ \begin{matrix} b \\ s \end{matrix} \middle| \begin{matrix} i, j \\ r_1, r_2 \end{matrix} \right\} \left\{ \begin{matrix} b \\ s \end{matrix} \middle| \begin{matrix} k, l \\ r_3, r_4 \end{matrix} \right\}$$

$$\text{For } \alpha_{s s 1} = \sqrt{\dim(s)}, \quad P_{i j k l}^{s,(\text{aut})} = \sum_b \left\langle \begin{matrix} b, i, j \\ s, r_1, r_2 \end{matrix} \right\rangle \left\langle \begin{matrix} b, k, j \\ s, r_3, r_4 \end{matrix} \right\rangle$$

$$\text{OPEs : } \alpha_{\text{abc, sim}}^2 = \sqrt{\dim(c)} \alpha_{\text{abc, prj}}^2, \quad \alpha_{\text{abc, aut}}^2 = \dim(c) \alpha_{\text{abc, prj}}^2, \quad \alpha_{\text{abc, sim}}^2 = \frac{1}{\sqrt{\dim(c)}} \alpha_{\text{abc, aut}}^2$$

$$\alpha_{\phi_1 \phi_2 \phi_3}^{\text{aut}} = \alpha_{\phi_1 \phi_2 \phi_3}^{\text{sim}} \dim(\phi_1)^{1/4} \dim(\phi_2)^{1/4} \dim(\phi_3)^{1/4}$$



CFT : OPE

Define $C^{\nu_1 \dots \nu_\ell}(x, -\partial_z)$ by

$$\frac{Z^{\mu_1 \dots \mu_\ell} \text{-trace}}{|x|^{\Delta_i + \Delta_j - (\Delta_k - \ell)} |z|^{\Delta_j + (\Delta_k - \ell) - \Delta_i} |z-x|^{\Delta_k - \ell + \Delta_i - \Delta_j}} = C^{\nu_1 \dots \nu_\ell}(x, -\partial_z) \frac{I_1^{\mu_1} \dots I_{\nu_\ell}^{\mu_\ell} \text{-trace}}{|z|^{2 \Delta_k}}, \quad I_\nu^\mu = \delta_\nu^\mu - 2 \frac{(y-z)^\mu (y-z)_\nu}{(y-z)^2}$$

$$\text{OPE : } \phi_{1, r_1[a_1]}(x_1) \times \phi_{2, r_2[a_2]}(x_2) = \sum_r \sum_{O:r} \sum_n \frac{\sqrt{\dim(O)}}{\alpha_{OO1}} \alpha_{\phi_1 \phi_2 O}^{(n)} \sum_a \left\langle \frac{a, a_1, a_2}{r, r_1, r_2} \right\rangle_n C_{\phi_1 \phi_2 O}^{\mu_1 \dots \mu_k}(x_1 - x_2, \partial_{y_1}) O_{r[a]}^{\mu_1 \dots \mu_k}(y_1)$$

(group decomposition $r_1 \otimes r_2 \rightarrow r \oplus \dots$)

Exercise : show $\alpha_{\phi_1 \phi_2 O}$ in the OPE is the same $\alpha_{\phi_1 \phi_2 O}$ in the definition of 3pt.

↔

Bootstrap equations

$$r_1 \otimes r_2 \rightarrow R \oplus \dots$$

Define crossing matrix “ M ” by $P_{r_3 r_2 r_1 r_4}^R = \sum_S M_{RS} P_{r_1 r_2 r_3 r_4}^S$.

Define convolved block : $F_{\pm,0}^{12;34}(u, v) = v^{\frac{\Delta_2 + \Delta_3}{2}} g_{0}^{\Delta_{12}, \Delta_{34}}(u, v) \pm u^{\frac{\Delta_1 + \Delta_2}{2}} g_{0}^{\Delta_{32}, \Delta_{14}}(v, u)$

The bootstrap equation is

$$\pm \sum_R \sum_{O:R} \alpha_{\phi_3 \phi_2 O} \alpha_{\phi_1 \phi_4 O} (M^T)_{S,R} F_{\pm,0}^{32;14}(u, v) - \sum_{O:S} \alpha_{\phi_1 \phi_2 O} \alpha_{\phi_3 \phi_4 O} F_{\pm,0}^{12;34}(u, v) = 0$$

Special case $\phi_1 = \phi_3$: $\sum_R \sum_{O:R} \alpha_{\phi_1 \phi_2 O} \alpha_{\phi_1 \phi_4 O} (\pm M^T - \mathbb{1})_{S,R} F_{\pm,0}^{12;14}(u, v) = 0$

Note : M_{RS} depends on the choice of $\alpha_{\phi\phi_1}$

Exercise : derive above equations

CFT : 2-point, 3-point function

We define 3pt coefficient λ_{ijk} as

$$\langle \phi_{1,r|a|}(x) \phi_{2,s|b|}(y) \phi_{3,t|c|}^{\mu_1 \dots \mu_\ell}(z) \rangle = \lambda_{123} \left\{ \frac{c}{t} \middle| \frac{a,b}{r,s} \right\} \frac{Z^{\mu_1 \dots \mu_\ell} \text{-trace}}{|x-y|^{\Delta_i + \Delta_j - (\Delta_k - \ell)} |y-z|^{\Delta_j + (\Delta_k - \ell) - \Delta_i} |z-x|^{\Delta_k - \ell + \Delta_i - \Delta_j}}, \quad Z^\mu = \frac{(x-z)^\mu}{(x-z)^2} - \frac{(y-z)^\mu}{(y-z)^2}$$

$\phi_{1,r|a|}(x)$: ϕ is in representation r with indices a .

We define 3pt coefficient α_{ijk} as $\lambda_{ijk} \left\{ \frac{c}{t} \middle| \frac{a,b}{r,s} \right\} = \alpha_{ijk} \left\langle \frac{a,b,c}{r,s,t} \right\rangle$, i.e. $\alpha_{ijk} = \lambda_{ijk} \sqrt{\dim(\phi_k)}$

2pt function :

$$\langle \phi_{A|a|}(x) \phi_{B|b|}(y) \rangle = \langle \phi_{A|a|}(x) \phi_{B|b|}(y) \mathbb{1} \rangle = \lambda_{\phi\phi\mathbb{1}} \left\{ \frac{\mathbb{1}}{S} \middle| \frac{a,b}{A,B} \right\} \frac{\delta_{AB}}{|x-y|^{2\Delta}} = \frac{\alpha_{\phi\phi\mathbb{1}}}{\sqrt{\dim(A)}} \frac{\delta_{AB} \delta_a^{\{A\}}}{|x-y|^{2\Delta}}$$

(difference choices of $\alpha_{\phi\phi\mathbb{1}}$ lead to different conventions.)

OPE coefficient conventions

Let's now fix $\alpha_{\phi\phi 1}$. Different choices of $\alpha_{\phi\phi 1}$ correspond to different OPE coefficient conventions.

$$\text{For } \alpha_{s s 1} = 1, \quad P_{i j k l}^{s,(\text{sim})} = \frac{1}{\sqrt{\dim(s)}} \sum_b \left\langle \frac{b}{s} \left| \frac{i, j}{r_1, r_2} \right. \right\rangle \left\langle \frac{b}{s} \left| \frac{k, l}{r_3, r_4} \right. \right\rangle = \sqrt{\dim(s)} \sum_b \left\langle \frac{b, i, j}{s, r_1, r_2} \right\rangle \left\langle \frac{b, k, l}{s, r_3, r_4} \right\rangle$$

$$\text{For } \alpha_{s s 1} = \frac{1}{\sqrt{\dim(s)}}, \quad P_{i j k l}^{s,(\text{prj})} = \sum_b \left\langle \frac{b}{s} \left| \frac{i, j}{r_1, r_2} \right. \right\rangle \left\langle \frac{b}{s} \left| \frac{k, l}{r_3, r_4} \right. \right\rangle$$

$$\text{For } \alpha_{s s 1} = \sqrt{\dim(s)}, \quad P_{i j k l}^{s,(\text{aut})} = \sum_b \left\langle \frac{b, i, j}{s, r_1, r_2} \right\rangle \left\langle \frac{b, k, j}{s, r_3, r_4} \right\rangle$$

$$\text{OPEs : } \alpha_{\text{abc, sim}}^2 = \sqrt{\dim(c)} \alpha_{\text{abc, prj}}^2, \quad \alpha_{\text{abc, aut}}^2 = \dim(c) \alpha_{\text{abc, prj}}^2, \quad \alpha_{\text{abc, sim}}^2 = \frac{1}{\sqrt{\dim(c)}} \alpha_{\text{abc, aut}}^2$$

$$\alpha_{\phi_1 \phi_2 \phi_3}^{\text{aut}} = \alpha_{\phi_1 \phi_2 \phi_3}^{\text{sim}} \dim(\phi_1)^{1/4} \dim(\phi_2)^{1/4} \dim(\phi_3)^{1/4}$$



autoboot

$\left\langle \frac{a,b,c}{r,s,t} \right\rangle$ is an invariant tensor :

$$\begin{array}{ccc}
 \phi_a^{(r)} \otimes \phi_b^{(s)} & \xrightarrow{\left\langle \frac{a,b,c}{r,s,t} \right\rangle} & \phi_c^{(t)} \\
 \downarrow g_{aa'}^{(r)} & & \downarrow g_{cc'}^{(t)} \\
 \phi_{a'}^{(r)} \otimes \phi_{b'}^{(s)} & \xrightarrow{\left\langle \frac{a',b',c'}{r,s,t} \right\rangle} & \phi_{c'}^{(t)} \\
 \downarrow g_{bb'}^{(s)} & & \downarrow g_{cc'}^{(t)}
 \end{array}$$

Example : $\phi^{(S)} = \delta_{a,b} \phi^{(V)}_a \phi^{(V)}_b$

$$g_{a,a'}^{(r)} g_{b,b'}^{(s)} g_{c,c'}^{(t)} \left\langle \frac{a',b',c'}{r,s,t} \right\rangle = \left\langle \frac{a,b,c}{r,s,t} \right\rangle \quad (g_{a,a'}^{(r)} : \text{matrix representation of any group element in rep } r)$$

$$G_{a,a'}^{(r)} \left\langle \frac{a',b',c'}{r,s,t} \right\rangle = 0 \quad (G_{a,a'}^{(r)} : \text{matrix representation of any generators in rep } r)$$

Autoboot : treat $\left\langle \frac{a,b,c}{r,s,t} \right\rangle$ as explicitly $\dim(r) \times \dim(s) \times \dim(t)$ rank-3 tensor. Solve above equation explicitly.

Work well for small number of generator and small rep dimension.

I

Good for discrete group and small lie group $O(2)$, $O(3)$.

Actually computation of V_R : tutorial this afternoon

Compute 3j symbols : a simple example

Take $O(n)$ $v \otimes v \rightarrow t$ as an example

Step 1 : write down generalized delta function $\delta_{i,j}^{(V)}$, $\delta_{(i j), (k l)}^{(T)}$

$$\delta_{i,j}^{(V)} = \delta_{ij} \quad , \quad \delta_{(i j), (k l)}^{(T)} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{1}{n} \delta_{ij} \delta_{kl}$$

For generic reps R : $\delta_{i,j}^{(R)} = Y_j \circ Y_i \circ \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \dots$ where Y_i is Young symmetrizer operating on i indices and subtract traces.

Example : $\delta_{(i j), (k l)}^{(T)} = \delta_{(i}^{(k} \delta_{j)}^{l)} - x \delta_{ij} \delta_{kl}$, where $x = \frac{1}{n}$ is fixed by $\delta^{ij} \delta_{(i j), (k l)}^{(T)} = 0$

Exercise : proof for traceless symmetric rank- ℓ tensor, the generalized delta is

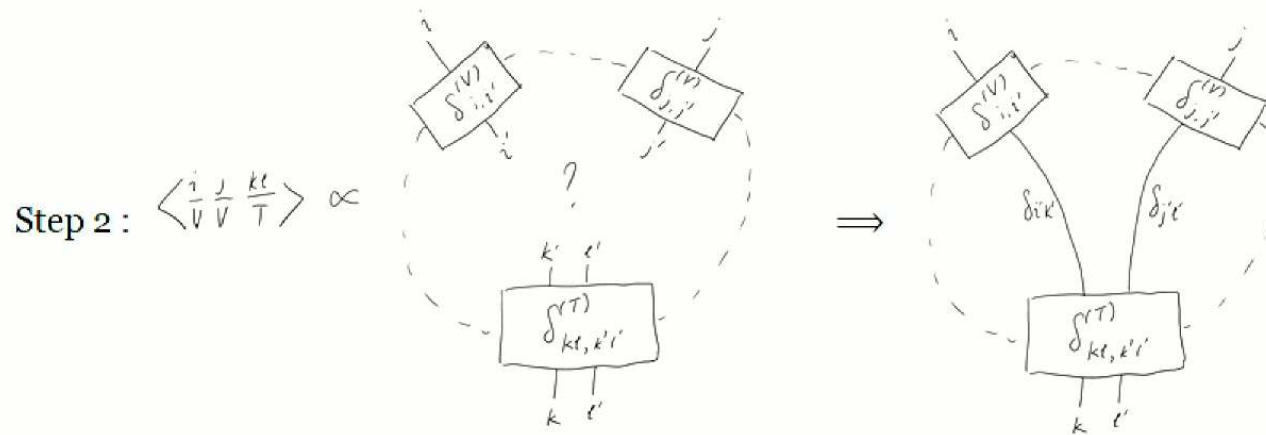
$$\delta_{a_1 \dots a_\ell}^{b_1 \dots b_\ell} = \delta_{a_1}^{b_1} \dots \delta_{a_\ell}^{b_\ell} + c_1 \delta_{(a_1 a_2} \delta^{(b_1 b_2} \delta_{a_3}^{b_3} \dots \delta_{a_\ell}^{b_\ell)} + c_2 \delta_{(a_1 a_2 a_3 a_4} \delta^{(b_1 b_2 b_3 b_4} \delta_{a_5}^{b_5} \dots \delta_{a_\ell}^{b_\ell)} + \dots$$

$$\text{where } c_m = \frac{(-1)^m \left(\frac{1}{2} - \frac{\ell}{2}\right)_m \left(-\frac{\ell}{2}\right)_m}{\Gamma(m+1) \left(-\frac{n}{2} - \ell + 2\right)_m}$$

⋮

Compute 3j symbols : a simple example

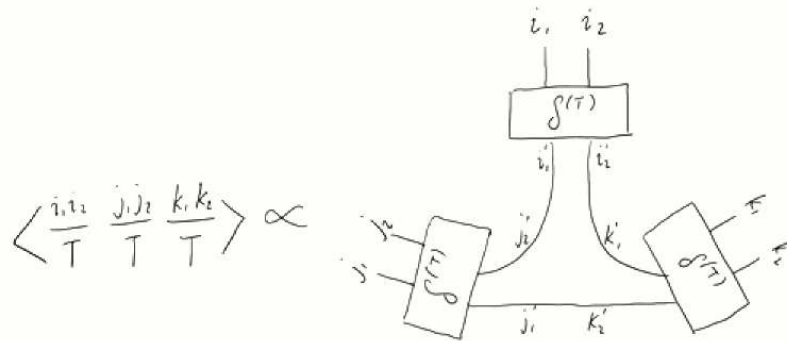
Take $O(n)$ $v \otimes v \rightarrow t$ as an example



$$\left\langle \frac{i}{V} \frac{j}{V} \frac{kl}{T} \right\rangle = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{1}{n} \delta_{ij} \delta_{kl} \quad (\text{overall factor is fixed by } \sum_{a,b,c} \left\langle \frac{a,b,c}{r,s,t} \right\rangle \left\langle \frac{a,b,c}{r,s,t} \right\rangle = 1)$$

⊖

Compute 3j symbols : $\langle T T T \rangle$



Exercise: compute $\langle \frac{i_1 i_2}{T} \frac{j_1 j_2}{T} \frac{k_1 k_2}{T} \rangle$

Answer :

$$\sqrt{\frac{(-2+n)(-1+n)(2+n)(4+n)}{8n}} \langle \frac{i_1 i_2}{T} \frac{j_1 j_2}{T} \frac{k_1 k_2}{T} \rangle = -\frac{1}{2n} \left((\sigma_{i_1, i_2} \sigma_{j_2, k_1} + \sigma_{i_1, k_1} \sigma_{i_2, k_2}) \sigma_{j_1, j_2} + \sigma_{i_1, i_2} (\sigma_{j_1, k_2} \sigma_{j_2, k_1} + \sigma_{j_1, k_1} \sigma_{j_2, k_2}) + (\sigma_{i_1, j_2} \sigma_{i_2, j_1} + \sigma_{i_1, j_1} \sigma_{i_2, j_2}) \sigma_{k_1, k_2} \right) +$$

$$\frac{1}{8} \left(\sigma_{i_1, j_2} (\sigma_{i_2, k_2} \sigma_{j_1, k_1} + \sigma_{i_2, k_1} \sigma_{j_1, k_2}) + \sigma_{i_1, k_2} (\sigma_{i_2, j_2} \sigma_{j_1, k_1} + \sigma_{i_2, j_1} \sigma_{j_2, k_1}) + \sigma_{i_1, k_1} (\sigma_{i_2, j_2} \sigma_{j_1, k_2} + \sigma_{i_2, j_1} \sigma_{j_2, k_2}) + \sigma_{i_1, j_1} (\sigma_{i_2, k_2} \sigma_{j_2, k_1} + \sigma_{i_2, k_1} \sigma_{j_2, k_2}) \right) +$$

$$\frac{2}{n^2} \sigma_{i_1, i_2} \sigma_{j_1, j_2} \sigma_{k_1, k_2}$$

⊖

Exercise : current and central charge

Traditionally we define the $O(n)$ current through the Ward identity :

$$\epsilon^{im} \langle \phi^m(x_1) \phi^j(x_2) \rangle = -\frac{\epsilon_{\alpha\beta}}{2} \oint_{x_1} dS_\mu \langle J_\mu^{\alpha\beta}(x) \phi^i(x_1) \phi^j(x_2) \rangle, \text{ where } \epsilon^{im} \text{ is } O(n) \text{ generator}$$

The normalization of $J_\mu^{\alpha\beta}$ is related to the current central charge through

$$\langle J_\mu^{ij}(x) J_\nu^{kl}(y) \rangle = \frac{\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}}{2} \frac{2 C_J}{(S_{d-1})^2} \frac{I_{\mu\nu}}{|x-y|^{2\Delta_J}} \text{ with } S_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

However in conformal bootstrap numerics, we normalization

$$\langle J_\mu^{ij}(x) J_\nu^{kl}(y) \rangle = \frac{\alpha_{JJ}}{\sqrt{\dim(J)}} \frac{\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}}{2} \frac{I_{\mu\nu}}{|x-y|^{2\Delta_J}}$$

Exercise :

- 1, Workout the coefficient (...) in $J_\mu^{ij} = (...) J_\mu^{ij}$.
- 2, Workout the relation between $\alpha_{\phi\phi J}$, $\alpha_{\tau\tau J}$ and C_J



Exercise : current and central charge

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- 1, Workout the coefficient (...) in $J_\mu^{ij} = (...) J_\mu^{ij}$.
- 2, Workout the relation between $\alpha_{\phi\phi J}$, $\alpha_{\mathbb{T}T J}$ and C_J



$O(N)$ $\{v,s,t\}$ bootstrap equations

All correlators :

$$\langle \phi\phi\phi\phi \rangle, \langle tttt \rangle, \langle ssss \rangle, \langle \phi s\phi s \rangle, \langle \phi\phi ts \rangle, \langle \phi\phi tt \rangle, \langle \phi\phi ss \rangle, \langle sstt \rangle, \langle sttt \rangle, \langle \phi\phi ts \rangle$$

Rep decomposition:

$$V \otimes V = S \oplus T \oplus A$$

$$T \otimes T = S \oplus T \oplus A \oplus Y_{2,2} \oplus Y_4 \oplus Y_{3,1}$$

$$T \otimes V = V \oplus Y_3 \oplus Y_{2,1}$$

3j symbols:

$$\langle V V S \rangle, \langle V V A \rangle, \langle V V T \rangle, \langle S S S \rangle,$$

$$\langle T T S \rangle, \langle T T T \rangle, \langle T T A \rangle, \langle T T Y_{2,2} \rangle, \langle T T Y_4 \rangle, \langle T T Y_{3,1} \rangle$$

$$\langle T V Y_3 \rangle, \langle T V Y_{2,1} \rangle$$

Bootstrap equations for $O(N)$:

$$V_{\text{identity}} + (\lambda_{\phi\phi t}, \lambda_{tts}, \lambda_{ttt}, \lambda_{sss}, \lambda_{\phi\phi s}) \cdot V_{\theta}(\lambda_{\phi\phi t}, \lambda_{tts}, \lambda_{ttt}, \lambda_{sss}, \lambda_{\phi\phi s}) + \sum_r \sum_{O:r} V_r = 0$$

Actually computation of V_R : tutorial this afternoon

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Correlators and theory space

Correlators:

$$O(2) : \langle \phi\phi\phi\phi \rangle, \langle tttt \rangle, \langle ssss \rangle, \langle \phi s\phi s \rangle, \langle \phi\phi ts \rangle, \langle \phi\phi tt \rangle, \langle \phi\phi ss \rangle, \langle sstt \rangle$$

$$O(3), O(4), \dots : \langle \phi\phi\phi\phi \rangle, \langle tttt \rangle, \langle ssss \rangle, \langle \phi s\phi s \rangle, \langle \phi\phi ts \rangle, \langle \phi\phi tt \rangle, \langle \phi\phi ss \rangle, \langle sstt \rangle, \langle sstt \rangle, \langle \phi\phi ts \rangle$$

Parameters to scan:

$$O(2) : \{\Delta_\phi, \Delta_s, \Delta_t\} \text{ and } \{\lambda_{\phi\phi t}, \lambda_{tts}, \lambda_{sss}\} / \lambda_{\phi\phi s}$$

$$O(3), O(4), \dots : \{\Delta_\phi, \Delta_s, \Delta_t\} \text{ and } \{\lambda_{\phi\phi t}, \lambda_{tts}, \lambda_{sss}, \lambda_{ttt}\} / \lambda_{\phi\phi s}$$

It's important to scan those OPEs to ensure $\{\phi, s, t\}$ are really the only one relevant operators

Naive scan : cost $\sim e^{\text{dimension}}$ (the curse of dimensionality)

Plan : Scan $\{\Delta_\phi, \Delta_s, \Delta_t\}$ using Delaunay search. Scan $\{\lambda_{\phi\phi t}, \lambda_{tts}, \lambda_{sss}, \lambda_{ttt}\}$ using cutting surface algorithm.

⌘

$O(N)$ $\{v,s,t\}$ bootstrap equations

All correlators :

$$\langle \phi\phi\phi\phi \rangle, \langle tttt \rangle, \langle ssss \rangle, \langle \phi s\phi s \rangle, \langle \phi\phi ts \rangle, \langle \phi\phi tt \rangle, \langle \phi\phi ss \rangle, \langle sstt \rangle, \langle sstt \rangle, \langle \phi\phi ts \rangle$$

Rep decomposition:

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3j symbols:

$$\langle V V S \rangle, \langle V V A \rangle, \langle V V T \rangle, \langle S S S \rangle,$$

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Actually computation of V_R : tutorial this afternoon

Correlators and theory space

Correlators:

$$O(2) : \langle \phi\phi\phi\phi \rangle, \langle tttt \rangle, \langle ssss \rangle, \langle \phi s\phi s \rangle, \langle \phi\phi ts \rangle, \langle \phi\phi tt \rangle, \langle \phi\phi ss \rangle, \langle sstt \rangle$$

$$O(3), O(4), \dots : \langle \phi\phi\phi\phi \rangle, \langle tttt \rangle, \langle ssss \rangle, \langle \phi s\phi s \rangle, \langle \phi\phi ts \rangle, \langle \phi\phi tt \rangle, \langle \phi\phi ss \rangle, \langle sstt \rangle, \langle sttt \rangle, \langle \phi\phi ts \rangle$$

Parameters to scan:

I

$$O(2) : \{\Delta_\phi, \Delta_s, \Delta_t\} \text{ and } \{\lambda_{\phi\phi t}, \lambda_{tts}, \lambda_{sss}\} / \lambda_{\phi\phi s}$$

$$O(3), O(4), \dots : \{\Delta_\phi, \Delta_s, \Delta_t\} \text{ and } \{\lambda_{\phi\phi t}, \lambda_{tts}, \lambda_{sss}, \lambda_{ttt}\} / \lambda_{\phi\phi s}$$

It's important to scan those OPEs to ensure $\{\phi, s, t\}$ are really the only one relevant operators

Naive scan : cost $\sim e^{\text{dimension}}$ (the curse of dimensionality)

Plan : Scan $\{\Delta_\phi, \Delta_s, \Delta_t\}$ using Delaunay search. Scan $\{\lambda_{\phi\phi t}, \lambda_{tts}, \lambda_{sss}, \lambda_{ttt}\}$ using cutting surface algorithm.

Why OPE scan

$$V_{\text{identity}} + (\lambda_{\phi\theta t}, \lambda_{tts}, \lambda_{ttt}, \lambda_{sss}, \lambda_{\phi\theta s}) \cdot V_{\theta}(\lambda_{\phi\theta t}, \lambda_{tts}, \lambda_{ttt}, \lambda_{sss}, \lambda_{\phi\theta s}) + \sum_r \sum_{O:r} V_r = 0$$

Without OPE scan: $\alpha \cdot V_{\text{identity}} = 1$, $\alpha \cdot V_{\theta} \geq 0$, $\alpha \cdot V_r \geq 0$

allowed region : $\{(\Delta_{\phi}, \Delta_t, \Delta_s) \mid \text{cannot find } \alpha \text{ at } (\Delta_{\phi}, \Delta_t, \Delta_s)\}$

With OPE scan: $\alpha \cdot V_{\text{identity}} = 1$, $\vec{\lambda} \cdot (\alpha \cdot V_{\theta}) \cdot \vec{\lambda} \geq 0$, $\alpha \cdot V_r \geq 0$

allowed region : $\{(\Delta_{\phi}, \Delta_t, \Delta_s, \vec{\lambda}) \mid \text{cannot find } \alpha \text{ at } (\Delta_{\phi}, \Delta_t, \Delta_s, \vec{\lambda})\}$

allowed in Δs : $\{(\Delta_{\phi}, \Delta_t, \Delta_s) \mid (\Delta_{\phi}, \Delta_t, \Delta_s, \vec{\lambda}) \text{ is allowed for some } \vec{\lambda}\}$

⊖

Why OPE scan

$$V_{\text{identity}} + \vec{\lambda} \cdot V_{\theta} \cdot \vec{\lambda} + \sum_r \sum_{O:r} V_r = 0$$

- OPE scan demands ϕ , s , t are really the only relevant operators. It's a stronger condition.

Without OPE scan: $\alpha \cdot V_{\text{identity}} = 1$, $\alpha \cdot V_{\theta} \geq 0$, $\alpha \cdot V_r \geq 0$

For some $\{\Delta_{\phi}, \Delta_s, \Delta_t\}$, there are solutions with two ϕ_1, ϕ_2 , $\Delta_{\phi_1} = \Delta_{\phi_2}$, $\lambda_{\phi_1 \phi_2 t} \neq \lambda_{\phi_1 \phi_2 t}$, $\lambda_{\phi_1 \phi_2 s} \neq \lambda_{\phi_1 \phi_2 s}$,

but no solution for $\Delta_{\phi_1} = \Delta_{\phi_2}$, $\lambda_{\phi_1 \phi_2 t} = \lambda_{\phi_1 \phi_2 t}$, $\lambda_{\phi_1 \phi_2 s} = \lambda_{\phi_1 \phi_2 s}$, i.e. $\phi_1 = \phi_2$

Without OPE scan: cannot find α

With OPE scan : can find α

- ... (Thursday tutorial)



Cutting surface algorithm : a prototype example

Parameters : $\{\Delta_\sigma, \Delta_\epsilon, x = \lambda_{\sigma\sigma\epsilon}/\lambda_{\epsilon\epsilon\epsilon}\}$

SDP in feasibility mode : find a linear functional $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ such that

$$(1) : \alpha \cdot \left((1 \ 1) \cdot V_{\text{even}}(\Delta = 0, \ell = 0) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = 1$$

$$\alpha \cdot V_{\text{even}}(\Delta, \ell) \geq 0 \text{ for } \Delta \geq 3, \ell = 0$$

$$\alpha \cdot V_{\text{even}}(\Delta, \ell) \geq 0 \text{ for } \Delta \geq \Delta_{\text{unitary}}, \ell = 2, 4, 6, \dots$$

$$\alpha \cdot V_{\text{odd}}(\Delta, \ell) \geq 0 \text{ for } \Delta \geq 3, \ell = 0$$

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$$\alpha \cdot V_{\text{odd}}(\Delta, \ell) \geq 0 \text{ for } \Delta \geq \Delta_{\text{unitary}}, \ell = 1, 2, 3, 4, \dots$$

$$(2) : \alpha \cdot \left((x \ 1) \cdot V_{\text{even}}(\Delta = \Delta_\epsilon, \ell = 0) \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} + x^2 V_{\text{odd}}(\Delta = \Delta_\sigma, \ell = 0) \right) \geq 0$$

Assuming we find such α at $\{\Delta_{\sigma,0}, \Delta_{\epsilon,0}, x_0\}$, constraint (2) has the form $(x \ 1) \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \geq 0$.

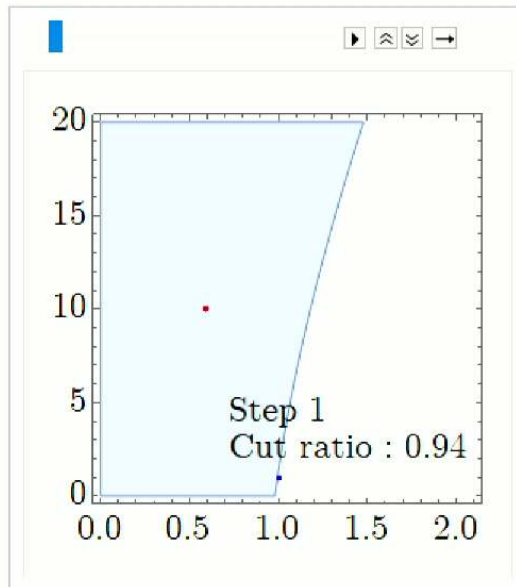
(m depends on $\Delta_{\sigma,0}, \Delta_{\epsilon,0}$ and α , but not x_0)

Solve this constraints, one find α concludes a wide range of points in x !

Cutting surface algorithm : visualization for 2 OPE coeffs

Fix $\{\Delta_i\}$. Scan $\vec{x} = \{\lambda_1, \dots, \lambda_n\}$, assuming a bounding box $\lambda_{\phi\phi t}^{(\min)} \leq \lambda_{\phi\phi t} \leq \lambda_{\phi\phi t}^{(\max)}$

Given a functional α at $\vec{x} = (\lambda_1, \dots, \lambda_n) \rightarrow$ quadratic constraints $\vec{x} \cdot (\alpha \cdot V_\theta) \cdot \vec{x} \leq 0$ ($\alpha \cdot V_\theta : n \times n$ matrix)



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White region : ruled out

Blue region : undetermined (yet to be scanned)

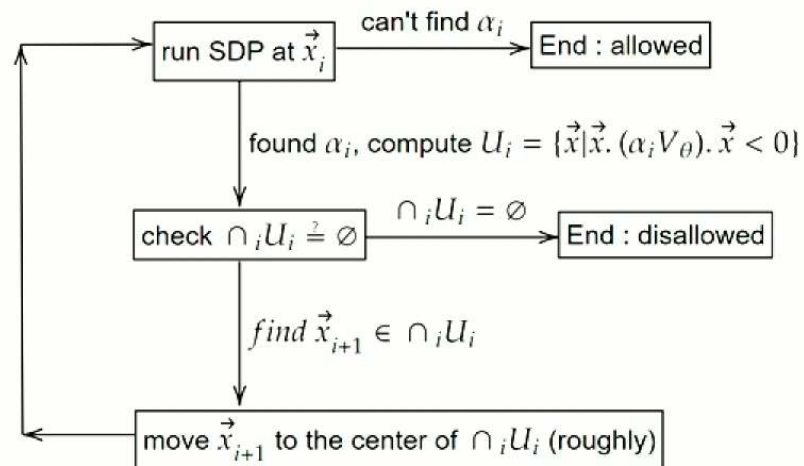
Cut ratio ~ 0.5

Steps $\propto \log(\text{volume}) \propto \text{dim of the space}$

Cutting surface algorithm

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Given a functional α_i at the i -th step, defined $U_i = \{\vec{x} \mid \vec{x} \cdot (\alpha \cdot V_\theta) \cdot \vec{x} \leq 0\}$ (region allowed by α_i)



Cutting surface algorithm : subtleties

- Unfortunately checking $\bigcap U_i \stackrel{?}{=} \emptyset$ is NP-hard!

In practice : with small bounding box, we have effective heuristics to find a point in $\bigcap U_i$

For example : \vec{x}_i often sit slightly outside the boundary of $\bigcap U_i$, i.e.

Let $f^{(n)}(\vec{x}) = \vec{x} \cdot (\alpha^{(n)} \cdot V_\theta) \cdot \vec{x}$, $f^{(1)}(\vec{x}_i) \leq 0$, $f^{(2)}(\vec{x}_i) \leq 0$, ... except $f^{(j)}(\vec{x}_i) = \epsilon$ for small ϵ .

We may try to minimize $f^{(j)}(\vec{x}_i)$ while keep the rest $f^{(m)}(\vec{x}_i) \leq 0$

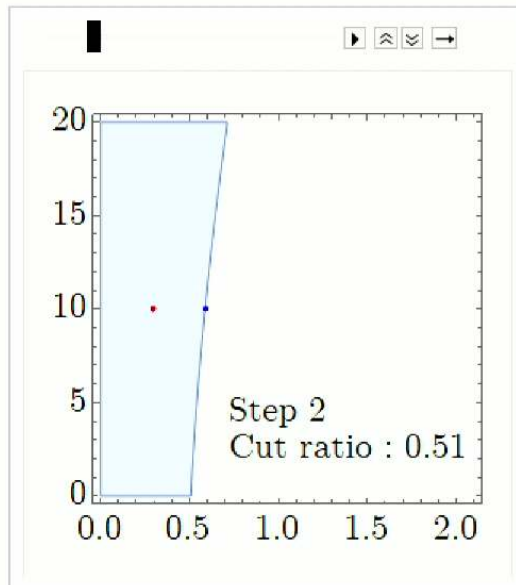
- Given $\vec{x}_{i+1} \in \bigcap_i U_i$, how to move \vec{x}_{i+1} to the center of $\bigcap_i U_i$?

starting with $\vec{x}_{i+1}^{(0)} \in \bigcap U_i$, one may draw a line interval over $\vec{x}_{i+1}^{(0)}$ and ends on the boundary of $\bigcap U_i$. Move $\vec{x}_{i+1}^{(0)}$ to $\vec{x}_{i+1}^{(1)}$ at the center of the interval.

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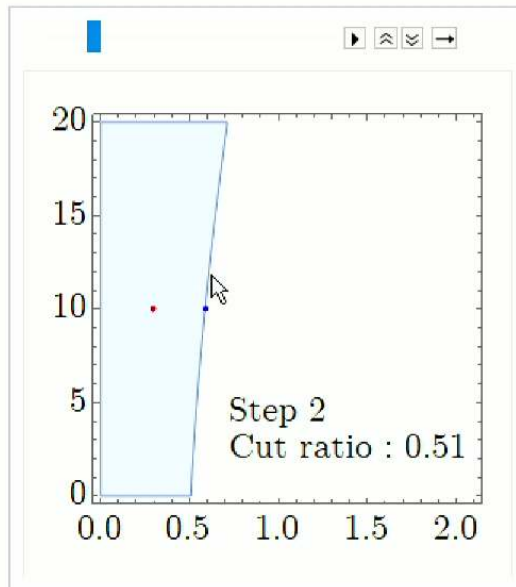
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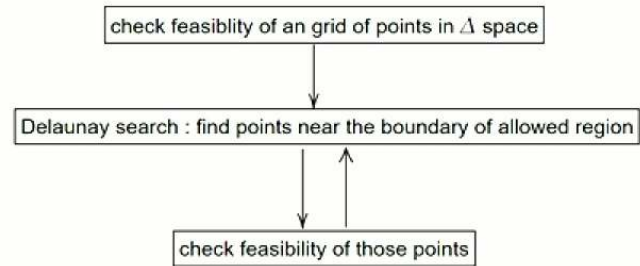
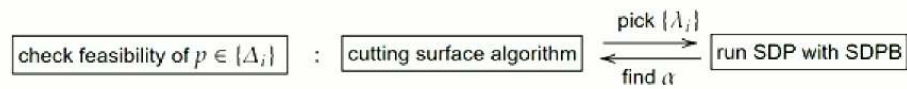
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Summary of the search algorithm

Theory space : $\{\Delta_i\}, \{\lambda_i\}$



Thank you

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