Title: Lecture 2: Bootstrapping global symmetries. Cutting surface algorithm

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Collection: Mini-Course of Numerical Conformal Bootstrap

Date: April 25, 2023 - 10:00 AM

URL: https://pirsa.org/23040139

Plan for today

- Derivation of O(N) bootstrap equations for all four-point functions involving $\{v = \phi_i,\ s = \phi^2,\ t = \phi_i\,\phi_j\}$
- Cutting surface algorithm
- Tutorial A : Learn how to use autoboot, how to use projectors, run cutting surface algorithm
- Tutorial B : Using Hyperion to bootstrap 3d Ising

Group theory: CG coeff, 3j symbol

[Reference : autoboot arXiv:1903.10522, <<Birdtracks, Lie's, and Exceptional Groups>> Predrag Cvitanovic]

Consider a group with real representations generic reps $s,\ t,\ r,\ \ldots$

Tensor product decomposition : $s \otimes t = r_1{}^{n_1} \oplus r_2{}^{n_2} \oplus r_3{}^{n_3} \oplus \ldots$, (n_i : multiplicity)

Clebsch Gordan coefficient : decomposition $r \otimes s \to t^n$ is given explicitly by $\left\{ \frac{c}{t} \middle| \frac{a,b}{r,s} \right\}_{i=1,\ldots n}$, (a : indices of rep r, etc.) i.e. $\phi_c^{(t)} = \left\{ \frac{c}{t} \middle| \frac{a,b}{r,s} \right\} \phi_a^{(r)} \phi_b^{(s)}$

Normalization of CG coeff : $\sum_{a,b} \left\{ \frac{c}{t} \middle| \frac{a,b}{r,s} \right\}_n \left\{ \frac{c'}{t'} \middle| \frac{a,b}{r,s} \right\}_{n'} = \delta_{tt'} \delta_{nn'} \delta_{cc'}$

Define the 3j symbol : $\left\langle \frac{a,b,c}{r,s,t} \right\rangle_n = \frac{1}{\sqrt{\dim(t)}} \left\{ \frac{c}{t} \middle| \frac{a,b}{r,s} \right\}_n$, normalization: $\sum_{a,b,c} \left\langle \frac{a,b,c}{r,s,t} \right\rangle_n \left\langle \frac{a,b,c}{r,s,t} \right\rangle_n = 1$

Example : $O(n)$ group with S (singlet), V (vector) reps:

CG coefficient: $\left\{ \frac{i}{V}\ \frac{j}{V} \middle| \frac{\square}{S} \right\} = \frac{1}{\sqrt{n}} \delta_{ij}$, 3j symbol : $\left\langle \frac{i\ \ j\ \ \square}{V\ V\ V} \right\rangle = \frac{1}{\sqrt{n}} \delta_{ij}$, $i,\ j$: $O(n)$ vector indices

Exercise : check $\left\langle \frac{i\ \ j\ \ \square}{V\ V\ V} \right\rangle$ is symmetric (In general, 3j symbol is symmetric / antisymmetric)

Group theory: CG coeff, 3j symbol

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Exercise : check $\left\langle \frac{i\ j\ \square}{V\ V\ S} \right\rangle$ is symmetric (In general, 3j symbol is symmetric / antisymmetric)

Group theory: generalized delta

Let $i,\ j,\ k,\ \ldots$ be the $O(n)$ vector indices. We can use the vector indices to label an arbitrary rep R in $O(n)$

"generalized delta function" $\delta^{(R)}_{i,j}$: $\delta^{(R)}_{i,j}\,\phi^{(R)}_j = \phi^{(R)}_j$, $\boldsymbol{i} = \{i_1,\ \ldots\ i_n\}$ and $\sum_j\ = \sum_{j_1}\sum_{j_2}\ldots$

Properties : $\delta^{(R)}_{i,j}\,\delta^{(R)}_{j,k} = \delta^{(R)}_{i,k}$ and $\delta^{(R)}_{j,j} = \dim(R)$

$\delta^{(R)}_{i,j}$: projection $V\otimes V\otimes V\ldots\to R$

Example : traceless symmetric rank-2 tensor T : $\delta^{(T)}_{(i\,j),(k\,l)} = \frac{1}{2}\left(\delta_{i\,k}\,\delta_{j\,l} + \delta_{i\,l}\,\delta_{j\,k}\right) - \frac{1}{n}\,\delta_{i\,j}\,\delta_{k\,l}$

Example : $\delta^{(T)}_{(i\,j),(k\,l)}\,\phi^i\,\phi^j = \phi_k\,\phi_l - \frac{1}{n}\,\phi^i\,\phi_i\,\delta_{k\,l}$ is a traceless symmetric rank-2 tensor

Exercise : $\left\{\frac{i}{R}\ \frac{j}{R}\,\middle|\,\frac{\square}{S}\right\} = \frac{1}{\sqrt{\dim(R)}}\,\delta^{(R)}_{i\,j}$, $\left\langle\frac{i\ j\ \square}{R\,R\,S}\right\rangle = \frac{1}{\sqrt{\dim(R)}}\,\delta^{(R)}_{i\,j}$ for any rep R

CFT : 2-point, 3-point function

We define 3pt coefficient $\lambda_{i\,j\,k}$ as

$$\left\langle \phi_{1,r|a|}(x)\,\phi_{2,s|b|}(y)\,\phi_{3,t|c|}^{\mu_1\cdots\mu_\ell}(z)\right\rangle = \lambda_{123}\left\{\frac{c}{t}\,\middle|\,\frac{a,b}{r,s}\right\}\frac{Z^{\mu_1}\dots Z^{\mu_\ell}-\text{trace}}{|x-y|^{\Delta_i+\Delta_j-(\Delta_k-\ell)}\,|y-z|^{\Delta_j+(\Delta_k-\ell)-\Delta_i}\,|z-x|^{(\Delta_k-\ell)+\Delta_i-\Delta_j}}\;,\; Z^\mu = \frac{(x-z)^\mu}{(x-z)^2}-\frac{(y-z)^\mu}{(y-z)^2}$$

$\phi_{1,r|a|}(x)$: ϕ is in representation r with indices a.

We define 3pt coefficient $\alpha_{i\,j\,k}$ as $\lambda_{i\,j\,k}\left\{\frac{c}{t}\,\middle|\,\frac{a,b}{r,s}\right\} = \alpha_{i\,j\,k}\left\langle\frac{a,b,c}{r,s,t}\right\rangle$, i.e. $\alpha_{i\,j\,k} = \lambda_{i\,j\,k}\sqrt{\dim(\phi_k)}$

2pt function :

$$\langle\phi_{A|a|}(x)\,\phi_{B|b|}(y)\rangle = \langle\phi_{A|a|}(x)\,\phi_{B|b|}(y)\,\mathbb{1}\rangle = \lambda_{\phi\phi 1}\left\{\frac{\square}{S}\,\middle|\,\frac{a,b}{A,B}\right\}\frac{\delta_{AB}}{|x-y|^{2\Delta}} = \frac{\alpha_{\phi\phi 1}}{\sqrt{\dim(\phi)}}\,\frac{\delta_{AB}\,\delta^{(A)}_{a\,b}}{|x-y|^{2\Delta}}$$

(difference choices of $\alpha_{\phi\phi 1}$ lead to different conventions.)

CFT : OPE

Define $C^{\nu_1 \cdots \nu_\ell}(x, -\partial_z)$ by

$$\frac{Z^{\mu_1} \dots Z^{\mu_\ell} - \text{trace}}{|x|^{\Delta_i+\Delta_j-(\Delta_k-\ell)}\, |z|^{\Delta_j+(\Delta_k-\ell)-\Delta_i}\, |z-x|^{(\Delta_k-\ell)+\Delta_i-\Delta_j}} = C^{\nu_1 \cdots \nu_\ell}(x, -\partial_z)\, \frac{I^{(\mu_1}_{\nu_1} \dots I^{\mu_\ell)}_{\nu_\ell} - \text{trace}}{|z|^{2\Delta_k}} \quad , \quad I^\mu_\nu = \delta^\mu_\nu - 2\,\frac{(y-z)^\mu (y-z)_\nu}{(y-z)^2}$$

$$\text{OPE} : \phi_{1,r_1[a_1]}(x_1) \times \phi_{2,r_2[a_2]}(x_2) = \sum_r \sum_{O:r} \sum_n \frac{\sqrt{\dim(O)}}{\alpha_{OO1}}\, \alpha^{(n)}_{\phi_1\,\phi_2\,O} \sum_a \left\langle \frac{a,\, a_1,\, a_2}{r,\, r_1,\, r_2} \right\rangle_n C^{\mu_1 \cdots \mu_k}_{\phi_1\,\phi_2\,O}\left(x_1 - x_2,\, \partial_{y_1}\right) O^{\mu_1 \cdots \mu_k}_{r[a]}(y_1)$$

(group decomposition $r_1 \otimes r_2 \to r \oplus \dots$)

Exercise : show $\alpha_{\phi_1\,\phi_2\,O}$ in the OPE is the same $\alpha_{\phi_1\,\phi_2\,O}$ in the definition of 3pt.

Group theory : 4pt structure

For simplicity, let's assume there is no multiplicity.

Define 4pt structure as

$$P^{s}_{r_1[a_1]\,r_2[a_2]\,r_3[a_3]\,r_4[a_4]} := \sum_b \left\{ \begin{smallmatrix} b \\ s \end{smallmatrix} \middle| \begin{smallmatrix} a_1, a_2 \\ r_1, r_2 \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} b \\ s \end{smallmatrix} \middle| \begin{smallmatrix} a_3, a_4 \\ r_3, r_4 \end{smallmatrix} \right\} \frac{1}{\sqrt{\dim(s)}} \frac{1}{\alpha_{s\,s\,1}}$$

Taking $\alpha_{s\,s\,1} = \frac{1}{\sqrt{\dim(s)}}$ (in this case P is called 4pt projector), P has the following properties:

1, (completeness), $\left(P^{s_1}_{r_1\,r_2\,r_1\,r_2} + P^{s_2}_{r_1\,r_2\,r_1\,r_2} + \ldots P^{s_N}_{r_1\,r_2\,r_1\,r_2}\right)_{a_1\,a_2\,a_3\,a_4} = \delta^{(r_1)}_{a_1\,a_3}\,\delta^{(r_2)}_{a_2\,a_4}$

2, (projection), $\sum_{a_3\,a_4} \left(P^{r}_{r_1\,r_2\,r_3\,r_4}\right)_{a_1\,a_2\,a_3\,a_4} \left(P^{s}_{r_3\,r_4\,r_5\,r_6}\right)_{a_3\,a_4\,a_5\,a_6} = \delta_{r\,s} \left(P^{s}_{r_1\,r_2\,r_5\,r_6}\right)_{a_1\,a_2\,a_5\,a_6}$

3, (orthogonality), $\sum_{a_1\,a_2\,a_3\,a_4} \left(P^{r}_{r_1\,r_2\,r_3\,r_4}\right)_{a_1\,a_2\,a_3\,a_4} \left(P^{s}_{r_1\,r_2\,r_3\,r_4}\right)_{a_1\,a_2\,a_3\,a_4} = \delta_{r\,s}\,\dim(r)$

Exercise : proof above properties using the properties of $\left\{ \begin{smallmatrix} b \\ s \end{smallmatrix} \middle| \begin{smallmatrix} a_1, a_2 \\ r_1, r_2 \end{smallmatrix} \right\}$

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Normalization of CG coeff : $\sum_{a,b} \left\{ \begin{smallmatrix} c \\ t \end{smallmatrix} \middle| \begin{smallmatrix} a,b \\ r,s \end{smallmatrix} \right\}_n \left\{ \begin{smallmatrix} c' \\ t' \end{smallmatrix} \middle| \begin{smallmatrix} a,b \\ r,s \end{smallmatrix} \right\}_{n'} = \delta_{tt'}\, \delta_{nn'}\, \delta_{cc'}$

Define the 3j symbol : $\left\langle \begin{smallmatrix} a,b,c \\ r,s,t \end{smallmatrix} \right\rangle_n = \frac{1}{\sqrt{\dim(t)}} \left\{ \begin{smallmatrix} c \\ t \end{smallmatrix} \middle| \begin{smallmatrix} a,b \\ r,s \end{smallmatrix} \right\}_n$, normalization: $\sum_{a,b,c} \left\langle \begin{smallmatrix} a,b,c \\ r,s,t \end{smallmatrix} \right\rangle_n \left\langle \begin{smallmatrix} a,b,c \\ r,s,t \end{smallmatrix} \right\rangle_n = 1$

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Exercise : check $\left\langle \begin{smallmatrix} i\ j\ \square \\ V\ V\ S \end{smallmatrix} \right\rangle$ is symmetric (In general, 3j symbol is symmetric / antisymmetric)

CFT : 4pt function

4pt function:

$\langle \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3)\, \phi_4(x_4) \rangle =$

$$\frac{1}{|x_{12}|^{\Delta_1+\Delta_2}\, |x_{34}|^{\Delta_3+\Delta_4}} \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{12}} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{34}} \sum_s \sum_{\mathcal{O}:s} \sum_{n\,m}\, \alpha^{n}_{\phi_1\,\phi_2\,\mathcal{O}}\, \alpha^{m}_{\phi_3\,\phi_4\,\mathcal{O}} \sum_b \left\langle \frac{b, a_1, a_2}{s, r_1, r_2} \right\rangle \left\langle \frac{b, a_3, a_4}{s, r_3, r_4} \right\rangle \frac{\sqrt{\dim(\mathcal{O})}}{\alpha_{\mathcal{O}\mathcal{O}1}}\, g_{\mathcal{O}}^{\Delta_{12},\Delta_{34}}(u,\, v)$$

where

$$g_{\mathcal{O}}^{\Delta_{12},\Delta_{34}}(u,\, v) = \left|\frac{x_{24}}{x_{14}}\right|^{-\Delta_{12}} \left|\frac{x_{14}}{x_{13}}\right|^{-\Delta_{34}} C_{\phi_1\,\phi_2\,\mathcal{O},k}\big(x_{12},\, \partial_{y_1}\big)\, C_{\phi_3\,\phi_4\,\mathcal{O},j}\big(x_{34},\, \partial_{y_2}\big)\, \frac{I^{i\,j}(y_{12})}{|y_{12}|^{2\,\Delta}} \quad , \; x_{i\,j} = x_i - x_j$$

(Convention : Poland, Rychkov, Vichi (2018) Row 1 of Table I)

$\mathcal{N}_{d,\ell}$	Reference
$\frac{\ell!}{(-2)^\ell (d/2-1)_\ell}$	Dolan and Osborn (2001b, 2004), Rattazzi *et al.* (2008), Penedones *et al.* (2016), **this review** ★
$\frac{\ell!}{(d-2)_\ell}$	Dolan and Osborn (2011), Hogervorst and Rychkov (2013), El-Showk *et al.* (2012, 2014b), Costa *et al.* (2016b), JuliBoots (Paulos, 2014b), cboot (Ohtsuki, 2016)
$\frac{(-1)^\ell \ell!}{4^\Delta (d/2-1)_\ell}$	Kos *et al.* (2014a, 2015b, 2016), Li *et al.* (2017b) PyCFTBoot (Behan, 2017a)
$\frac{\ell!}{(d/2-1)_\ell}$	Poland *et al.* (2012), Poland and Stergiou (2015)
$\frac{\ell!}{4^\Delta (d-2)_\ell}$	Kos *et al.* (2014b) Mathematica notebook (Simmons-Duffin, 2015b)
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TABLE I Summary of various conformal block normalizations $\mathcal{N}_{d,\ell}$, Eqs. (52, 62), used in the literature.

Exercise : plug in OPE and derive above expansion.

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4pt function:

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\,\phi_4(x_4)\rangle =$$

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where

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Define $C^{\nu_1 \cdots \nu_\ell}(x, -\partial_z)$ by

$$\frac{Z^{\mu_1} \dots Z^{\mu_\ell} - \text{trace}}{|x|^{\Delta_i+\Delta_j-(\Delta_k-\ell)} \, |z|^{\Delta_j+(\Delta_k-\ell)-\Delta_i} \, |z-x|^{(\Delta_k-\ell)+\Delta_i-\Delta_j}} = C^{\nu_1 \cdots \nu_\ell}(x, -\partial_z) \, \frac{I_{\nu_1}^{(\mu_1} \dots I_{\nu_\ell}^{\mu_\ell)} - \text{trace}}{|z|^{2\Delta_k}} \quad , \quad I_\nu^\mu = \delta_\nu^\mu - 2\,\frac{(y-z)^\mu (y-z)_\nu}{(y-z)^2}$$

OPE : $\phi_{1,r_1[a_1]}(x_1) \times \phi_{2,r_2[a_2]}(x_2) = \sum_r \sum_{O:r} \sum_n \frac{\sqrt{\dim(O)}}{\alpha_{OO1}} \, \alpha^{(n)}_{\phi_1 \phi_2 O} \sum_a \left\langle \frac{a,\, a_1, a_2}{r,\, r_1, r_2} \right\rangle_n C^{\mu_1 \cdots \mu_k}_{\phi_1 \phi_2 O}\big(x_1 - x_2,\, \partial_{y_1}\big)\, O^{\mu_1 \cdots \mu_k}_{r[a]}(y_1)$

(group decomposition $r_1 \otimes r_2 \to r \oplus \dots$)

Exercise : show $\alpha_{\phi_1 \phi_2 O}$ in the OPE is the same $\alpha_{\phi_1 \phi_2 O}$ in the definition of 3pt.

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Bootstrap equations

$r_1 \otimes r_2 \to R \oplus \ldots$

Define crossing matrix "M" by $P^S_{r_3\, r_2\, r_1\, r_4} = \sum_R M_{R\,S}\, P^S_{r_1\, r_2\, r_3\, r_4}$.

Define convolved block : $F^{12;34}_{\pm,O}(u,\ v) = v^{\frac{\Delta_2+\Delta_3}{2}}\ g_O^{\Delta_{12},\Delta_{34}}(u,\ v) \pm u^{\frac{\Delta_1+\Delta_2}{2}}\ g_O^{\Delta_{32},\Delta_{14}}(v,\ u)$

The bootstrap equation is

$\pm \sum_R \sum_{O:R} \alpha_{\phi_3\,\phi_2\,O}\, \alpha_{\phi_1\,\phi_4\,O}\, (M^T)_{S,R}\, F^{32;14}_{\pm,O}(u,\ v) - \sum_{O:S} \alpha_{\phi_1\,\phi_2\,O}\, \alpha_{\phi_3\,\phi_4\,O}\, F^{12;34}_{\pm,O}(u,\ v) = 0$

Special case $\phi_1 = \phi_3$: $\sum_R \sum_{O:R} \alpha_{\phi_1\,\phi_2\,O}\, \alpha_{\phi_1\,\phi_4\,O}\, (\pm M^T - \mathbb{1})_{S,R}\, F^{12;14}_{\pm,O}(u,\ v) = 0$

Note : $M_{R\,S}$ depends on the choice of $\alpha_{\phi\phi 1}$

Exercise : derive above equations

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OPE coefficient conventions

Let's now fix $\alpha_{\phi\phi 1}$. Different choices of $\alpha_{\phi\phi 1}$ correspond to different OPE coefficient conventions.

For $\alpha_{s\,s\,1} = 1$, $P^{s,(\mathrm{sim})}_{i\,j\,k\,l} = \frac{1}{\sqrt{\dim(s)}} \sum_b \left\{ \frac{b}{s} \middle| \frac{i,j}{r_1,r_2} \right\} \left\{ \frac{b}{s} \middle| \frac{k,l}{r_3,r_4} \right\} = \sqrt{\dim(s)} \sum_b \left\langle \frac{b,i,j}{s,r_1,r_2} \right\rangle \left\langle \frac{b,k,l}{s,r_3,r_4} \right\rangle$

For $\alpha_{s\,s\,1} = \frac{1}{\sqrt{\dim(s)}}$, $P^{s,(\mathrm{prj})}_{i\,j\,k\,l} = \sum_b \left\{ \frac{b}{s} \middle| \frac{i,j}{r_1,r_2} \right\} \left\{ \frac{b}{s} \middle| \frac{k,l}{r_3,r_4} \right\}$

For $\alpha_{s\,s\,1} = \sqrt{\dim(s)}$, $P^{s,(\mathrm{aut})}_{i\,j\,k\,l} = \sum_b \left\langle \frac{b,i,j}{s,r_1,r_2} \right\rangle \left\langle \frac{b,k,j}{s,r_3,r_4} \right\rangle$

OPEs : $\alpha^2_{\mathrm{abc,\,sim}} = \sqrt{\dim(c)}\;\alpha^2_{\mathrm{abc,prj}}$, $\alpha^2_{\mathrm{abc,\,aut}} = \dim(c)\,\alpha^2_{\mathrm{abc,prj}}$, $\alpha^2_{\mathrm{abc,\,sim}} = \frac{1}{\sqrt{\dim(c)}}\,\alpha^2_{\mathrm{abc,\,aut}}$

$\alpha^{\mathrm{aut}}_{\phi_1\,\phi_2\,\phi_3} = \alpha^{\mathrm{sim}}_{\phi_1\,\phi_2\,\phi_3}\,\dim(\phi_1)^{1/4}\,\dim(\phi_2)^{1/4}\,\dim(\phi_3)^{1/4}$

CFT : OPE

Define $C^{\nu_1 \cdots \nu_\ell}(x, -\partial_z)$ by

$$\frac{Z^{\mu_1} \ldots Z^{\mu_\ell} - \text{trace}}{|x|^{\Delta_i+\Delta_j-(\Delta_k-\ell)}\, |z|^{\Delta_j+(\Delta_k-\ell)-\Delta_i}\, |z-x|^{(\Delta_k-\ell)+\Delta_i-\Delta_j}} = C^{\nu_1 \cdots \nu_\ell}(x, -\partial_z)\, \frac{I^{(\mu_1}_{\nu_1} \ldots I^{\mu_\ell)}_{\nu_\ell} - \text{trace}}{|z|^{2\Delta_k}} \quad , \quad I^\mu_\nu = \delta^\mu_\nu - 2\, \frac{(y-z)^\mu (y-z)_\nu}{(y-z)^2}$$

OPE : $\phi_{1,r_1[a_1]}(x_1) \times \phi_{2,r_2[a_2]}(x_2) = \sum_r \sum_{O:r} \sum_n \frac{\sqrt{\dim(O)}}{\alpha_{OO1}}\, \alpha^{(n)}_{\phi_1\, \phi_2\, O} \sum_a \left\langle \frac{a,\, a_1,\, a_2}{r,\, r_1,\, r_2} \right\rangle_n C^{\mu_1 \cdots \mu_k}_{\phi_1\, \phi_2\, O}\left(x_1 - x_2,\, \partial_{y_1}\right) O^{\mu_1 \cdots \mu_k}_{r[a]}(y_1)$

(group decomposition $r_1 \otimes r_2 \to r \oplus \ldots$)

Exercise : show $\alpha_{\phi_1\, \phi_2\, O}$ in the OPE is the same $\alpha_{\phi_1\, \phi_2\, O}$ in the definition of 3pt.

Bootstrap equations

$r_1 \otimes r_2 \to R \oplus \dots$

Define crossing matrix "M" by $P^{R}_{r_3\, r_2\, r_1\, r_4} = \sum_S M_{R\,S}\, P^{S}_{r_1\, r_2\, r_3\, r_4}$.

Define convolved block : $F^{12;34}_{\pm,O}(u,\ v) = v^{\frac{\Delta_2+\Delta_3}{2}}\ g_O^{\Delta_{12},\Delta_{34}}(u,\ v) \pm u^{\frac{\Delta_1+\Delta_2}{2}}\ g_O^{\Delta_{32},\Delta_{14}}(v,\ u)$

The bootstrap equation is

$\pm \sum_R \sum_{O:R} \alpha_{\phi_3\,\phi_2\,O}\, \alpha_{\phi_1\,\phi_4\,O}\, (M^T)_{S,R}\, F^{32;14}_{\pm,O}(u,\ v) - \sum_{O:S} \alpha_{\phi_1\,\phi_2\,O}\, \alpha_{\phi_3\,\phi_4\,O}\, F^{12;34}_{\pm,O}(u,\ v) = 0$

Special case $\phi_1 = \phi_3$: $\sum_R \sum_{O:R} \alpha_{\phi_1\,\phi_2\,O}\, \alpha_{\phi_1\,\phi_4\,O}\, (\pm M^T - \mathbb{1})_{S,R}\, F^{12;14}_{\pm,O}(u,\ v) = 0$

Note : $M_{R\,S}$ depends on the choice of $\alpha_{\phi\phi 1}$

Exercise : derive above equations

CFT : 2-point, 3-point function

We define 3pt coefficient $\lambda_{i\,j\,k}$ as

$$\left\langle \phi_{1,r|a|}(x)\,\phi_{2,s|b|}(y)\,\phi_{3,t|c|}^{\mu_1\cdots\mu_\ell}(z)\right\rangle = \lambda_{123}\left\{\frac{c}{t}\,\middle|\,\frac{a,b}{r,s}\right\}\frac{Z^{\mu_1}\ldots Z^{\mu_\ell}-\text{trace}}{|x-y|^{\Delta_i+\Delta_j-(\Delta_k-\ell)}\,|y-z|^{\Delta_j+(\Delta_k-\ell)-\Delta_i}\,|z-x|^{(\Delta_k-\ell)+\Delta_i-\Delta_j}}\ ,\ Z^\mu = \frac{(x-z)^\mu}{(x-z)^2}-\frac{(y-z)^\mu}{(y-z)^2}$$

$\phi_{1,r|a|}(x)$: ϕ is in representation r with indices a.

We define 3pt coefficient $\alpha_{i\,j\,k}$ as $\lambda_{i\,j\,k}\left\{\frac{c}{t}\,\middle|\,\frac{a,b}{r,s}\right\} = \alpha_{i\,j\,k}\left\langle\frac{a,b,c}{r,s,t}\right\rangle$, i.e. $\alpha_{i\,j\,k} = \lambda_{i\,j\,k}\sqrt{\dim(\phi_k)}$

2pt function :

$$\langle\phi_{A|a|}(x)\,\phi_{B|b|}(y)\rangle = \langle\phi_{A|a|}(x)\,\phi_{B|b|}(y)\,\mathbb{1}\rangle = \lambda_{\phi\phi 1}\left\{\frac{\square}{S}\,\middle|\,\frac{a,b}{A,B}\right\}\frac{\delta_{AB}}{|x-y|^{2\Delta}} = \frac{\alpha_{\phi\phi 1}}{\sqrt{\dim(A)}}\,\frac{\delta_{AB}\,\delta^{(A)}_{a\,b}}{|x-y|^{2\Delta}}$$

(difference choices of $\alpha_{\phi\phi 1}$ lead to different conventions.)

OPE coefficient conventions

Let's now fix $\alpha_{\phi\phi 1}$. Different choices of $\alpha_{\phi\phi 1}$ correspond to different OPE coefficient conventions.

For $\alpha_{s\,s\,1} = 1$, $P^{s,(\mathrm{sim})}_{i\,j\,k\,l} = \frac{1}{\sqrt{\dim(s)}} \sum_b \left\{ \frac{b}{s} \middle| \frac{i,j}{r_1,r_2} \right\} \left\{ \frac{b}{s} \middle| \frac{k,l}{r_3,r_4} \right\} = \sqrt{\dim(s)} \sum_b \left\langle \frac{b,i,j}{s,r_1,r_2} \right\rangle \left\langle \frac{b,k,l}{s,r_3,r_4} \right\rangle$

For $\alpha_{s\,s\,1} = \frac{1}{\sqrt{\dim(s)}}$, $P^{s,(\mathrm{prj})}_{i\,j\,k\,l} = \sum_b \left\{ \frac{b}{s} \middle| \frac{i,j}{r_1,r_2} \right\} \left\{ \frac{b}{s} \middle| \frac{k,l}{r_3,r_4} \right\}$

For $\alpha_{s\,s\,1} = \sqrt{\dim(s)}$, $P^{s,(\mathrm{aut})}_{i\,j\,k\,l} = \sum_b \left\langle \frac{b,i,j}{s,r_1,r_2} \right\rangle \left\langle \frac{b,k,j}{s,r_3,r_4} \right\rangle$

OPEs : $\alpha^2_{\mathrm{abc,\,sim}} = \sqrt{\dim(c)}\ \alpha^2_{\mathrm{abc,prj}}$, $\alpha^2_{\mathrm{abc,\,aut}} = \dim(c)\, \alpha^2_{\mathrm{abc,prj}}$, $\alpha^2_{\mathrm{abc,\,sim}} = \frac{1}{\sqrt{\dim(c)}}\ \alpha^2_{\mathrm{abc,\,aut}}$

$\alpha^{\mathrm{aut}}_{\phi_1\,\phi_2\,\phi_3} = \alpha^{\mathrm{sim}}_{\phi_1\,\phi_2\,\phi_3} \dim(\phi_1)^{1/4} \dim(\phi_2)^{1/4} \dim(\phi_3)^{1/4}$

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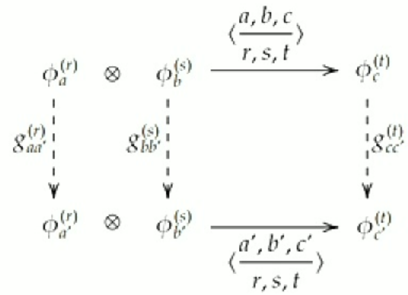
autoboot

$\left\langle \frac{a,b,c}{r,s,t} \right\rangle$ is an invariant tensor :



Example : $\phi^{(S)} = \delta_{a,b}\, \phi^{(V)}{}_a\, \phi^{(V)}{}_b$

$g^{(r)}_{a,a'}\, g^{(s)}_{b,b'}\, g^{(t)}_{c,c'} \left\langle \frac{a',b',c'}{r,s,t} \right\rangle = \left\langle \frac{a,b,c}{r,s,t} \right\rangle$ $\quad$ ($g^{(r)}_{a,a'}$: matrix representation of any group element in rep r)

$G^{(r)}_{a,a'} \left\langle \frac{a',b',c'}{r,s,t} \right\rangle = 0$ ($G^{(r)}_{a,a'}$: matrix representation of any generators in rep r)

Autoboot : treat $\left\langle \frac{a,b,c}{r,s,t} \right\rangle$ as explicitly $\dim(r) \times \dim(s) \times \dim(t)$ rank-3 tensor. Solve above equation explicitly.

Work well for small number of generator and small rep dimension.

Good for discrete group and small lie group $O(2),\ O(3)$.

Actually computation of V_R : tutorial this afternoon

Compute 3j symbols : a simple example

Take $O(n)$ $v \otimes v \to t$ as an example

Step 1 : write down generalized delta function $\delta^{(V)}_{i,j}$, $\delta^{(T)}_{(i\,j),(k\,l)}$

$\delta^{(V)}_{i,j} = \delta_{i\,j}$, $\delta^{(T)}_{(i\,j),(k\,l)} = \frac{1}{2}\,(\delta_{i\,k}\,\delta_{j\,l} + \delta_{i\,l}\,\delta_{j\,k}) - \frac{1}{n}\,\delta_{i\,j}\,\delta_{k\,l}$

For generic reps R : $\delta^{(R)}_{i,\,j} = Y_j \circ Y_i \circ \delta^{j_1}_{i_1}\,\delta^{j_2}_{i_2}\,\ldots$ where Y_i is Young symmetrizer operating on i indices and subtract traces.

Example : $\delta^{(T)}_{(i\,j),(k\,l)} = \delta^{(k}_{(i}\,\delta^{l)}_{j)} - x\,\delta_{i\,j}\,\delta_{k\,l}$, where $x = \frac{1}{n}$ is fixed by $\delta^{i\,j}\,\delta^{(T)}_{(i\,j),(k\,l)} = 0$

Exercise : proof for traceless symmetric rank-ℓ tensor, the generalized delta is

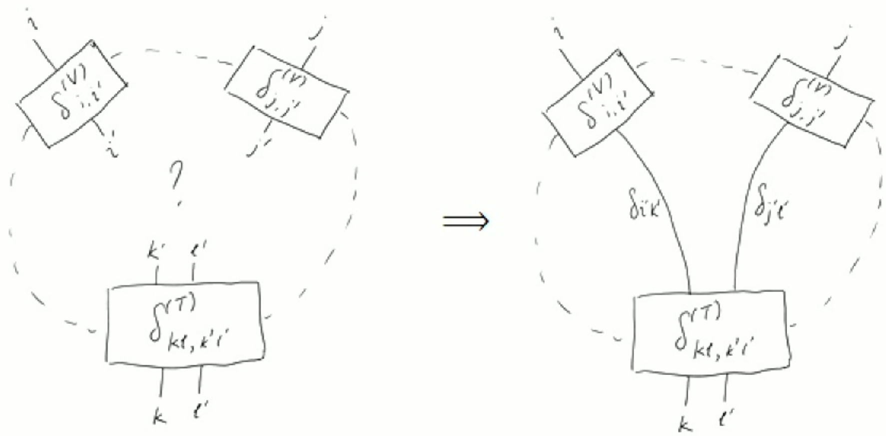
$\delta^{b_1\ldots\,b_\ell}_{a_1\ldots\,a_\ell} = \delta^{b_1}_{(a_1}\ldots\delta^{b_\ell}_{a_\ell)} + c_1\,\delta_{(a_1\,a_2}\,\delta^{(b_1\,b_2}\,\delta^{b_3}_{a_3}\ldots\delta^{b_\ell)}_{a_\ell)} + c_2\,\delta_{(a_1\,a_2}\,\delta_{a_3\,a_4}\,\delta^{(b_1\,b_2}\,\delta^{b_3\,b_4}\,\delta^{b_5}_{a_5}\ldots\delta^{b_\ell)}_{a_\ell)} + \ldots$

where $c_m = \frac{(-1)^m\left(\frac{1}{2}-\frac{\ell}{2}\right)_m\left(-\frac{\ell}{2}\right)_m}{\Gamma(m+1)\left(-\frac{n}{2}-\ell+2\right)_m}$

Compute 3j symbols : a simple example

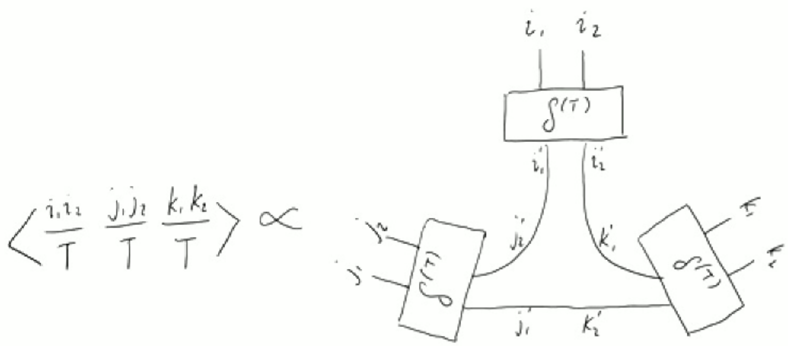
Take $O(n)$ $v \otimes v \rightarrow t$ as an example

Step 2 : $\left\langle \frac{i}{V} \frac{j}{V} \frac{kl}{T} \right\rangle \propto$



$$\left\langle \frac{i}{V} \; \frac{j}{V} \; \frac{k\,l}{T} \right\rangle = \frac{1}{2}\left(\delta_{i\,k}\,\delta_{j\,l} + \delta_{i\,l}\,\delta_{j\,k}\right) - \frac{1}{n}\,\delta_{i\,j}\,\delta_{k\,l}$$ (overall factor is fixed by $\sum_{a,b,c} \left\langle \frac{a,b,c}{r,s,t} \right\rangle \left\langle \frac{a,b,c}{r,s,t} \right\rangle = 1$)

Compute 3j symbols : ⟨T T T⟩



Exercise: compute $\left\langle \frac{i_1\, i_2}{T}\ \frac{j_1\, j_2}{T}\ \frac{k_1\, k_2}{T} \right\rangle$

Answer :

$$\sqrt{\frac{(-2+n)(-1+n)(2+n)(4+n)}{8\,n}}\left\langle \frac{i_1\, i_2}{T}\ \frac{j_1\, j_2}{T}\ \frac{k_1\, k_2}{T} \right\rangle = -\frac{1}{2\,n}\Big(\big(\delta_{i_1,k_2}\,\delta_{i_2,k_1}+\delta_{i_1,k_1}\,\delta_{i_2,k_2}\big)\delta_{j_1,j_2}+\delta_{i_1,i_2}\big(\delta_{j_1,k_2}\,\delta_{j_2,k_1}+\delta_{j_1,k_1}\,\delta_{j_2,k_2}\big)+\big(\delta_{i_1,j_2}\,\delta_{i_2,j_1}+\delta_{i_1,j_1}\,\delta_{i_2,j_2}\big)\delta_{k_1,k_2}\Big)+$$

$$\frac{1}{8}\Big(\delta_{i_1,j_2}\big(\delta_{i_2,k_2}\,\delta_{j_1,k_1}+\delta_{i_2,k_1}\,\delta_{j_1,k_2}\big)+\delta_{i_1,k_2}\big(\delta_{i_2,j_2}\,\delta_{j_1,k_1}+\delta_{i_2,j_1}\,\delta_{j_2,k_1}\big)+\delta_{i_1,k_1}\big(\delta_{i_2,j_2}\,\delta_{j_1,k_2}+\delta_{i_2,j_1}\,\delta_{j_2,k_2}\big)+\delta_{i_1,j_1}\big(\delta_{i_2,k_2}\,\delta_{j_2,k_1}+\delta_{i_2,k_1}\,\delta_{j_2,k_2}\big)\Big)+$$

$$\frac{2}{n^2}\,\delta_{i_1,i_2}\,\delta_{j_1,j_2}\,\delta_{k_1,k_2}$$

Exercise : current and central charge

Traditionally we define the $O(n)$ current through the Ward identity :

$$\epsilon^{i\,m}\langle\phi^m(x_1)\,\phi^j(x_2)\rangle = -\frac{\epsilon_{\alpha\,\beta}}{2}\oint_{x_1} d\,S_\mu\left\langle J_\mu^{\alpha\,\beta}(x)\,\phi^i(x_1)\,\phi^j(x_2)\right\rangle$$, where $\epsilon^{i\,m}$ is $O(n)$ generator

The normalization of $J_\mu^{\alpha\,\beta}$ is related to the current central charge through

$$\left\langle J_\mu^{i\,j}(x)\,J_\nu^{k\,l}(y)\right\rangle = \frac{\delta_{i\,k}\,\delta_{j\,l}-\delta_{i\,l}\,\delta_{j\,k}}{2}\,\frac{2\,C_J}{(S_{d-1})^2}\,\frac{I_{\mu\nu}}{|x-y|^{2\,\Delta_J}}$$ with $S_{d-1} = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}$

However in conformal bootstrap numerics, we normalization

$$\left\langle J_\mu^{i\,j}(x)\,J_\nu^{k\,l}(y)\right\rangle = \frac{\alpha_{J\,J\,1}}{\sqrt{\dim(J)}}\,\frac{\delta_{i\,k}\,\delta_{j\,l}-\delta_{i\,l}\,\delta_{j\,k}}{2}\,\frac{I_{\mu\nu}}{|x-y|^{2\,\Delta_J}}$$

Exercise :

1, Workout the coefficient (...) in $J_\mu^{i\,j} = (\,...\,)\,J_\mu^{i\,j}$.

2, Workout the relation between $\alpha_{\phi\phi\,J}$, $\alpha_{tt\,J}$ and C_J

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However in conformal bootstrap numerics, we normalization

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1, Workout the coefficient (...) in $J_\mu^{i\,j}$ = (...) $J_\mu^{i\,j}$.

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O(N) {v,s,t} bootstrap equations

All correlators :
$\langle \phi\phi\phi\phi \rangle, \langle \mathrm{tttt} \rangle, \langle \mathrm{ssss} \rangle, \langle \phi \mathrm{s} \phi \mathrm{s} \rangle, \langle \phi\phi \mathrm{ts} \rangle, \langle \phi\phi \mathrm{tt} \rangle, \langle \phi\phi \mathrm{ss} \rangle, \langle \mathrm{sstt} \rangle, \ \langle \mathrm{sstt} \rangle, \ \langle \phi\phi \mathrm{ts} \rangle$

Rep decomposition:
$V \otimes V = S \oplus T \oplus A$

$T \otimes T = S \oplus T \oplus A \oplus Y_{2,2} \oplus Y_4 \oplus Y_{3,1}$

$T \otimes V = V \oplus Y_3 \oplus Y_{2,1}$

3j symbols:
$\langle V \ V \ S \rangle, \langle V \ V \ A \rangle, \langle V \ V \ T \rangle, \langle S \ S \ S \rangle,$

$\langle T \ T \ S \rangle, \ \langle T \ T \ T \rangle, \ \langle T \ T \ A \rangle, \ \langle T \ T \ Y_{2,2} \rangle, \ \langle T \ T \ Y_4 \rangle, \ \langle T \ T \ Y_{3,1} \rangle$

$\langle T \ V \ Y_3 \rangle, \ \langle T \ V \ Y_{2,1} \rangle$

Bootstrap equations for $O(N)$:
$V_{\text{identity}} + (\lambda_{\phi\phi t}, \lambda_{tts}, \lambda_{ttt}, \lambda_{sss}, \lambda_{\phi\phi s}).V_{\theta}.(\lambda_{\phi\phi t}, \lambda_{tts}, \lambda_{ttt}, \lambda_{sss}, \lambda_{\phi\phi s}) + \sum_r \sum_{O:r} V_r = 0$

Actually computation of V_R : tutorial this afternoon

Correlators and theory space

Correlators:

$$O(2) : \langle\phi\phi\phi\phi\rangle,\ \langle \mathrm{tttt}\rangle,\ \langle \mathrm{ssss}\rangle,\ \langle\phi \mathrm{s}\phi \mathrm{s}\rangle,\ \langle\phi\phi \mathrm{ts}\rangle,\ \langle\phi\phi \mathrm{tt}\rangle,\ \langle\phi\phi \mathrm{ss}\rangle,\ \langle \mathrm{sstt}\rangle$$

$$O(3),\ O(4),\ \ldots : \langle\phi\phi\phi\phi\rangle,\ \langle \mathrm{tttt}\rangle,\ \langle \mathrm{ssss}\rangle,\ \langle\phi \mathrm{s}\phi \mathrm{s}\rangle,\ \langle\phi\phi \mathrm{ts}\rangle,\ \langle\phi\phi \mathrm{tt}\rangle,\ \langle\phi\phi \mathrm{ss}\rangle,\ \langle \mathrm{sstt}\rangle,\ \langle \mathrm{sstt}\rangle,\ \langle\phi\phi \mathrm{ts}\rangle$$

Parameters to scan:

$$O(2) : \{\Delta_\phi,\ \Delta_s,\ \Delta_t\} \text{ and } \{\lambda_{\phi\phi\mathrm{t}},\ \lambda_{\mathrm{tts}},\ \lambda_{\mathrm{sss}}\}/\lambda_{\phi\phi\mathrm{s}}$$

$$O(3),\ O(4),\ \ldots : \{\Delta_\phi,\ \Delta_s,\ \Delta_t\} \text{ and } \{\lambda_{\phi\phi\mathrm{t}},\ \lambda_{\mathrm{tts}},\ \lambda_{\mathrm{sss}},\ \lambda_{\mathrm{ttt}}\}/\lambda_{\phi\phi\mathrm{s}}$$

It's important to scan those OPEs to ensure $\{\phi,\ s,\ t\}$ are really the only one relevant operators

Naive scan : cost ~ $e^{\text{dimension}}$ (the curse of dimensionality)

Plan : Scan $\{\Delta_\phi,\ \Delta_s,\ \Delta_t\}$ using Delaunay search. Scan $\{\lambda_{\phi\phi\mathrm{t}},\ \lambda_{\mathrm{tts}},\ \lambda_{\mathrm{sss}},\ \lambda_{\mathrm{ttt}}\}$ using cutting surface algorithm.

O(N) {v,s,t} bootstrap equations

All correlators :

$\langle\phi\phi\phi\phi\rangle$, $\langle\mathrm{tttt}\rangle$, $\langle\mathrm{ssss}\rangle$, $\langle\phi\mathrm{s}\phi\mathrm{s}\rangle$, $\langle\phi\phi\mathrm{ts}\rangle$, $\langle\phi\phi\mathrm{tt}\rangle$, $\langle\phi\phi\mathrm{ss}\rangle$, $\langle\mathrm{sstt}\rangle$, $\langle\mathrm{sstt}\rangle$, $\langle\phi\phi\mathrm{ts}\rangle$

Rep decomposition:

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$\langle V\ V\ S\rangle$, $\langle V\ V\ A\rangle$, $\langle V\ V\ T\rangle$, $\langle S\ S\ S\rangle$,

$\langle T\ T\ S\rangle$, $\langle T\ T\ T\rangle$, $\langle T\ T\ A\rangle$, $\langle T\ T\ Y_{2,2}\rangle$, $\langle T\ T\ Y_4\rangle$, $\langle T\ T\ Y_{3,1}\rangle$

$\langle T\ V\ Y_3\rangle$, $\langle T\ V\ Y_{2,1}\rangle$

Bootstrap equations for $O(N)$:

$V_{\mathrm{identity}} + (\lambda_{\phi\phi\mathrm{t}}, \lambda_{\mathrm{tts}}, \lambda_{\mathrm{ttt}}, \lambda_{\mathrm{sss}}, \lambda_{\phi\phi\mathrm{s}}).V_\theta.(\lambda_{\phi\phi\mathrm{t}}, \lambda_{\mathrm{tts}}, \lambda_{\mathrm{ttt}}, \lambda_{\mathrm{sss}}, \lambda_{\phi\phi\mathrm{s}}) + \sum_r \sum_{O:r} V_r = 0$

Actually computation of V_R : tutorial this afternoon

Correlators and theory space

Correlators:

$$O(2): \langle\phi\phi\phi\phi\rangle, \langle \mathrm{tttt}\rangle, \langle \mathrm{ssss}\rangle, \langle\phi \mathrm{s}\phi \mathrm{s}\rangle, \langle\phi\phi \mathrm{ts}\rangle, \langle\phi\phi \mathrm{tt}\rangle, \langle\phi\phi \mathrm{ss}\rangle, \langle \mathrm{sstt}\rangle$$

$$O(3),\ O(4),\ \ldots: \langle\phi\phi\phi\phi\rangle, \langle \mathrm{tttt}\rangle, \langle \mathrm{ssss}\rangle, \langle\phi \mathrm{s}\phi \mathrm{s}\rangle, \langle\phi\phi \mathrm{ts}\rangle, \langle\phi\phi \mathrm{tt}\rangle, \langle\phi\phi \mathrm{ss}\rangle, \langle \mathrm{sstt}\rangle, \langle \mathrm{sttt}\rangle, \langle\phi\phi \mathrm{ts}\rangle$$

Parameters to scan:

$$O(2): \{\Delta_\phi, \Delta_s, \Delta_t\} \text{ and } \{\lambda_{\phi\phi\mathrm{t}}, \lambda_{\mathrm{tts}}, \lambda_{\mathrm{sss}}\}/\lambda_{\phi\phi\mathrm{s}}$$

$$O(3),\ O(4),\ \ldots: \{\Delta_\phi, \Delta_s, \Delta_t\} \text{ and } \{\lambda_{\phi\phi\mathrm{t}}, \lambda_{\mathrm{tts}}, \lambda_{\mathrm{sss}}, \lambda_{\mathrm{ttt}}\}/\lambda_{\phi\phi\mathrm{s}}$$

It's important to scan those OPEs to ensure $\{\phi, s, t\}$ are really the only one relevant operators

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Why OPE scan

$$V_{\text{identity}} + (\lambda_{\phi\phi t},\ \lambda_{\text{tts}},\ \lambda_{\text{ttt}},\ \lambda_{\text{sss}},\ \lambda_{\phi\phi s}).\ V_{\theta}.(\lambda_{\phi\phi t},\ \lambda_{\text{tts}},\ \lambda_{\text{ttt}},\ \lambda_{\text{sss}},\ \lambda_{\phi\phi s}) + \sum_r \sum_{O:r} V_r = 0$$

Without OPE scan: $\alpha \cdot V_{\text{identity}} = 1,\ \ \alpha \cdot V_{\theta} \geq 0,\ \ \alpha \cdot V_r \geq 0$

allowed region : $\{(\Delta_\phi,\ \Delta_t,\ \Delta_s) \mid \text{cannot find } \alpha \text{ at } (\Delta_\phi,\ \Delta_t,\ \Delta_s)\}$

With OPE scan: $\alpha \cdot V_{\text{identity}} = 1,\ \ \vec{\lambda}.(\alpha \cdot V_{\theta}).\vec{\lambda} \geq 0,\ \ \alpha \cdot V_r \geq 0$

allowed region : $\left\{\left(\Delta_\phi,\ \Delta_t,\ \Delta_s,\ \vec{\lambda}\right) \middle| \text{ cannot find } \alpha \text{ at } \left(\Delta_\phi,\ \Delta_t,\ \Delta_s,\ \vec{\lambda}\right)\right\}$

allowed in Δs : $\left\{(\Delta_\phi,\ \Delta_t,\ \Delta_s) \middle| \left(\Delta_\phi,\ \Delta_t,\ \Delta_s,\ \vec{\lambda}\right) \text{ is allowed for some } \vec{\lambda}\right\}$

Why OPE scan

$$V_{\text{identity}} + \vec{\lambda}.\,V_\theta.\vec{\lambda} + \sum_r \sum_{O:r} V_r = 0$$

- OPE scan demands $\phi,\ s,\ t$ are really the only relevant operators. It's a stronger condition.

  Without OPE scan: $\alpha \cdot V_{\text{identity}} = 1,\ \alpha \cdot V_\theta \geq 0,\ \alpha \cdot V_r \geq 0$
  For some $\{\Delta_\phi,\ \Delta_s,\ \Delta_t\}$, there are solutions with two $\phi_1,\ \phi_2,\ \Delta_{\phi_1} = \Delta_{\phi_2},\ \lambda_{\phi_1 \phi_2 t} \neq \lambda_{\phi_1 \phi_2 t}, \lambda_{\phi_1 \phi_2 s} \neq \lambda_{\phi_1 \phi_2 s}$,
  but no solution for $\Delta_{\phi_1} = \Delta_{\phi_2},\ \lambda_{\phi_1 \phi_2 t} = \lambda_{\phi_1 \phi_2 t}, \lambda_{\phi_1 \phi_2 s} = \lambda_{\phi_1 \phi_2 s}$, i.e. $\phi_1 = \phi_2$
  Without OPE scan: cannot find α
  With OPE scan : can find α

- ... (Thursday tutorial)

Cutting surface algorithm : a prototype example

Parameters : $\{\Delta_\sigma,\ \Delta_\epsilon,\ x = \lambda_{\sigma\sigma\epsilon}/\lambda_{\epsilon\epsilon\epsilon}\}$

SDP in feasibility mode : find a linear functional $\vec{\alpha} = (\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5)$ such that

(1) : $\alpha\cdot\left((1\ \ 1)\cdot V_{\text{even}}(\Delta=0,\ \ell=0)\cdot\begin{pmatrix}1\\1\end{pmatrix}\right) = 1$

$\alpha\cdot V_{\text{even}}(\Delta,\ell) \geq 0$ for $\Delta \geq 3,\ \ell = 0$

$\alpha\cdot V_{\text{even}}(\Delta,\ell) \geq 0$ for $\Delta \geq \Delta_{\text{unitary}},\ \ell = 2, 4, 6, \ldots$

$\alpha\cdot V_{\text{odd}}(\Delta,\ell) \geq 0$ for $\Delta \geq 3,\ \ell = 0$

$\alpha\cdot V_{\text{odd}}(\Delta,\ell) \geq 0$ for $\Delta \geq \Delta_{\text{unitary}},\ \ell = 1, 2, 3, 4, \ldots$

(2) : $\alpha\cdot\left((x\ \ 1)\cdot V_{\text{even}}(\Delta=\Delta_\epsilon,\ \ell=0)\cdot\begin{pmatrix}x\\1\end{pmatrix} + x^2\ V_{\text{odd}}(\Delta=\Delta_\sigma,\ \ell=0)\right) \geq 0$

Assuming we find such α at $\{\Delta_{\sigma,0},\ \Delta_{\epsilon,0},\ x_0\}$, constraint (2) has the form $(x\ \ 1)\begin{pmatrix}m_{11} & m_{12}\\ m_{12} & m_{22}\end{pmatrix}\begin{pmatrix}x\\1\end{pmatrix} \geq 0$.

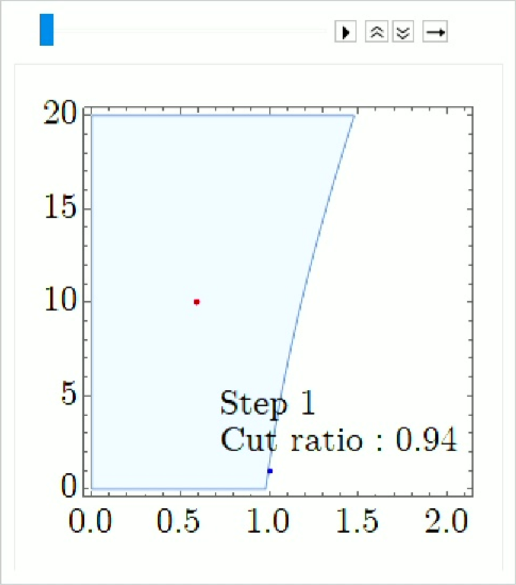
(m depends on $\Delta_{\sigma,0},\ \Delta_{\epsilon,0}$ and α, but not x_0)

Solve this constraints, one find α concludes a wide range of points in x!

Cutting surface algorithm : visualization for 2 OPE coeffs

Fix $\{\Delta_i\}$. Scan $\vec{x} = \{\lambda_1, \ ..., \ \lambda_n\}$, assuming a bounding box $\lambda_{\phi\phi t}^{(\min)} \leq \lambda_{\phi\phi t} \leq \lambda_{\phi\phi t}^{(\max)}$

Given a functional α at $\vec{x} = (\lambda_1, \ ... \lambda_n) \longrightarrow$ quadratic constraints $\vec{x} \cdot (\alpha \cdot V_\theta) \cdot \vec{x} \leq 0$ ($\alpha \cdot V_\theta : n \times n$ matrix)



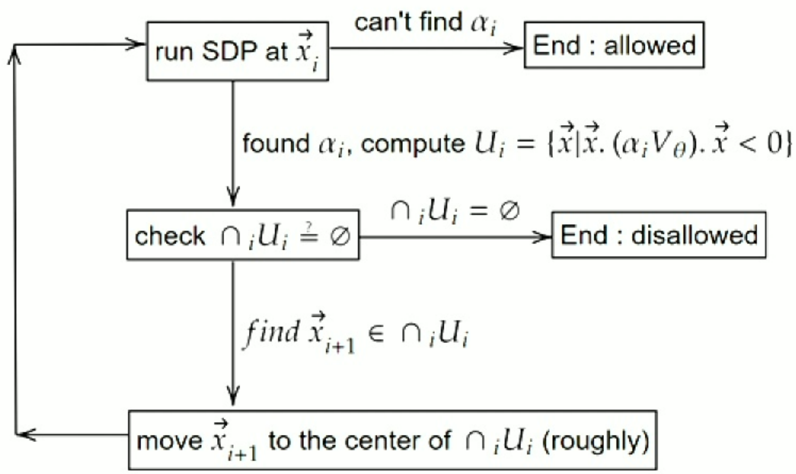
White region : ruled out
Blue region : undetermined (yet to be scanned)

Cut ratio ~ 0.5
Steps ∝ log(volume) ∝ dim of the space

Cutting surface algorithm

Fix $\{\Delta_i\}$. Scan $\vec{x} = \{\lambda_1, \ \ldots, \ \lambda_n\}$, assuming a bounding box $\lambda^{(\min)}_{\phi\phi t} \leq \lambda_{\phi\phi t} \leq \lambda^{(\max)}_{\phi\phi t}$

Given a functional α_i at the i-th step, defined $U_i = \left\{\vec{x} \,\middle|\, \vec{x} \cdot (\alpha \cdot V_\theta) \cdot \vec{x} \leq 0\right\}$ (region allowed by α_i)



Cutting surface algorithm : subtleties

- Unfortunately checking $\bigcap U_i \stackrel{?}{=} \emptyset$ is NP-hard!

  In practice : with small bounding box, we have effective heuristics to find a point in $\bigcap U_i$

  For example : $\vec{x}_i$ often sit slightly outside the boundary of $\bigcap U_i$, i.e.
  Let $f^{(n)}(\vec{x}) = \vec{x} \cdot (\alpha^{(n)} \cdot V_\theta) \cdot \vec{x}$, $f^{(1)}(\vec{x}_i) \le 0,\ f^{(2)}(\vec{x}_i) \le 0,\ \ldots$ except $f^{(j)}(\vec{x}_i) = \epsilon$ for small ϵ.
  We may try to minimize $f^{(j)}(\vec{x}_i)$ while keep the rest $f^{(n)}(\vec{x}_i) \le 0$

- Given $\vec{x}_{i+1} \in \bigcap_i U_i$, how to move $\vec{x}_{i+1}$ to the center of $\bigcap_i U_i$?

  starting with $\vec{x}^{(0)}_{i+1} \in \bigcap U_i$, one may draw a line interval over $\vec{x}^{(0)}_{i+1}$ and ends on the boundary of $\bigcap U_i$. Move $\vec{x}^{(0)}_{i+1}$ to $\vec{x}^{(1)}_{i+1}$ at the center of the interval.

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Cutting surface algorithm : visualization for 2 OPE coeffs

Fix $\{\Delta_i\}$. Scan $\vec{x} = \{\lambda_1, \ ..., \ \lambda_n\}$, assuming a bounding box $\lambda_{\phi\phi t}^{(\mathrm{min})} \leq \lambda_{\phi\phi t} \leq \lambda_{\phi\phi t}^{(\mathrm{max})}$

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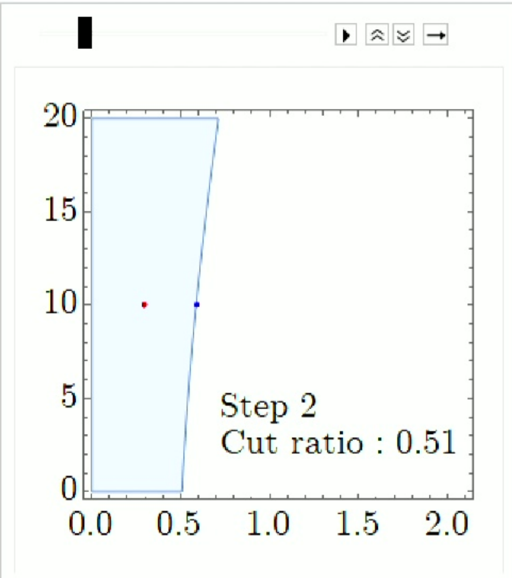

White region : ruled out

Blue region : undetermined (yet to be scanned)

Cut ratio ~ 0.5

Steps ∝ log(volume) ∝ dim of the space

Cutting surface algorithm : visualization for 2 OPE coeffs

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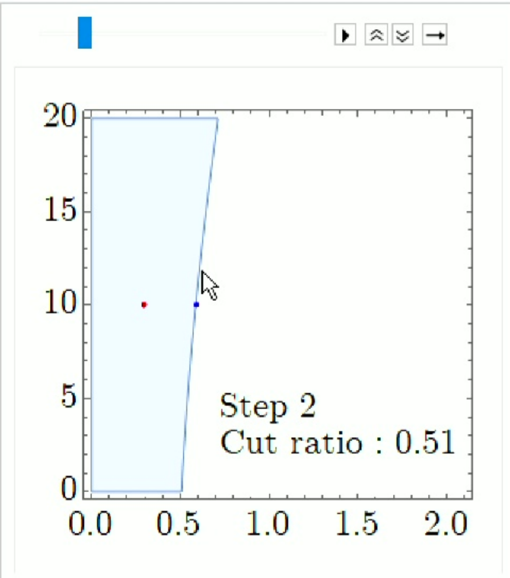

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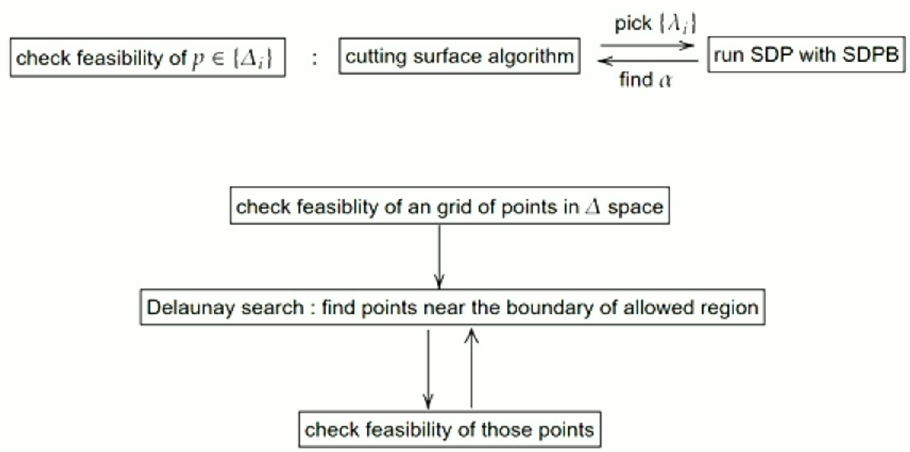
Summary of the search algorithm

Theory space : $\{\Delta_i\}, \{\lambda_i\}$



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Thank you