Title: Lecture 1: Introduction and Overview;   Bootstrapping Ising mixed correlator

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Collection: Mini-Course of Numerical Conformal Bootstrap

Date: April 24, 2023 - 10:05 AM

URL: https://pirsa.org/23040135

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Introduction; Bootstrapping Ising mixed correlator

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24/04/2023 Perimeter Bootstrap Minicourse

Constraints for conformal field theory

1, Operators as conformal rep: $\mathcal{O}_{\Delta,\ell}$ ← spin; ↑ scaling dimension

Primary : $K_\mu O(0) = 0$

Descendant : $\partial_{\mu_1} \ldots \partial_{\mu_2} O$

Scaling dimensions $\longleftrightarrow$ critical exponents

$$\alpha = 2 - d/(d - \Delta_\epsilon)$$
$$\eta = 2\Delta_\sigma - d + 2$$

$$\left\langle O^{\mu_1\cdots\mu_\ell}(x)\, O_{\nu_1\ldots\nu_\ell}(y)\right\rangle = \frac{I_{\nu_1}^{(\mu_1}\ldots I_{\nu_\ell}^{\mu_\ell)} - \text{trace}}{|x-y|^{2\Delta_k}}\ , I_\nu^\mu = \delta_\nu^\mu - 2\,\frac{(x-y)^\mu\,(x-y)_\nu}{(x-y)^2}$$

$$\langle \phi_1(x)\,\phi_2(y)\,O^{\mu_1\cdots\mu_\ell}(z)\rangle = \lambda_{\phi_1\phi_2 O}\,\frac{Z^{\mu_1}\ldots Z^{\mu_\ell} - \text{trace}}{|x-y|^{\Delta_i+\Delta_j-(\Delta_k-\ell)}\,|y-z|^{\Delta_j+(\Delta_k-\ell)-\Delta_i}\,|z-x|^{(\Delta_k-\ell)+\Delta_i-\Delta_j}} \qquad Z^{\mu_1} = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}$$

$$\langle \phi(x)\,\phi(y)\,O^{\mu_1\cdots\mu_\ell}(z)\rangle = 0 \ \text{ for odd } \ell$$

Constraints for conformal field theory

2, Operator Product Expansion : $\phi_i(x)\,\phi_j(y) = \sum_k \lambda_{ijk}\, C_k(x-y,\,\partial_y)\, O_k(y)$

Operator Product Expansion (OPE) coefficients

conformal symmetry fix $C_k(x-y,\,\partial_y)$: $\langle \phi_i(x)\,\phi_j(y)\, O_k(z)\rangle = \lambda_{ijk}\, C_k(x-y,\,\partial_y)\,\langle O_k(y)\, O_k(z)\rangle$

$$\langle \overline{\phi(x_1)\,\phi}(x_2)\,\overline{\phi(x_3)\,\phi}(x_4)\rangle = \sum \lambda_{12\,O}{}^2\, C_a(x_1-x_2,\,\partial_2)\, C_b(x_3-x_4,\,\partial_4)\,\langle O^a(x_2)\, O^b(x_4)\rangle$$

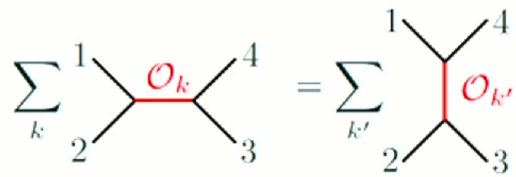
$$= \sum_O \lambda_{12\,O}{}^2\, x_{12}{}^{-\Delta_\phi}\, x_{34}{}^{-\Delta_\phi}\, g_{\Delta,\ell}(u,\,v)$$

$$u = z\,\overline{z} = \frac{x_{12}{}^2\, x_{34}{}^2}{x_{13}{}^2\, x_{24}{}^2}\,,\quad v = (1-z)(1-\overline{z}) = \frac{x_{14}{}^2\, x_{23}{}^2}{x_{13}{}^2\, x_{24}{}^2}$$

Constraints : crossing symmetry

3, Associativity : $(O_i O_j) O_k = O_i (O_j O_k)$

$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle \sim$



Bootstrap Equations : $u^{-\Delta_\phi} \sum_{\mathcal{O} \in \phi \times \phi} \lambda_{\phi\phi\mathcal{O}}{}^2 \, g_{\Delta,\ell}(u,\, v) = v^{-\Delta_\phi} \sum_{\mathcal{O} \in \phi \times \phi} \lambda_{\phi\phi\mathcal{O}}{}^2 \, g_{\Delta,\ell}(v,\, u)$

$\boxed{\sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} \, F_{\Delta,\ell}(u,\, v) = 0}$ with $F^{\phi\phi;\phi\phi}_{\Delta,\ell} = v^{\Delta_\phi} \, g^{\phi\phi;\phi\phi}_{\Delta,\ell}(u,\, v) - u^{\Delta_\phi} \, g^{\phi\phi;\phi\phi}_{\Delta,\ell}(v,\, u)$

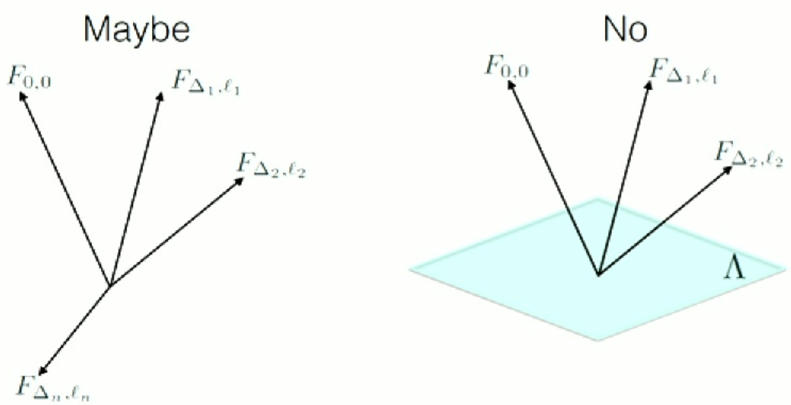
Numerical bootstrap

- 1, Pick up a set of correlators. Write down bootstrap equations. Truncate equations to finite dimension.

For example : $\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle$: $\sum_O \lambda^2_{\phi\phi O}\, F_{\Delta,\ell}(u,\, v) = 0$ (hold for various x_i)

$F_{\Delta,\ell}(z,\, \bar{z}) = \sum c_{mn}\, \partial_u^m \partial_v^n F_{\Delta,\ell}(u,\, v)\,|_{z=\bar{z}=1/2}\, (z-1/2)^m\, (\bar{z}-1/2)^n$ $\quad F_{\Delta,\ell}(z,\, \bar{z}) \rightarrow$ vector $(c_{01},\, c_{12},\, c_{23}\, \ldots)$

- 2, Unitarity : $\lambda^2_{\phi\phi O} \geq 0$

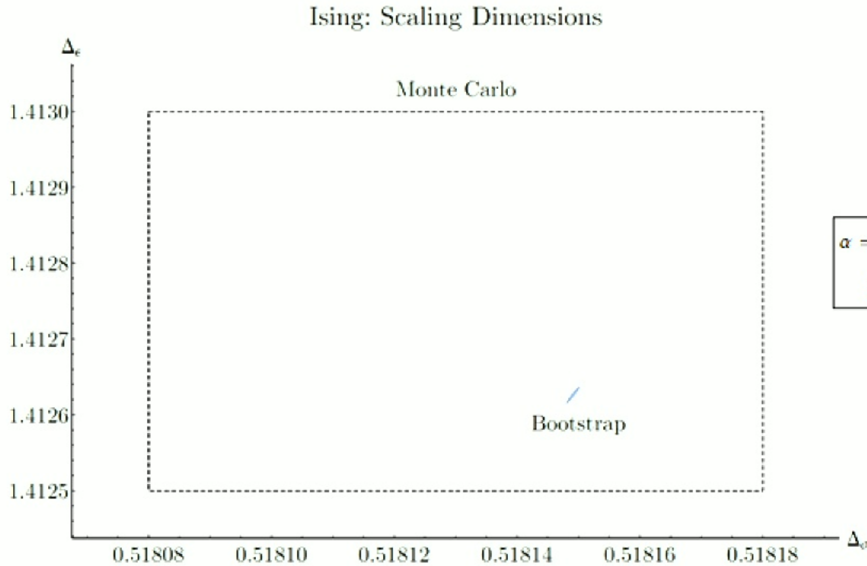


Semidefinite Program (SDP) $\Longrightarrow \begin{cases} \text{plane exist : disallowed} \\ \text{not exist : allowed} \end{cases}$

Bootstrap 3D Ising CFT

Correlators : $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\epsilon\epsilon\epsilon\epsilon\rangle$, $\langle\epsilon\epsilon\sigma\sigma\rangle$

Assumptions : σ, ϵ are the only two relevant scalars.



$\alpha = 2 - 3/(3 - \Delta_\epsilon)$

$\eta = 2\Delta_\sigma - 1$

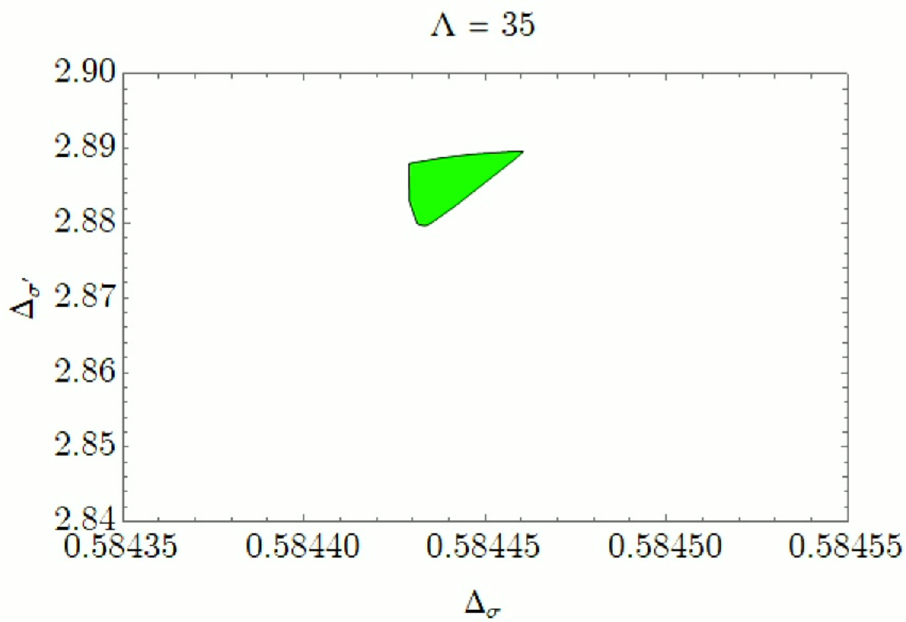
$\eta = 0.036298\,(\mathbf{2})$, $\alpha = 0.11008\,(\mathbf{1})$

(Kos, Poland, Simmons-Duffin, Vichi 2016)

Bootstrap 3D super-Ising

$$\mathcal{L}_{\text{SuperIsing}} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}\,\overline{\psi}\,\not{\partial}\,\psi + \frac{\lambda}{2}\,\sigma\,\overline{\psi}\,\not{\partial}\,\psi + \frac{\lambda^2}{8}\,\sigma^4$$

Bootstrapping $\langle\sigma\sigma\sigma\sigma\rangle$ with 3D $\mathcal{N} = 1$ SUSY (Rong, NS 2018)



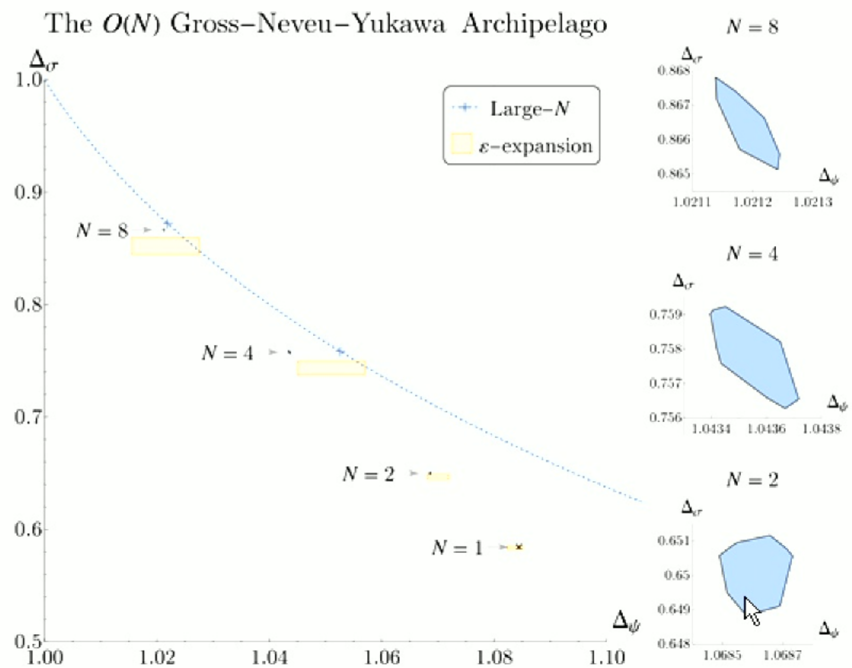
$\Delta_\sigma = 0.584444\,(30)$

$\eta = 0.168888\,(60)$

Bootstrap 3D GNY

$$\mathcal{L}_{\rm GNY} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi + \frac{\lambda}{4}\phi^4 + \frac{g}{2}\phi\bar{\psi}\partial\!\!\!/\psi$$

Bootstrapping all correlators involving $\{\psi, \phi, \phi^2\}$



[Erramilli, Iliesiu, Kravchuk, Liu, Poland, Simmons-Duffin, 2022]

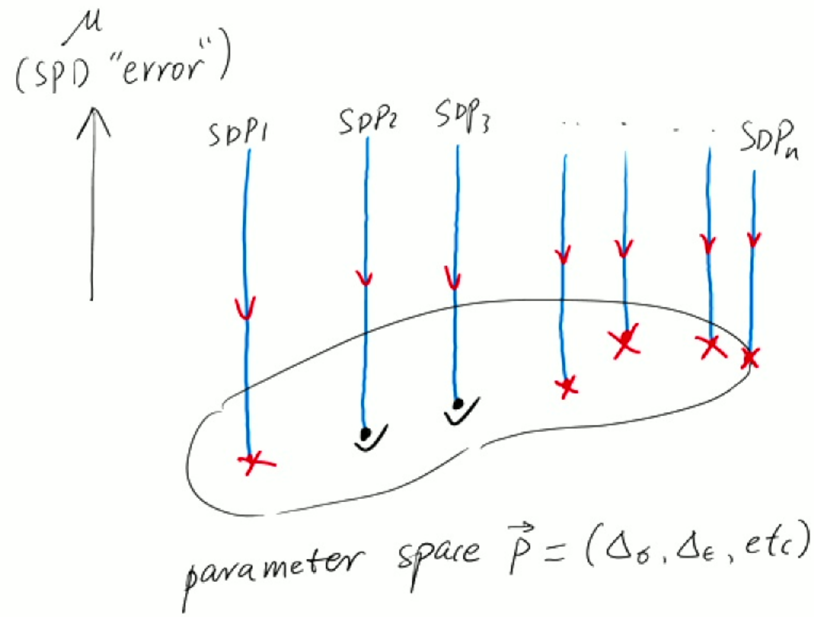
Computational perspective

The formal theory of CFT : constraints are easy to understand, but hard to compute

- 1960s : "Bootstrap" ideas by Polyakov
- 1980s : Many results for 2D CFTs
- 2001 : Conformal blocks $F_{\Delta,\ell}(u,\ v)$ in 2D and 4D are solved by Dolan, Osborn
- 2008-present : Effective numerical method for $F_{\Delta,\ell}(u,\ v)$ in arbitrary dimension
  Effective numerical method to check crossing symmetry (SDP approach)
  Effective numerical method to scan theory space
  ...
  \+ Greatly improved analytical understanding of bootstrap equation, especially at large spin

Computational perspective : bootstrap algorithms

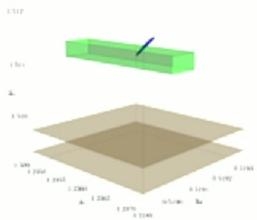
Choose a point in theory space (such as Δ_σ) $\Longrightarrow$ Solving SDP $\Longrightarrow$ Repeat and carving out theory space



Algorithms : 1, Scan algorithm; 2, SDP solver algorithm

Majors challenges in bootstrap numerics

- 1, The dimension of parameters could be high



$\longrightarrow$ Scan over $\left\{\Delta_\phi,\ \Delta_s,\ \Delta_t,\ \frac{\lambda_{\phi\phi t}}{\lambda_{\phi\phi s}},\ \frac{\lambda_{tts}}{\lambda_{\phi\phi s}},\ \frac{\lambda_{sss}}{\lambda_{\phi\phi s}}\right\}$

Naive scan : cost ~ $e^{\text{dimension}}$ (the curse of dimensionality)

- 2, Solving a SDP is slow, if SDP problem is too big

Ising : $\langle\sigma\sigma\sigma\sigma\rangle,\ \langle\epsilon\epsilon\epsilon\epsilon\rangle,\ \langle\epsilon\epsilon\sigma\sigma\rangle$

$O(2)$: $\langle\phi\phi\phi\phi\rangle,\ \langle tttt\rangle,\ \langle ssss\rangle,\ \langle\phi s\phi s\rangle,\ \langle\phi\phi ts\rangle,\ \langle\phi\phi tt\rangle,\ \langle\phi\phi ss\rangle,\ \langle sstt\rangle$

$O(3)$: $\langle\phi\phi\phi\phi\rangle,\ \langle tttt\rangle,\ \langle ssss\rangle,\ \langle\phi s\phi s\rangle,\ \langle\phi\phi ts\rangle,\ \langle\phi\phi tt\rangle,\ \langle\phi\phi ss\rangle,\ \langle sstt\rangle,\ \langle sttt\rangle,\ \langle\phi\phi ts\rangle$

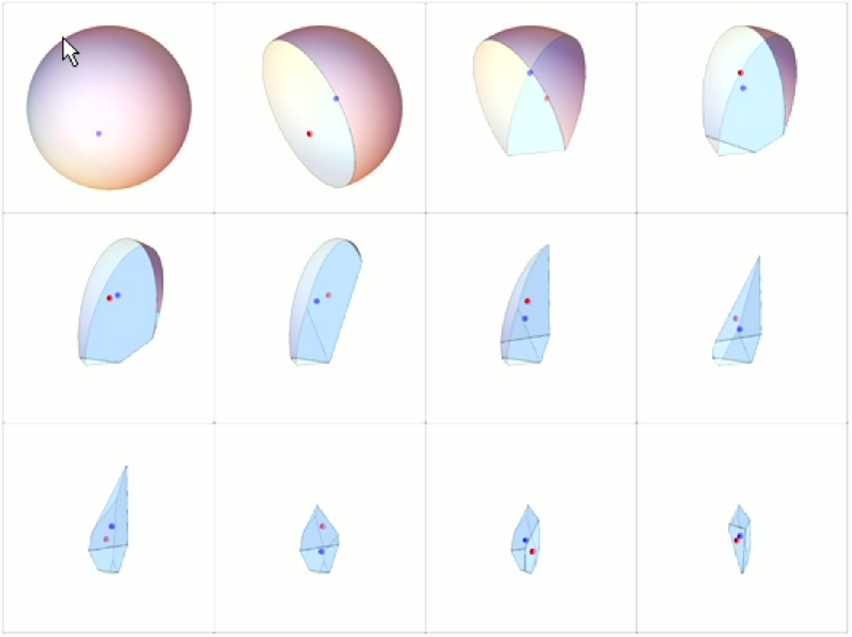
More complicated CFTs : need more correlators

More powerful computational method $\longrightarrow$ explore bigger set of constraints $\longrightarrow$ access to more CFTs and data

Cutting surface algorithm

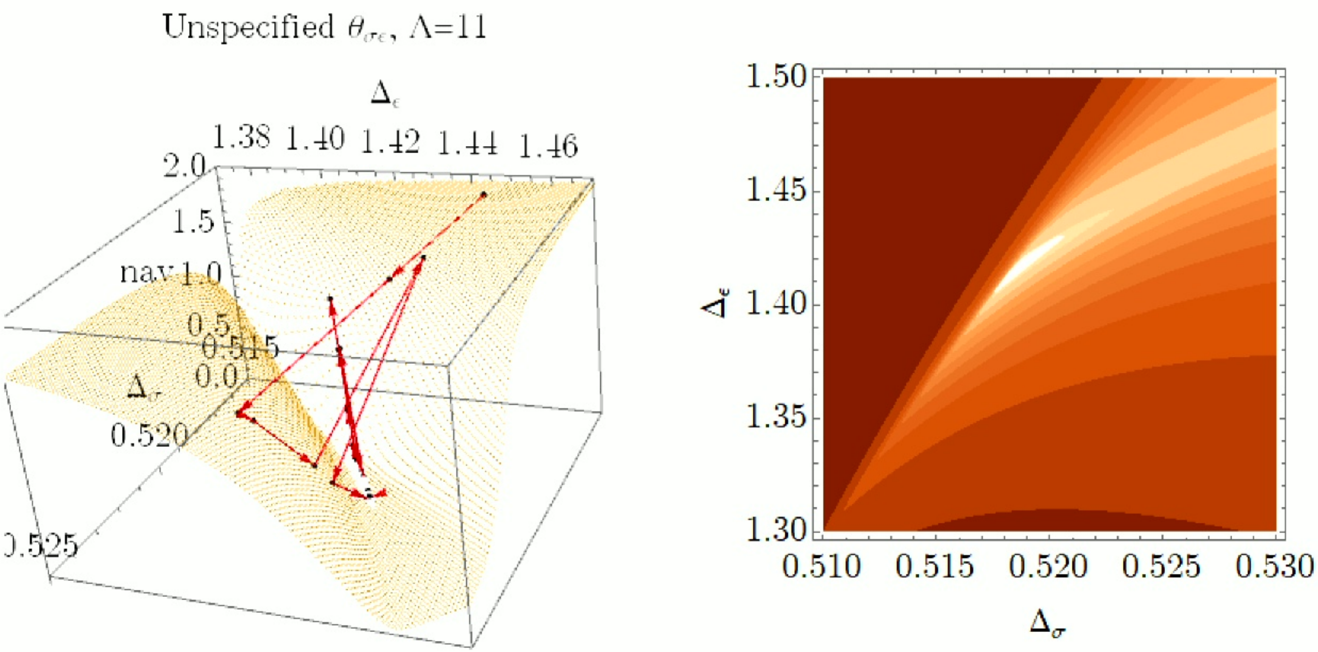
Scan $\left\{\frac{\lambda_{\phi\phi t}}{\lambda_{\phi\phi s}}, \frac{\lambda_{tts}}{\lambda_{\phi\phi s}}, \frac{\lambda_{sss}}{\lambda_{\phi\phi s}}\right\}$: cutting surface algorithm : cost linear to number of OPE coeff

$\longrightarrow$ success in O(2), O(3), GNY



The navigator function

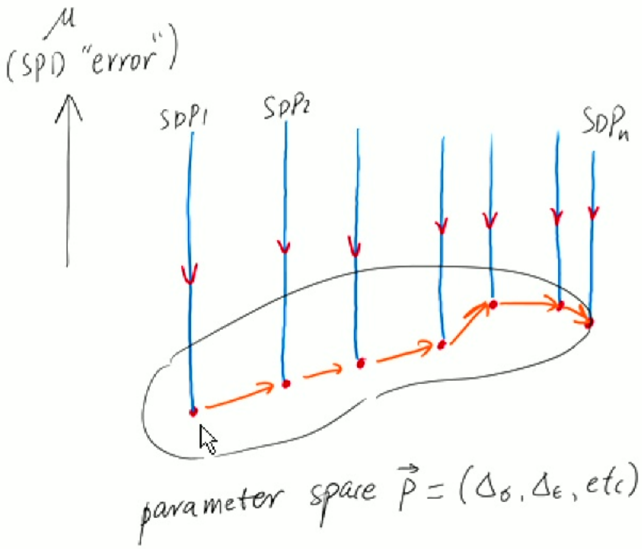
"allowed/disallowed" → a continuous measure of success (navigator function)



[Reehorst, Rychkov, Simmons-Duffin, Sirois, SN, van Rees 2021]

Algorithm in navigator computation

A section of SDPs over a parameter space. We solve SDPs one by one.

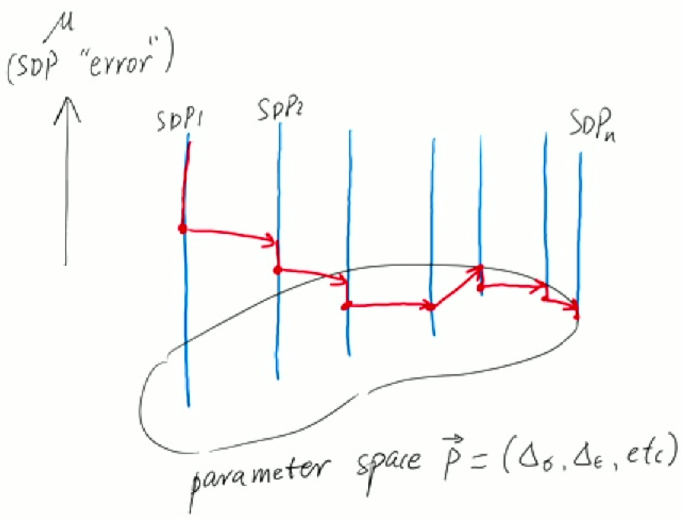


Typical Newtonian iterations per SDP : 100 to 400

Skydiving algorithm

[Aike Liu, David Simmons-Duffin, NS, Balt van Rees, to appear]

The new numerical method: treat two optimizations (in $\vec{p}$ and μ) as a single optimization problem.



Solving the optimization in the parameter $\vec{p}$

and the optimization of SDP ($\mu \to 0$) simultaneously

The software

scalar_blocks, **block3d** : generate conformal blocks

SDPB, **SDP_skydiving** : solvers for SDP, dynamical SDP problems.

autoboot : derive bootstrap equation for many global symmetries

Hyperion, **simpleboot** : bootstrap frameworks that integrate scalarblocks, block3d, SDPB, SDP_skydiving and various scan algorithms.

other software : **Juliboot**, **pyCFTboot**, **qboot**

Plan for this week

- Morning : Lectures
- Afternoon : 1h Tutorial A

  1h Break : running sample code, doing exercises and enjoying coffee

  1h Tutorial B

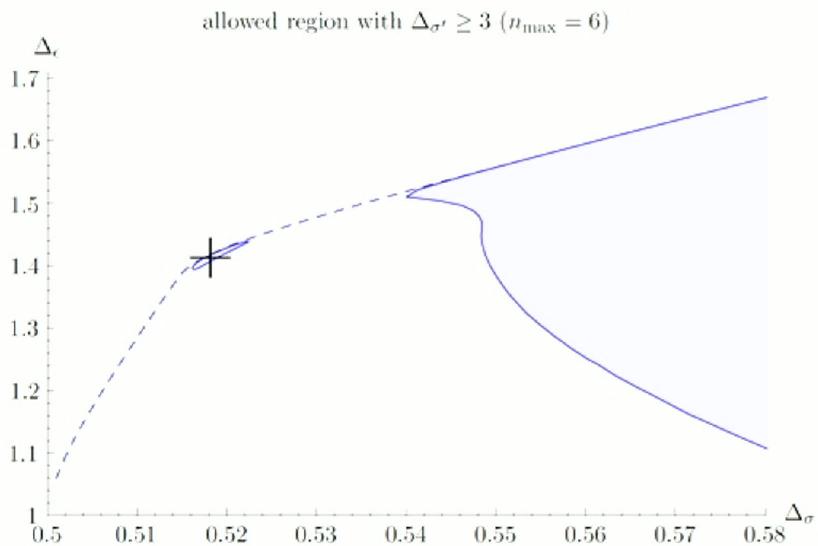
  1h Break : running sample code, doing exercises and enjoying coffee

Plan for this week

- Monday : 3D Ising $\{\sigma, \epsilon\}$ system
- Tuesday : 3D O(N) $\{v, s, t\}$ system, cutting Surface algorithm
- Wednesday : 3D GNY model $\{\psi, \phi, \epsilon\}$
- Thursday : Navigator method
- Friday : Skydiving algorithm

Today's goal : bootstrap 3D Ising

Bootstrap 3D Ising $\{\sigma, \epsilon\}$ system



Reference : Weizmann Lectures on the Numerical Conformal Bootstrap. Shai M. Chester

Bootstrap equations for scalars

Consider four different scalar $\phi_1, \phi_2, \phi_3, \phi_4$, the conformal partial wave decomposition is

$$\langle \overline{\phi_1(x_1)\,\phi_2}\,(x_2)\,\overline{\phi_3(x_3)\,\phi_4}\,(x_4)\rangle = \frac{1}{x_{12}{}^{\Delta_1+\Delta_2}\,x_{34}{}^{\Delta_3+\Delta_4}}\left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{12}}\left(\frac{x_{14}}{x_{13}}\right)^{\Delta_{34}}\sum_O \lambda_{i\,j\,O}\,\lambda_{k\,l\,O}\,g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(u,\,v)\ ,\quad \Delta_{i\,j} = \Delta_i - \Delta_j\ .$$

Crossing : $\langle \overline{\phi_1\,\phi_2}\,\overline{\phi_3\,\phi_4}\rangle_{u,v} - \langle \overline{\phi_3\,\phi_2}\,\overline{\phi_1\,\phi_4}\rangle_{u,v} = 0$. Under $1 \leftrightarrow 3$ exchange, $u \leftrightarrow v$, $z \leftrightarrow 1-z$, $\overline{z} \leftrightarrow 1-\overline{z}$

$$\sum_O \lambda_{12\,O}\,\lambda_{34\,O}\,F_{\pm,\Delta,\ell}^{12,\,34} \mp \lambda_{32\,O}\,\lambda_{14\,O}\,F_{\pm,\Delta,\ell}^{32,\,14} = 0 \text{ where } F_{\pm,O}^{12;34}(u,\,v) = v^{\frac{\Delta_2+\Delta_3}{2}}\ g_O^{\Delta_{12},\Delta_{34}}(u,\,v) \pm u^{\frac{\Delta_2+\Delta_3}{2}}\ g_O^{\Delta_{12},\Delta_{34}}(v,\,u)$$

Denote $\langle \overline{\phi_1\,\phi_2}\,\overline{\phi_3\,\phi_4}\rangle = \langle \overline{\phi_3\,\phi_2}\,\overline{\phi_1\,\phi_4}\rangle$ simply as $\langle 1234\rangle$. Only $\langle 1234\rangle$, $\langle 1324\rangle$, $\langle 1243\rangle$ are independent equations.
If $\phi_1 = \phi_2,\ \phi_3 = \phi_4$, only $\langle 1144\rangle,\ \langle 1414\rangle$ are independent equations.

Consider CFT with Z_2 symmetry. Assuming ϕ_1 is Z_2 odd, ϕ_2 is Z_2 even.

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Bootstrap equations for Z_2 even/odd scalars

Consider CFT with Z_2 symmetry. Assuming ϕ_1 is Z_2 odd, ϕ_2 is Z_2 even. Independents bootstrap equations :

(1), $\langle \phi_1 \phi_1 \phi_1 \phi_1 \rangle$:
$\sum_O \lambda_{11O}^2 F_{-,\Delta,\ell}^{11,11} = 0$

(2), $\langle \phi_2 \phi_2 \phi_2 \phi_2 \rangle$:
$\sum_O \lambda_{22O}^2 F_{-,\Delta,\ell}^{22,22} = 0$

(3), $\langle \phi_1 \phi_2 \phi_1 \phi_2 \rangle$:
$\sum_O \lambda_{12O}^2 F_{-,\Delta,\ell}^{12,12} = 0$

(4)(5), $\langle \phi_1 \phi_1 \phi_2 \phi_2 \rangle$:
$\sum_O \lambda_{21O} \lambda_{12O} F_{\mp,\Delta,\ell}^{21,12} \pm \lambda_{11O} \lambda_{22O} F_{\mp,\Delta,\ell}^{11,22} = 0$

$(\lambda_{21O} = (-1)^\ell \lambda_{12O})$

$$V_{\text{even}} = \begin{pmatrix} \begin{pmatrix} F_{-,\Delta,\ell}^{11,11} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & F_{-,\Delta,\ell}^{22,22} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2} F_{-,\Delta,\ell}^{11,22} \\ \frac{1}{2} F_{-,\Delta,\ell}^{11,22} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2} F_{+,\Delta,\ell}^{11,22} \\ \frac{1}{2} F_{+,\Delta,\ell}^{11,22} & 0 \end{pmatrix} \end{pmatrix}, \quad V_{\text{odd}} = \begin{pmatrix} 0 \\ 0 \\ F_{-,\Delta,\ell}^{12,12} \\ (-1)^\ell F_{-,\Delta,\ell}^{21,12} \\ -(-1)^\ell F_{-,\Delta,\ell}^{21,12} \end{pmatrix}$$

$$\sum_{O^+} (\lambda_{11O} \;\; \lambda_{22O}) \cdot V_{\text{even}} \cdot \begin{pmatrix} \lambda_{11O} \\ \lambda_{22O} \end{pmatrix} + \sum_{O^-} \lambda_{12O}{}^2 \, V_{\text{odd}} = 0$$

Bootstrap equations for Z_2 even/odd scalars

The feasibility test for $\sum_{O^+} (\lambda_{11O} \;\; \lambda_{22O}) \cdot V_{\text{even},\Delta_{O^+},\ell_{O^+}} \cdot \begin{pmatrix} \lambda_{11O} \\ \lambda_{22O} \end{pmatrix} + \sum_{O^-} \lambda_{12O}{}^2 \, V_{\text{odd},\Delta_{O^-},\ell_{O^-}} = 0$

Let's make some assumptions about spectrum (thus $V_{\text{even}}, \; V_{\text{odd}}$ vectors), and test whether our assumptions is true.

Positivity test : find a linear functional $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ such that

$\alpha \cdot V_{\text{even}} \geq 0$

$\alpha \cdot V_{\text{odd}} \geq 0$

Note: $\alpha \cdot V_{\text{even}}$ is a 2×2 matrix. $\alpha \cdot V_{\text{even}} \geq 0$ means the matrix is positive semi-definite, i.e.

$(\lambda_{11O} \;\; \lambda_{22O}) \, \alpha \cdot V_{\text{even}} \begin{pmatrix} \lambda_{11O} \\ \lambda_{22O} \end{pmatrix} \geq 0$ for arbitrary $\lambda_{11O}, \lambda_{22O}$

Bootstrap 3D Ising CFT $\{\sigma,\epsilon\}$

$$\sum_{O^+} (\lambda_{11O} \;\; \lambda_{22O}) \cdot V_{\text{even}} \cdot \begin{pmatrix} \lambda_{11O} \\ \lambda_{22O} \end{pmatrix} + \sum_{O^-} \lambda_{12O}^2 \; V_{\text{odd}} = 0$$

$$\xRightarrow{\text{isolate identity},\sigma,\epsilon}$$

$$(1 \;\; 1) \cdot V_{\text{even},\Delta=0,\ell=0} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (\lambda_{\sigma\sigma\epsilon} \;\; \lambda_{\epsilon\epsilon\epsilon}) \cdot V_{\text{even},\Delta=\Delta_\epsilon,\ell=0} \cdot \begin{pmatrix} \lambda_{\sigma\sigma\epsilon} \\ \lambda_{\epsilon\epsilon\epsilon} \end{pmatrix} + + \lambda_{\sigma\epsilon\sigma}^2 \; V_{\text{odd},\Delta=\Delta_\sigma,\ell=0}$$

$$\sum_{O^+ \neq 1,\epsilon} (\lambda_{\sigma\sigma O} \;\; \lambda_{\epsilon\epsilon O}) \cdot V_{\text{even}} \cdot \begin{pmatrix} \lambda_{\sigma\sigma O} \\ \lambda_{\epsilon\epsilon O} \end{pmatrix} + \sum_{O^- \neq \sigma} \lambda_{12O}^2 \; V_{\text{odd}} = 0$$

$$\xRightarrow{\text{Assume } \lambda_{\sigma\epsilon\sigma} = \lambda_{\sigma\sigma\epsilon}}$$

$$V_{\text{identity}} + (\lambda_{\sigma\sigma\epsilon} \;\; \lambda_{\epsilon\epsilon\epsilon}) \cdot V_\theta \cdot \begin{pmatrix} \lambda_{\sigma\sigma\epsilon} \\ \lambda_{\epsilon\epsilon\epsilon} \end{pmatrix} + \sum_{O^+ \neq 1,\epsilon} (\lambda_{\sigma\sigma O} \;\; \lambda_{\epsilon\epsilon O}) \cdot V_{\text{even}} \cdot \begin{pmatrix} \lambda_{\sigma\sigma O} \\ \lambda_{\epsilon\epsilon O} \end{pmatrix} + \sum_{O^- \neq \sigma} \lambda_{12O}^2 \; V_{\text{odd}} = 0$$

$$V_{\text{identity}} \equiv (1 \;\; 1) \cdot V_{\text{even},\Delta=0,\ell=0} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad V_\theta \equiv V_{\text{even},\Delta=\Delta_\epsilon,\ell=0} + \begin{pmatrix} V_{\text{odd},\Delta=\Delta_\sigma,\ell=0} & 0 \\ 0 & 0 \end{pmatrix}$$

Unitarity

Unitarity: all the states in Hilbert space have positive norm.

Consequences 1 : $\Delta \geq \Delta_{\text{unitary}}$, $\Delta_{\text{unitary}} = \begin{cases} d/2-1 & \text{for spin} = 0 \\ d-2+\ell & \text{for spin} = \ell \end{cases}$

by demanding all descendant state has positive norm as well.

Example : $\left\| P_\mu |O\rangle \right\|^2 = 2\Delta \left\| |O\rangle \right\|^2$ for scalar state $|O\rangle$, thus $\Delta \geq 0$

Consequences 2 : $\lambda_{\phi_1 \phi_2 O}$ is real

For Hermitian operators, the OPE coefficients are always real.

Proof: take Hermitian conjugation on $\langle \phi(x)\, \phi(y)\, O^{\mu_1 \cdots \mu_\ell}(z) \rangle = \lambda_{\phi\phi O} \frac{Z^{\mu_1} \ldots Z^{\mu_\ell} - \text{trace}}{|x-y|^{\Delta_i+\Delta_j-(\Delta_k-\ell)} |y-z|^{\Delta_j+(\Delta_k-\ell)-\Delta_i} |z-x|^{(\Delta_k-\ell)+\Delta_i-\Delta_j}}$

If some states have negative norm : $\langle \phi(x)\, \phi(y) \rangle = -\frac{1}{|x-y|^{2\Delta_\phi}}$. If we insist to normalize it as $\langle \phi(x)\, \phi(y) \rangle = \frac{1}{|x-y|^{2\Delta_\phi}}$ (as we did in defining conformal block), then $\phi \to i\,\phi$ makes ϕ anti-Hermitian and some OPE coefficients are imaginary.

Assumption for 3D Ising spectrum

We choose two numbers $\Delta_\sigma, \Delta_\epsilon$

Assumption on the CFT data:

1, σ has scaling dimension Δ_σ

2, ϵ has scaling dimension Δ_ϵ

3, all other spin 0 operators has dimension larger than 3

4, all spinning operators has dimension $\Delta \geq \Delta_{\text{unitary}}$, $\Delta_{\text{unitary}} = \begin{cases} d/2-1 & \text{for spin} = 0 \\ d-2+\ell & \text{for spin} = \ell \end{cases}$

5, all OPE coefficients are real

Problem to solve : does the choice of two numbers $\Delta_\sigma, \Delta_\epsilon$ + assumptions compatible with

$$V_{\text{identity}} + (\lambda_{\sigma\sigma\epsilon} \;\; \lambda_{\epsilon\epsilon\epsilon}) \cdot V_\theta \cdot \begin{pmatrix} \lambda_{\sigma\sigma\epsilon} \\ \lambda_{\epsilon\epsilon\epsilon} \end{pmatrix} + \sum_{O^+ \neq 1,\epsilon} (\lambda_{\sigma\sigma O} \;\; \lambda_{\epsilon\epsilon O}) \cdot V_{\text{even}} \cdot \begin{pmatrix} \lambda_{\sigma\sigma O} \\ \lambda_{\epsilon\epsilon O} \end{pmatrix} + \sum_{O^- \neq \sigma} \lambda_{12O}{}^2 \, V_{\text{odd}} = 0 \;\; ?$$

Numerical details

$$V_{\text{identity}} + (\lambda_{\sigma\sigma\epsilon}\ \ \lambda_{\epsilon\epsilon\epsilon}) \cdot V_\theta \cdot \begin{pmatrix} \lambda_{\sigma\sigma\epsilon} \\ \lambda_{\epsilon\epsilon\epsilon} \end{pmatrix} + \sum_{\mathcal{O}^+} (\lambda_{\sigma\sigma\mathcal{O}}\ \ \lambda_{\epsilon\epsilon\mathcal{O}}) \cdot V_{\text{even}} \cdot \begin{pmatrix} \lambda_{\sigma\sigma\mathcal{O}} \\ \lambda_{\epsilon\epsilon\mathcal{O}} \end{pmatrix} + \sum_{\mathcal{O}^-} \lambda_{12\mathcal{O}}{}^2\ V_{\text{odd}} = 0$$

Find a linear functional $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ such that

$\alpha \cdot (V_{\text{identity}}) = 1$

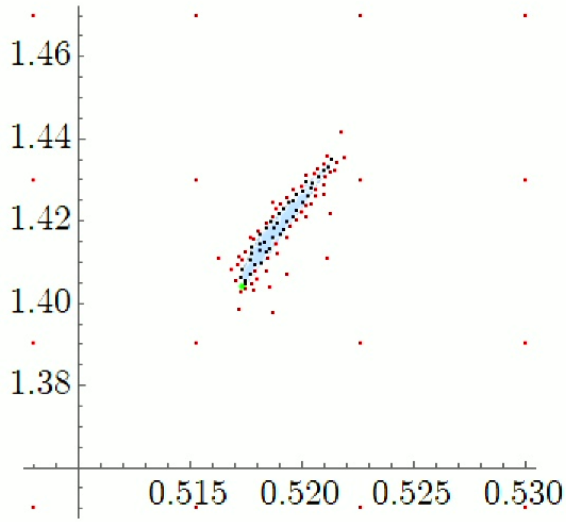
$\alpha \cdot (V_\theta) \geq 0$

$\alpha \cdot V_{\text{even}}(\Delta, \ell) \geq 0$ for $\Delta \geq 3,\ \ell = 0$

$\alpha \cdot V_{\text{odd}}(\Delta, \ell) \geq 0$ for $\Delta \geq 3,\ \ell = 0$

$\alpha \cdot V_{\text{even}}(\Delta, \ell) \geq 0$ for $\Delta \geq \Delta_{\text{unitary}},\ \ell = 2, 4, 6, \ldots$

$\alpha \cdot V_{\text{odd}}(\Delta, \ell) \geq 0$ for $\Delta \geq \Delta_{\text{unitary}},\ \ell = 1, 2, 3, 4, \ldots$



Numerical details

1, We truncate α as $\alpha = \sum_{n+m\leq\Lambda} \alpha_{m\,n}\, \partial_z^m \partial_{\bar z}^n|_{z=\bar z=\frac{1}{2}}$

2, $F_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}$ can be approximated as $\partial_z^m \partial_{\bar z}^n F_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(z,\bar z)\big|_{z=\bar z=\frac{1}{2}} \approx \chi_\ell(\Delta)\times P_{\Delta,\ell}^{m,n}(\Delta)$, $\chi_\ell(\Delta) = \frac{\left(3-2\sqrt{2}\right)^\Delta}{(\Delta-r_1)(\Delta-r_2)\ldots(\Delta-r_K)}$

$P_{\Delta,\ell}^{m,n}(\Delta)$: polynomial in Δ , can be computed using **scalar_blocks**

$\chi_\ell(\Delta)$ are the same for all $F_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}$ terms in V_{even}, V_{odd} . All poles are below unitary bound.

$$\alpha\cdot V_{\text{even},\Delta,\ell} = \chi_\ell(\Delta)\sum_{m,n}\alpha_{mn}\cdot\begin{pmatrix} \begin{pmatrix} \left(P_{-,\Delta,\ell}^{11,11}\right)^{mn} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & \left(P_{-,\Delta,\ell}^{22,22}\right)^{mn} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2}\left(P_{-,\Delta,\ell}^{11,22}\right)^{mn} \\ \frac{1}{2}\left(P_{-,\Delta,\ell}^{11,22}\right)^{mn} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2}\left(F_{+,\Delta,\ell}^{11,22}\right)^{mn} \\ \frac{1}{2}\left(P_{+,\Delta,\ell}^{11,22}\right)^{mn} & 0 \end{pmatrix} \end{pmatrix} \longrightarrow \alpha\cdot V_{\text{even},\Delta,\ell}^{(\text{poly})} \geq 0$$

Constraints for conformal field theory

2, Operator Product Expansion : $\phi_i(x)\,\phi_j(y) = \sum_k \lambda_{ijk}\, C_k(x-y,\, \partial_y)\, O_k(y)$

↓ Operator Product Expansion (OPE) coefficients

conformal symmetry fix $C_k(x-y,\,\partial_y)$: $\langle \phi_i(x)\,\phi_j(y)\, O_k(z)\rangle = \lambda_{ijk}\, C_k(x-y,\,\partial_y)\,\langle O_k(y)\, O_k(z)\rangle$

$$\langle \overline{\phi(x_1)\,\phi}(x_2)\,\overline{\phi(x_3)\,\phi}(x_4)\rangle = \sum \lambda_{12O}{}^2\, C_a(x_1-x_2,\,\partial_2)\, C_b(x_3-x_4,\,\partial_4)\,\langle O^a(x_2)\, O^b(x_4)\rangle$$

$$= \sum_O \lambda_{12O}{}^2\, x_{12}{}^{-\Delta_\phi}\, x_{34}{}^{-\Delta_\phi}\, g_{\Delta,\ell}(u,\, v)$$

$$u = z\,\bar{z} = \frac{x_{12}{}^2\, x_{34}{}^2}{x_{13}{}^2\, x_{24}{}^2}\,,\quad v = (1-z)(1-\bar{z}) = \frac{x_{14}{}^2\, x_{23}{}^2}{x_{13}{}^2\, x_{24}{}^2}$$

Next Slide

Numerical details

1, We truncate α as $\alpha = \sum_{n+m\leq\Lambda} \alpha_{m\,n}\, \partial_z^m\, \partial_{\bar z}^n |_{z=\bar z=\frac{1}{2}}$

2, $F_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}$ can be approximated as $\partial_z^m\, \partial_{\bar z}^n\, F_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(z,\, \bar z)\big|_{z=\bar z=\frac{1}{2}} \approx \chi_\ell(\Delta) \times P_{\Delta,\ell}^{m,n}(\Delta)$, $\chi_\ell(\Delta) = \frac{\left(3-2\sqrt{2}\right)^{\Delta}}{(\Delta-r_1)(\Delta-r_2)\ldots(\Delta-r_K)}$

$P_{\Delta,\ell}^{m,n}(\Delta)$: polynomial in Δ , can be computed using **scalar_blocks**

$\chi_\ell(\Delta)$ are the same for all $F_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}$ terms in V_{even}, V_{odd} . All poles are below unitary bound.

$$\alpha \cdot V_{\text{even},\Delta,\ell} = \chi_\ell(\Delta) \sum_{m,n} \alpha_{mn} \cdot \begin{pmatrix} \begin{pmatrix} \left(P^{11,11}_{-,\Delta,\ell}\right)^{mn} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & \left(P^{22,22}_{-,\Delta,\ell}\right)^{mn} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2}\left(P^{11,22}_{-,\Delta,\ell}\right)^{mn} \\ \frac{1}{2}\left(P^{11,22}_{-,\Delta,\ell}\right)^{mn} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2}\left(F^{11,22}_{+,\Delta,\ell}\right)^{mn} \\ \frac{1}{2}\left(P^{11,22}_{+,\Delta,\ell}\right)^{mn} & 0 \end{pmatrix} \end{pmatrix} \longrightarrow \alpha \cdot V^{(\text{poly})}_{\text{even},\Delta,\ell} \geq 0$$

Numerical details

3, For both $\alpha.V^{(\text{poly})}_{\text{even},\Delta,\ell} \geq 0$ and $\alpha.V^{(\text{poly})}_{\text{odd},\Delta,\ell} \geq 0$, we demand the condition for $\ell = 0, \ \dots \ell_{\max}$

The problem become polynomial programming, PMP

Find α such that

$\sum_k \alpha_k \, M^k_0(x) = 1$
$\sum_k \alpha_k \, M^k_j(x) > 0$ for $x > 0$ for $j = 1, \ \dots j_{\max}$

$M^k_j(x) \ \leftrightarrow \ V_{\text{even/odd},\Delta=\Delta_{\min}+x,\ell}$

SDPB : input

$sdp \equiv$ `SDP[`$\langle objective \rangle$`,` $\langle normalization \rangle$`,` $\langle positive\ matrices\ with\ prefactors \rangle$`]`

$objective \equiv \{a_0,\ \ldots,\ a_N\}$

$normalization \equiv \{n_0,\ \ldots,\ n_N\}$

$positive\ matrices\ with\ prefactors \equiv \{$
  $\langle positive\ matrix\ with\ prefactor\ 1 \rangle,$
  $\ldots$
  $\langle positive\ matrix\ with\ prefactor\ J \rangle,$
$\}$

$positive\ matrix\ with\ prefactor\ j \equiv$
  `PositiveMatrixWithPrefactor[`$\langle prefactor \rangle$`,`
    {
      {
        $\{Q^0_{j,11}(x),\ \ldots,\ Q^N_{j,11}(x)\},\ \ldots,\ \{Q^0_{j,m_j1}(x),\ \ldots,\ Q^N_{j,m_j1}(x)\}$
      },
      ...
      {
        $\{Q^0_{j,1m_j}(x),\ \ldots,\ Q^N_{j,1m_j}(x)\},\ \ldots,\ \{Q^0_{j,m_jm_j}(x),\ \ldots,\ Q^N_{j,m_jm_j}(x)\}$
      },
    }
  ]

Maximize α.objective, such that

α.normalization = 1

$\alpha . M_j(x) \geq 0$ for $x \geq 0$ for $j = 1, 2, \ldots J$

$$M_j = \begin{pmatrix} M_j^0 \\ M_j^1 \\ \vdots \\ M_j^N \end{pmatrix}$$

$$M_j^n = \begin{pmatrix} (M_j^n)_{11} & \ldots & (M_j^n)_{1\,m_j} \\ \vdots & \ddots & \vdots \\ (M_j^n)_{m_j\,1} & \ldots & (M_j^n)_{m_j\,m_j} \end{pmatrix}$$

$$(M_j^n)_{a\,b} = Q^n_{j,a\,b}(x)$$

SDPB in feasibility mode

Set $\text{objective} = 0$. Ask whether α exist for the condition

$\sum_k \alpha_k \, M_0^k(x) = 1$
$\sum_k \alpha_k \, M_j^k(x) > 0$ for $x > 0$ for $j = 1, \ \ldots j_{\max}$

Output :
"dual feasible jump detected", "find dual feasible solution" : find α
"primal feasible jump detected", "find primal feasible solution" : can't find α

SDPB in feasibility mode

Set $\mathrm{objective} = 0$. Ask whether α exist for the condition

$\sum_k \alpha_k \, M_0^k(x) = 1$
$\sum_k \alpha_k \, M_j^k(x) > 0$ for $x > 0$ for $j = 1, \ \dots \ j_{\max}$

Output :
"dual feasible jump detected", "find dual feasible solution" : find α
"primal feasible jump detected", "find primal feasible solution" : can't find α

SDPB in optimality mode : bound OPE coefficient

$\sum_{O\in\phi\times\phi}\lambda_{\phi\phi O}{}^2\, F_{\Delta,\ell}(u,\ v)=0\ \longrightarrow\ \ \lambda_{O_0}{}^2\, F_{\Delta_0,\ell_0}(u,\ v)=-F_{0,0}(u,\ v)-\sum_O\lambda_O{}^2\, F_{\Delta,\ell}(u,\ v)$

Find a linear functional α such that

$\alpha\big(F_{\Delta_0,\ell_0}(u,\ v)\big)=1$ (1)

$\alpha(F_{\Delta,\ell}(u,\ v))\geq 0$ for $\Delta>\Delta_{\min}$ (2)

If a α exist, we find an inequality

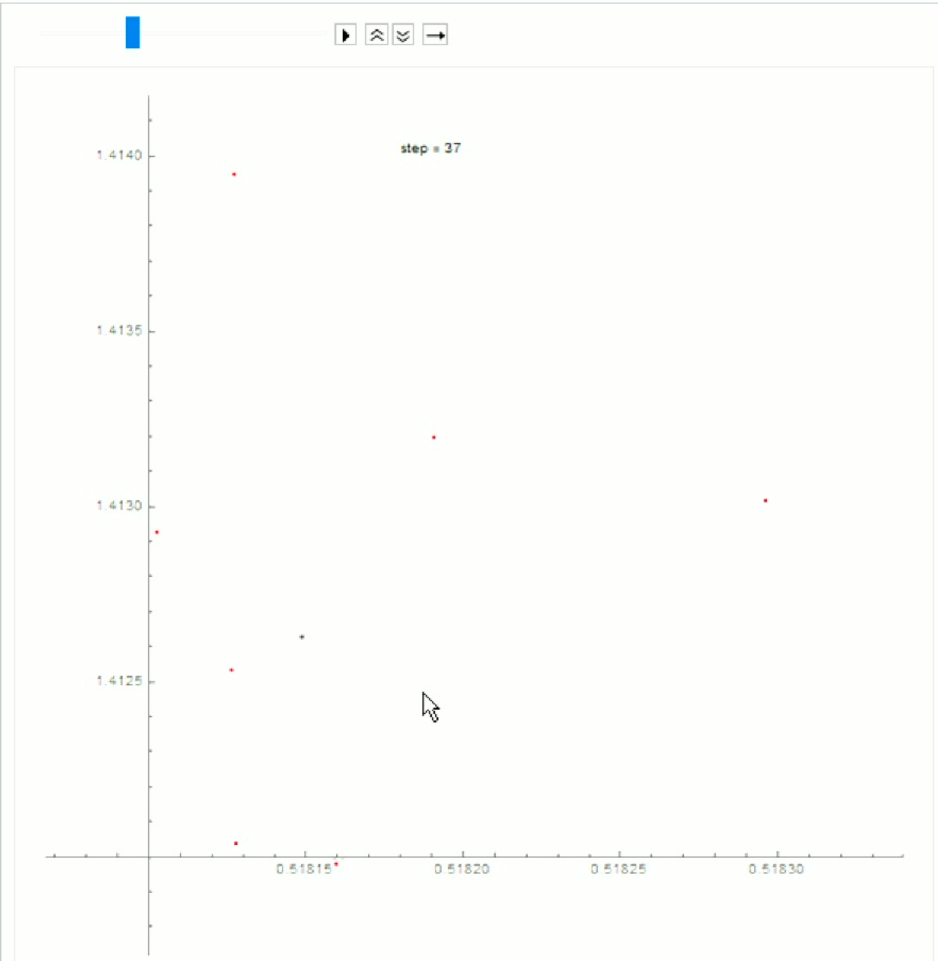
$\lambda_{O_0}{}^2=-\alpha(F_{0,0}(u,\ v))-\sum_O\lambda_O{}^2\,\alpha(F_{\Delta,\ell}(u,\ v))\leq-\alpha(F_{0,0}(u,\ v))$

To the most restrictive bound, we need α that minimizing $-\alpha(F_{0,0}(u,\ v))$ subject to (1), (2)

Such that α should satisfy $\alpha(F_{\Delta,\ell}(u,\ v))=0$ for $\Delta,\ \ell$ of each O in the $\sum_{O\in\phi\times\phi}\ \ldots$

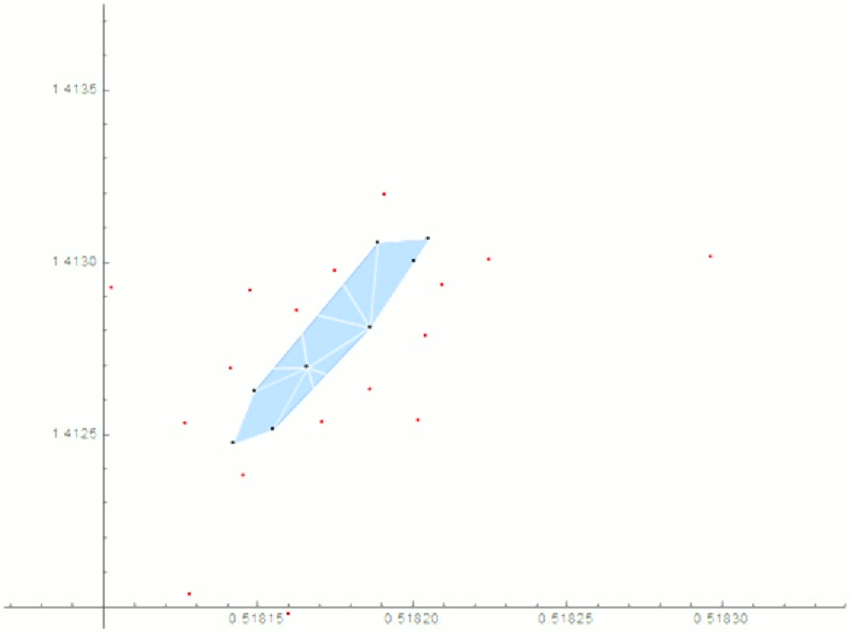
Zeros in $\alpha(F_{\Delta,\ell}(u,\ v))\longrightarrow$ physical spectrum Δ (**Extremal Functional Method**, EFM)

Delaunay search



Delaunay search

Delaunay search : an adaptive method to scan the boundary of a region
Assume we have a set allowed points and a set of disallowed points



1,Find a triangulation connecting all points
2,Find boundary triangles
3,Rank the triangles by areas;
find the largest n triangles;
scan their middle points.

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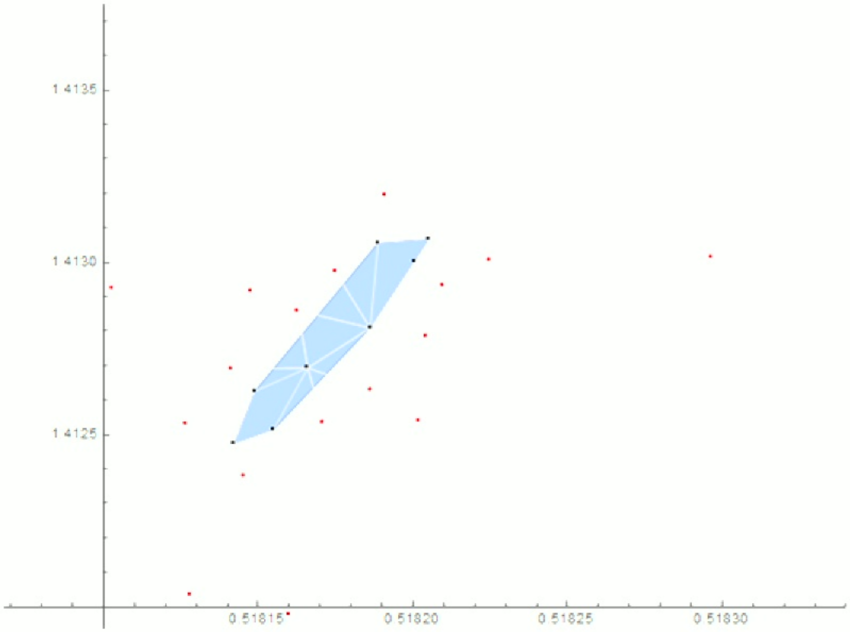

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