Title: Quantum causal inference in the presence of hidden common causes: An entropic approach

Speakers: Mohammad Ali Javidian

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Abstract: Quantum causality is an emerging field of study which has the potential to greatly advance our understanding of quantum systems. In this paper, we put forth a theoretical framework for merging quantum information science and causal inference by exploiting entropic principles. For this purpose, we leverage the tradeoff between the entropy of hidden cause and the conditional mutual information of observed variables to develop a scalable algorithmic approach for inferring causality in the presence of latent confounders (common causes) in quantum systems. As an application, we consider a system of three entangled qubits and transmit the second and third qubits over separate noisy quantum channels. In this model, we validate that the first qubit is a latent confounder and the common cause of the second and third qubits. In contrast, when two entangled qubits are prepared and one of them is sent over a noisy channel, there is no common confounder. We also demonstrate that the proposed approach outperforms the results of classical causal inference for the Tubingen database when the variables are classical by exploiting quantum dependence between variables through density matrices rather than joint probability distributions.

Quantum causal inference in the presence of hidden common causes: An entropic approach

Mohammad Ali Javidian

Appalachian State University

April 2023

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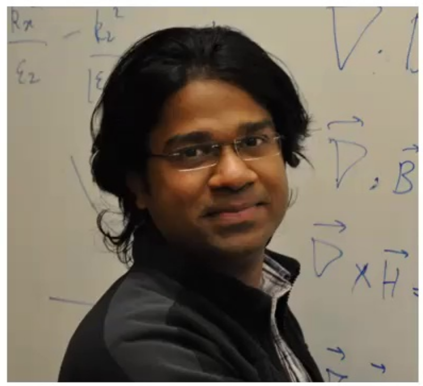
- Joint work with:

Vaneet Aggarwal

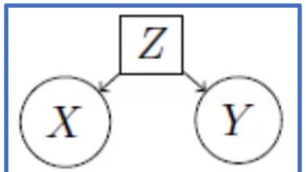


Zubin Jacob

Inspired by Murat Kocaoglu et al. (NeurIPS 2020)

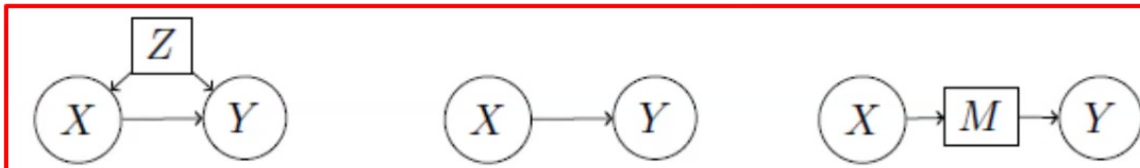
- Applications of Common Entropy for Causal Inference:
- **Common Entropy** $[G(X,Y) \coloneqq H(Z)]$: Given $P(X,Y)$, find Z with <u>minimum entropy</u> such that $X \perp\!\!\!\perp Y|Z$.
- **Goal**: Identifying Correlation without Causation via Rényi Common Entropy (latent graph (a)) using joint probability distribution $P(X,Y)$.
- **Assumption**: Suppose latent confounding is weak, i.e., $H(Z) < \theta$

$G(X,Y) < \theta$



(a) Latent Graph

In most cases: $G(X,Y) > \theta$



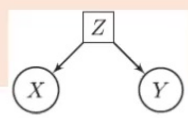
(b) Triangle Graph (c) Direct Graph (d) Mediator Graph

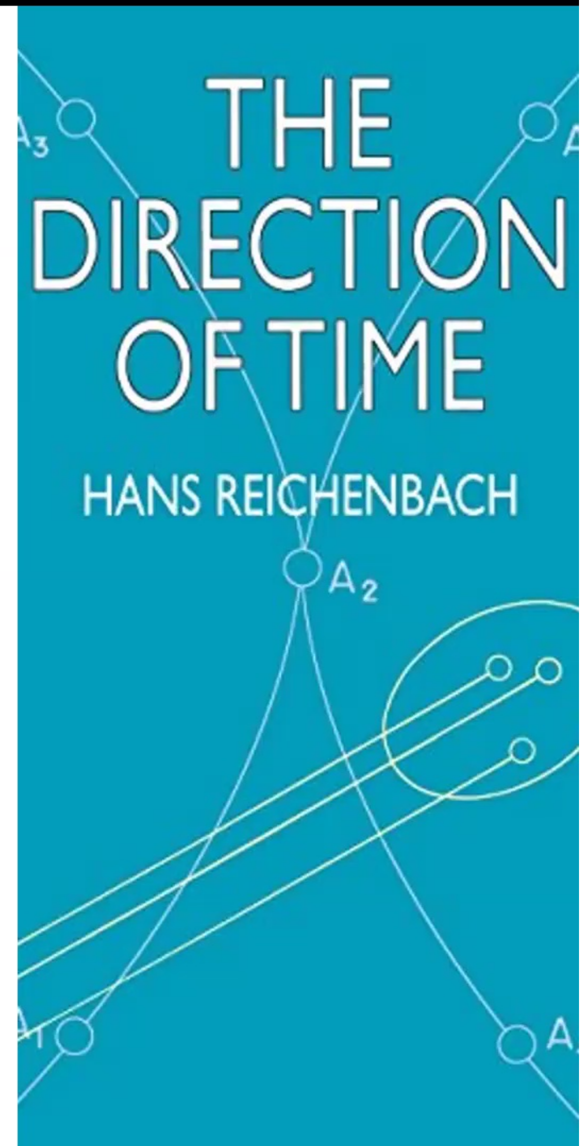
Our Contributions

- Identification of latent graphs in quantum systems
  - Conceptual challenges
  - Technical challenges
- Benefits of using quantum entropy in classical settings
  - Tubingen data set

Reichenbach's common cause principle

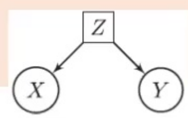
- If two random variables X and Y are statistically *dependent*, then there exists a third variable Z that causally affects both. As a special case, Z may coincide with either X or Y.

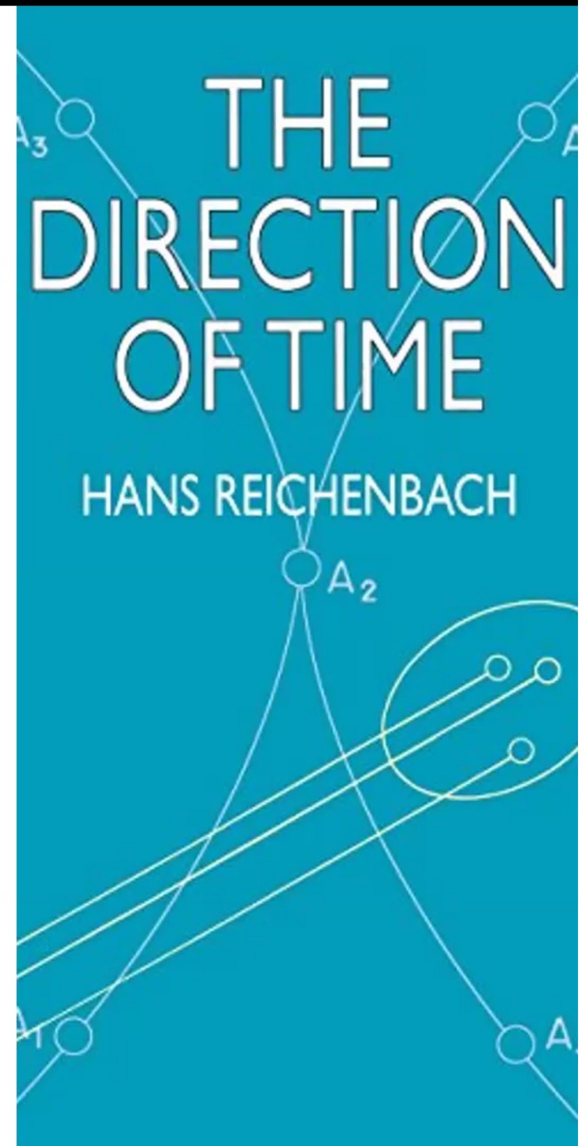




Reichenbach's common cause principle

- If two random variables X and Y are statistically *dependent*, then there exists a third variable Z that causally affects both. As a special case, Z may coincide with either X or Y.
- this variable *Z* makes *X* and *Y* conditionally independent





Conceptual challenges in quantum systems

- **Coexistence** of quantum systems:
  - For a given causal structure, a coexisting set of systems is one for which a joint state can be defined.
  - Because of the impossibility of **cloning**, the outcomes and the quantum systems that led to them do not exist *simultaneously*.
  - If a system X is measured to produce Y, then ρ_{XY} is not defined and hence neither is the entropy $S(\rho_{XY})$.
- In this presentation we assume that ρ_{XY} is given.

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- In this presentation we assume that ρ_{XY} is given.
- Quantum state tomography

Formalism of Classical probability vs Quantum theory

Probability distribution/Density operator (matrix)	$P(X)$	ρ_X
Joint distribution/Joint density	$P(X,Y)$	ρ_{XY}
Marginal distribution/Partial trace	$P(X) = \sum_Y P(X,Y)$	$\rho_X = \mathrm{Tr}_Y(\rho_{XY})$
Conditional probability/Conditional density matrix	$P(Y\|X) = \frac{P(X,Y)}{P(X)}$	$\rho_{Y\|X} = (\rho_X^{-\frac{1}{2}} \otimes I_Y)\rho_{XY}(\rho_X^{-\frac{1}{2}} \otimes I_Y)$
Shannon/von Neumann Entropy	$H(X) = -\sum_i P(x_i)\ln[P(x_i)]$	$S(Z) = -\mathrm{Tr}(\rho_Z \ln(\rho_Z))$
Conditional Entropy	$H(Y\|X) = H(X,Y) - H(X) \geq 0$	$S(Y\|X) = S(Y,X) - S(X) \in \mathbb{R}$
Mutual Information	$I(X;Y) = H(X) + H(Y) - H(X,Y)$	$I_Q(X;Y) = S(X) + S(Y) - S(X,Y)$

Inference of quantum common cause in the literature of quantum computing and information

- Wolfe et al., 2020: Quantifying Bell: the Resource Theory of Nonclassicality of Common-Cause Boxes
- Allen et al., 2017: Quantum Common Causes and Quantum Causal Models
- Chaves et al., 2014, 2015: in some cases (hidden) *common causes* can be distinguished from direct causation using information from the theoretical generalization of **Bell's inequalities** and causal directed acyclic graphs (DAGs).
- We introduce an alternative algorithmic approach to distinguish between a hidden common cause and direct (indirect) causal influences among two observed quantum systems via **quantum common entropy**.

Quantum Common Entropy for Identification of Latent Graphs

- **Quantum Common Entropy** $[G(X,Y) \coloneqq S(Z)]$: Given ρ_{XY}, find Z with <u>minimum entropy</u> such that $X \perp\!\!\!\perp Y|Z$.
- **Goal**: Identifying Correlation without Causation via quantum Common Entropy (latent graph (a)) using joint density matrix ρ_{XY}.
- **Assumption**: Suppose latent confounding is weak, i.e., $S(Z) < \theta$

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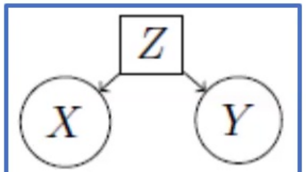
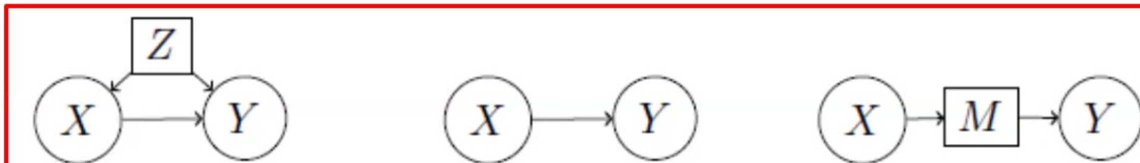


(a) Latent Graph



(b) Triangle Graph (c) Direct Graph (d) Mediator Graph

How to compute quantum common entropy?

- **Quantum Common Entropy** $[G(X,Y) \coloneqq S(Z)]$: Given ρ_{XY}, find Z with <u>minimum entropy</u> such that $X \perp\!\!\!\perp Y|Z$.
- Difficult to compute directly
- Instead of that we find the trade-off between the entropy of the unmeasured confounder and the quantum conditional mutual information of two observed quantum systems given the unmeasured confounder.

$$L = I_Q(X;Y|Z) + \beta S(Z)$$

- $I_Q(X;Y|Z) = 0$ implies the quantum conditional independence of X and Y given Z. And $S(Z)$ is small according to our assumption.
- Rather than searching over ρ_{XYZ} and enforcing the constraint $\rho_{XY} = \mathrm{Tr}_Z(\rho_{XYZ})$, we search over $\rho(Z|X,Y)$. Why and how?

How to write our objective loss function $L = I_Q(X;Y|Z) + \beta S(Z)$ based on $\rho(Z|X,Y)$

- $L = I_Q(X;Y|Z) + \beta S(Z)$
- $= S(XZ) + S(YZ) - S(Z) - S(XYZ) + \beta S(Z)$
- $= S(XZ) + S(YZ) - S(XYZ) + (\beta - 1)S(Z)$
- $= S(X) + S(Z|X) + S(Y) + S(Z|Y) - S(XY) - S(Z|X,Y) + (\beta - 1)S(Z)$
- $= S(Z|X) + S(Z|Y) - S(Z|X,Y) + (\beta - 1)S(Z) + I_Q(X;Y)$
- Due to our choice is quantum conditional states, we can rewrite all of terms based on $\rho(Z|X,Y)$. For example:
- $\rho(Z|X) = Tr_X\left[\left(\rho^{\frac{1}{2}}(X|Y)\otimes I_Z\right)\right]\rho(Z|X,Y)\left[\left(\rho^{\frac{1}{2}}(X|Y)\otimes I_Z\right)\right]$

How to find a stationary point of the optimization problem? An iterative algorithm

1) **Calculate Phase** (iteration i): In this phase, we use partial trace to get $\rho^i(Z|X)$ (lines 3 to 5), $\rho^i(Z|Y)$ (lines 6 to 8), and ρ_Z^i (line 9) from ρ_{XYZ}^i.
2) **Update Phase**: In this phase we update $\rho_{i+1}(Z|X,Y)$ to get ρ_{XYZ}^{i+1} (line 10) for the next iteration.
3) **Return**: ρ_{XYZ}

Algorithm 3. QLatentSearch: An iterative algorithm for computing exact quantum common entropy.

Input: Joint density matrix ρ_{XY}; Number of iterations N; β parameter in the loss function $L = I_Q(X;Y|Z) + \beta S(Z)$, Initialization of $\rho_1(Z|X,Y)$.

Output: Joint density matrix ρ_{XYZ}.

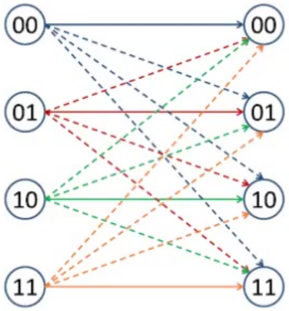
1 **for** $i = 1 : N$ **do**
  `/* Form the joint density matrix: */`
2 $\rho^i_{XYZ} = (\rho_{XY}^{1/2} \otimes I_Z)\rho_i(Z|X,Y)(\rho_{XY}^{1/2} \otimes I_Z)$;
  `/* Calculate Phase: */`
  `/* (i) Calculate` $\rho_i(Z|X)$: `*/`
3 $\rho^i_{XZ} = Tr_Y(\rho^i_{XYZ})$ `// Then, compute` $\rho^i_{XI_YZ}$ `by reordering the entries of` ρ^i_{XZ}
4 $\rho^i_X = Tr_Z(\rho^i_{XZ})$;
5 $\rho_i(Z|X) \leftarrow ((\rho^i_X)^{-1/2} \otimes I_{YZ})\rho^i_{XI_YZ}((\rho^i_X)^{-1/2} \otimes I_{YZ})$;
  `/* (ii) Calculate` $\rho_i(Z|Y)$: `*/`
6 $\rho^i_{YZ} = Tr_X(\rho^i_{XYZ})$ `// Then, compute` $\rho^i_{I_XYZ} = I_X \otimes \rho^i_{YZ}$
7 $\rho^i_Y = Tr_Z(\rho^i_{YZ})$;
8 $\rho_i(Z|Y) \leftarrow (I_X \otimes (\rho^i_Y)^{-1/2} \otimes I_Z)\rho^i_{I_XYZ}(I_X \otimes (\rho^i_Y)^{-1/2} \otimes I_Z)$;
  `/* (iii) Calculate` ρ^i_Z: `*/`
9 $\rho^i_Z = Tr_{XY}(\rho^i_{XYZ})$;
  `/* Update Phase: */`
10 $\rho_{i+1}(Z|X,Y) \leftarrow \exp\big(\log(\rho_i(Z|X)) + \log(\rho_i(Z|Y)) + (\beta - 1)\log(\rho^i_Z)\big)$;
11 **end**
12 **return** $\rho_{XYZ} := (\rho_{XY}^{1/2} \otimes I_Z)\rho_{N+1}(Z|X,Y)(\rho_{XY}^{1/2} \otimes I_Z)$.

QINFERGRAPH: An algorithm for the identification of latent confounders

- **Assumption**: Consider any causal model with observed quantum subsystems X and Y. Let Z represents the quantum system that captures all latent confounders between X and Y. Then $S(Z) < \theta$, where $S(Z) = -\mathrm{Tr}(\rho_Z \ln(\rho_Z))$.
- QINFERGRAPH calls QLATENTSEARCH N times to figure out if there exist a W, for which $I_Q(X;Y|W) < T$, i.e., W makes X and Y **conditionally independent**.
- Also, if the von Neumann entropy of W is enough small such that $S(W) < \alpha \min\{S(X), S(Y)\}$ for some $\alpha \in (0,1)$, then
- The algorithm declares W is a latent confounder. $L = I_Q(X;Y|Z) + \beta S(Z)$

Synthetic Setting of latent graphs in noisy channels

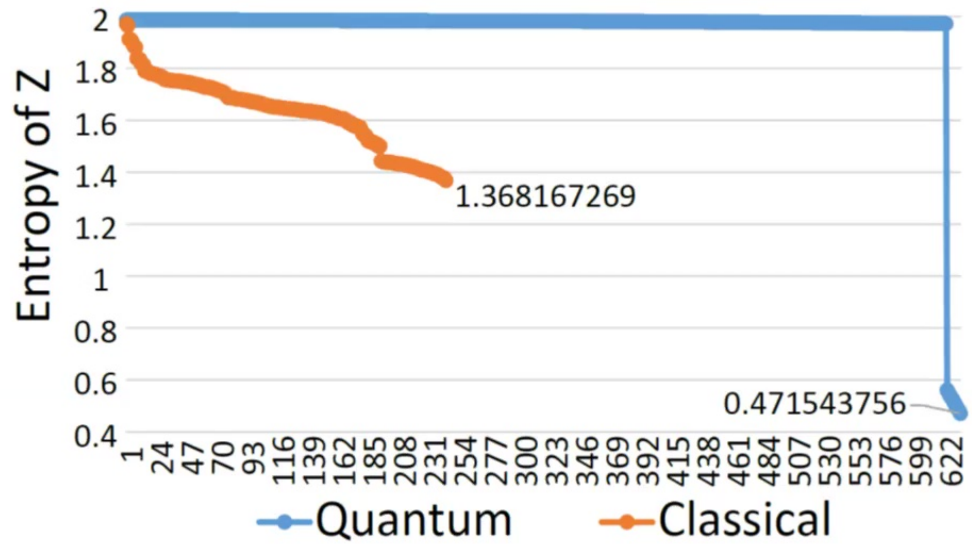
- **Model 1**: Assume a 2-bit input $Z \in \{00, 01, 10, 11\}$. Let each bit of Z be in the state 1 with probability q and $1-q$ otherwise, and independent of each other.
- Z is transmitted over a binary symmetric channel with independent bit-error probability of p_1 and is denoted X (See Figure with Z as input and X as output).
- A cloned version of Z is transmitted over a binary symmetric channel with independent bit-error probability of p_2 and is denoted Y.
- $p(X, Y, Z) = p(Z)p(X|Z)p(Y|Z)$.
- We marginalize out Z to obtain the joint probability distribution for the latent graph $X \leftrightarrow Y$.



Identification of latent graphs in noisy channels

- Validation of latent graph in **Model 1** via classical causal inference ($\alpha = 0.8$) vs quantum causal inference ($\alpha = 0.2$) for $T = 0.001$ and $\beta \in (0,1)$.

p_1 \ p_2	0.01	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99
0.01	F	F	F	T	T	T	T	T	F	F	F
0.1	F	F	F	T	T	T	T	T	F	F	F
0.2	F	F	T	T	T	T	T	T	T	F	F
0.3	T	T	T	T	T	T	T	T	T	T	T
0.4	T	T	T	T	T	T	T	T	T	T	T
0.5	T	T	T	T	T	T	T	T	T	T	T
0.6	T	T	T	T	T	T	T	T	T	T	T
0.7	T	T	T	T	T	T	T	T	T	T	T
0.8	F	F	T	T	T	T	T	T	T	F	F
0.9	F	F	F	T	T	T	T	T	F	F	F
0.99	F	F	F	T	T	T	T	T	F	F	F

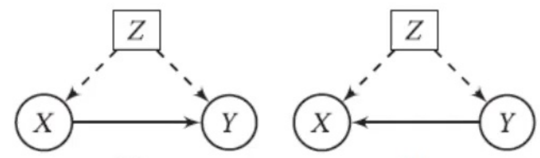


Some highlights for results in Model 1

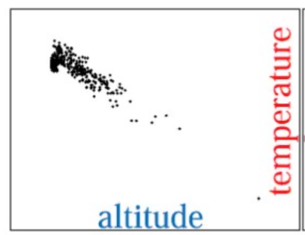
- When the probability of errors, i.e., p_1 and p_2 are very small, the latent confounder Z is hardly distinguishable from X (or Y).
- It seems that the classical causal inference algorithm is much more sensitive to the choice of hyperparameter α, while QINFERGRAPH is more robust to the choice of this parameter.
- QINFERGRAPH constantly returns local optima with lower entropy in comparison with the classical INFERGRAPH algorithm.

Real Database with cause-effect pairs: Tuebingen dataset

- Tubingen: Database with cause-effect pairs of the form (a) or (b).
- First cause-effect pair of data from Tubingen database: altitude causes temperature.



(a) (b)



Algorithm	True positive	False positive	False negative	Accuracy
QINFERGRAPH ($\alpha = 0.2, T = 0.005$)	**0.83**	0	**0.17**	**0.83**
Classical INFERGRAPH ($\alpha = 0.8, T = 0.001$)	0.32	0	0.68	0.32
Classical INFERGRAPH ($\alpha = 0.7, T = 0.001$)	0.49	0	0.51	0.49

Why should not map quantum to classical directly?

- We *lose* some quantum information due to the loss of **entanglement**.
- To verify this, we used a depolarizing channel as described in the paper: in all cases failed to obtain the correct results.

		p										
		0.01	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99
α	0.7	F	F	F	F	F	F	F	F	F	F	F
	0.8	F	F	F	F	F	F	F	F	F	F	F
	0.9	F	F	F	F	F	F	F	F	F	F	F
	1	F	F	F	F	F	F	F	F	F	F	F

Future Work

- Experiment on quantum simulators.
- The classical algorithm is fast even with large support sizes of X and Y. Since our algorithm uses matrices, it's quite slow when the support size of X or Y becomes large.

Summary

Quantum Common Entropy for Identification of Latent Graphs

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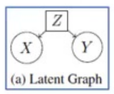

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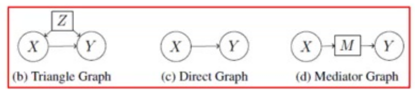

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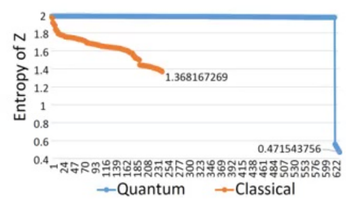

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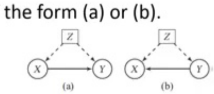

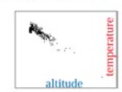

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