Title: On the computation of Poisson brackets in field theories

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Abstract: I will discuss several issues related to the computation of Poisson brackets in field theories. I will show that by choosing appropriate functional spaces as configuration manifolds, it is possible to avoid the use of distributions. This is relevant, for instance, in the context of Loop Quantum Gravity, where the basic holonomy/flux variables play a central role. I will illustrate the main ideas by using simple examples based on Sobolev spaces.

Zoom link: https://pitp.zoom.us/j/92794475242?pwd=T2Fjbk1lTXRCNnBxVDZabnJqWlAzUT09

On the computation of Poisson brackets in field theories

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Summary

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1. Motivation
2. Configuration spaces for field theories.
3. The phase space of a field theory.
4. Poisson brackets.
5. Differentiability.
6. Examples in Sobolev spaces.

Motivation

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- The **holonomy-flux algebra** in Loop Quantum Gravity involves distributional objects.
- In practice it must be defined by means of a relatively subtle **regularization** procedure.
- Naive, straightforward computations **quickly lead into trouble** (apparent violations of the Jacobi identities).
- In order to make computations rigorous, some **care with functional analytic issues** seems necessary.

This may be the case for all field theories

In particular when **boundaries** are present.

Configuration space

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- The configuration space Q for a mechanical system is, usually, a **finite-dimensional differentiable manifold**.
- The configuration spaces for field theories are "spaces of regular enough functions" defined on some spatial manifold Σ.
- The specification of what **regular** means often reduces to saying that the fields are **as smooth as needed**, so that they can satisfy dynamical equations involving differential operators.
- **More care is needed**. For instance, the specification of the topology of infinite dimensional manifolds (Banach manifolds, Hilbert manifolds,...) may be necessary and play an important role.
- Sometimes **it is not just a matter of mathematical rigor.**
- In this talk I will use **Sobolev spaces** as configuration spaces.

Phase space

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- Lagrangian dynamics is defined by a real function $L : TQ \to \mathbb{R}$ on the **tangent bundle** TQ of the configuration space Q from which the action is built as a **real function on a space of appropriate paths** in Q.

- The **stationary points of the action** correspond to the physical evolution of the system. They are given by the **Euler-Lagrange equations**.

- **Hamiltonian dynamics** takes place in the **phase space** T^*Q, the cotangent bundle of the configuration space Q. A fiber $T^*_q Q$ ($q \in Q$) is **dual** to the fiber $T_q Q$ (a topological vector space).

- Whereas in finite-dimensional manifolds the dual spaces are very simple, **the infinite dimensional case is richer and more subtle**.

The canonical symplectic form back

- If Q is finite-dim T^*Q is endowed with a **canonical symplectic form** Ω

  Closed: $\mathbb{d}\Omega = 0$.

  Non-degenerate: $\flat : \mathfrak{X}(Q) \to \Omega^1(Q) : \mathbb{X} \mapsto \Omega(\cdot, \mathbb{X})$ is an **isomorphism**.

- In the **finite dimensional** case if we have coordinates q_i on Q and write 1-forms as $\alpha = \sum_i p_i \mathrm{d}q^i$ we can use the q^i and p_i as canonical coordinates in phase space. In terms of them the symplectic form is

$$\Omega = \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i \,.$$

- This can be interpreted in the following way: given two vector fields in T^*Q, $\mathbb{X} = \sum_i X^i_q \partial_{q^i} + \sum_i X^p_i \partial_{p_i}$, $\mathbb{Y} = \sum_i Y^i_q \partial_{q^i} + \sum_i Y^p_i \partial_{p_i}$ we have

$$\Omega(\mathbb{X}, \mathbb{Y}) = \sum_i Y^p_i X^i_q - \sum_i X^p_i Y^i_q \,.$$

The canonical symplectic form back

- A point in T^*Q can be interpreted as a pair $(x, \mathbf{p})$ with $x \in Q$ and $\mathbf{p} \in T_x^*Q$, (i.e. a covector).
- A vector X in $T_{(x,\mathbf{p})}T^*Q$ can be interpreded as a pair $(X_q, \mathbf{X}_p)$ with $X_q \in T_xQ$ and $\mathbf{X}_p \in T_x^*Q$. We can then write

$$\Omega(\mathbb{X}, \mathbb{Y}) = \mathbf{Y}_p(X_q) - \mathbf{X}_p(Y_q) \qquad (*)$$

- **What about field theories?** One is tempted to "generalize as usual"

$$\sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i \rightsquigarrow \int_\Sigma \mathbb{d}q(x) \mathbin{\wedge\!\!\!\wedge} \mathbb{d}p(x)\, \mathrm{d}^n x\,,$$

where Σ is the manifold where the fields live.
- This, however, does not make much sense. For instance, what measure should be use in the integral?

The canonical symplectic form back

- The correct way to think about this problem is to use $(*)$ and **be careful with the duals**[1] (whose definition depends on the **functional details** of the configuration space).

$$\Omega(\mathbb{X}, \mathbb{Y}) = \sum_i \left(\mathbf{Y}_{p_i}(X_{q^i}) - \mathbf{X}_{p_i}(Y_{q^i}) \right)$$

- A concrete, **and properly justified**, mathematical description of the **covectors** in the fiber $T_q^* Q$ is necessary!

[1] See Marsden, *Applications of global analysis in Mathematical Physics.*

Hamiltonian vector fields and Poisson brackets back

- On Banach manifolds modeled on reflexive spaces (i.e. such that $\mathcal{H}^{**}$ and $\mathcal{H}$ are linearly isomorphic) the 2-form Ω is **closed** and **non-degenerate** (**strongly symplectic** or just **symplectic**).
- Interesting examples are **manifolds modeled on Hilbert spaces**.
- The **Hamiltonian vector field** $\mathbb{X}^f$ in T^*Q associated with a **differentiable function** $f : T^*Q \to \mathbb{R}$ is the (unique) field $\mathbb{X}^f$ satisfying

$$\imath_{\mathbb{X}^f}\Omega = \mathrm{d}f\,.$$

- If f and g are two real **differentiable** functions in T^*Q their **Poisson bracket** is defined as:

$$\{f,g\} := \mathrm{d}f(\mathbb{X}^g) = \mathbb{X}^g(f) = \mathcal{L}_{\mathbb{X}^g} f = \imath_{\mathbb{X}^g}\mathrm{d}f = \imath_{\mathbb{X}^g}\imath_{\mathbb{X}^f}\Omega = \Omega(\mathbb{X}^f, \mathbb{X}^g)$$

  where $\mathbb{X}^f$ and $\mathbb{X}^g$ are the Hamiltonian vector fields defined by f and g.
- Notice that $\{f,g\} = -\{g,f\}$.

Example of a Sobolev space

back

- In this talk I will use **Sobolev spaces**. Strictly speaking their elements **are not functions** but, rather, equivalence classes of functions, so I will work with a slight generalization of the physical concept of field.
- I will look at examples in **one dimension**.
- **Definition 1:** *Let I be an open interval of the real line $\mathbb{R}$, bounded or unbounded, and let $C^1_c(I)$ denote the space of continuously differentiable real functions with compact suport on I (test functions). We define*

$$H^1(I) := \{u \in L^2(I) : \exists g \in L^2(I) \ s.\,t. \ \int_I u\varphi' = -\int_I g\varphi \,, \forall \varphi \in C^1_c(I)\}$$

- The "functions" in $H^1(I)$ have **square integrable weak derivatives** defined on test functions by integration by parts. I will write $g = u'$.

Example of a Sobolev space

back

- **Theorem 1:** *With the scalar product*

$$\langle u, v\rangle_{H^1} := \int_I (uv + u'v')\,,$$

and its associated norm $\|u\|_{H^1} := \langle u, u\rangle_{H^1}$, *the space* $H^1(I)$ *is a separable Hilbert space.*[2]

- **Theorem 2:** *Let* $u \in H^1(I)$ *and* I *a bounded or unbounded interval of* $\mathbb{R}$; *then there exists a continuous function* $\tilde{u} \in C(\bar{I})$ *such that* $u = \tilde{u}$ *almost everywhere, and*

$$\tilde{u}(x) - \tilde{u}(y) = \int_y^x u'(t)\mathrm{d}t\,, \quad \forall x, y \in \bar{I}\,.$$

- This provides a **very useful characterization** of the elements of $H^1(I)$.

[2] See Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations.*

Example of a Sobolev space

back

Another useful (and representative) result:

- **Theorem 3:** *Let $u, v \in H^1(I)$. Then $uv \in H^1(I)$ and $(uv)' = u'v + uv'$. Furthermore, the formula for integration by parts holds:*

$$\int_a^b u'v = \tilde{u}(b)\tilde{v}(b) - \tilde{u}(a)\tilde{v}(a) - \int_a^b uv' , \quad \forall a, b \in \overline{I} .$$

- As a consequence $H^1(I)$ is **a Banach algebra**.
- The product of elements of $H^1(I)$ is well defined.

Sobolev spaces (comments)

back

- There are **more general examples** of Sobolev spaces (dimensions, number of derivatives, L^p,...)
- The elements of $H^1(I)$ have **continuous representatives** in $\overline{I}$. We can talk about the values of $u \in H^1(I)$ **at any point** $x \in \overline{I}$ despite the fact that the elements of $H^1(I)$ are defined only modulo zero measure sets.
- The **boundary values** (traces) of u are well defined if I is bounded.
- The continuous representative $\tilde{u}$ of $u \in H^1(I)$ is, actually, **differentiable** a.e. and **the classical derivative is equal to the weak derivative a.e.**
- In Hilbert spaces the **duals** are characterized by the **Riesz-Fréchet representation theorem:**

  Let $F : \mathcal{H} \to \mathbb{R}$ be a linear map on a real Hilbert space $\mathcal{H}$, then F is continuous if and only if there exists $\psi \in \mathcal{H}$ such that $F(v) = \langle \psi, v \rangle_{\mathcal{H}}$, for all $v \in \mathcal{H}$. Furthermore, ψ is unique and $\|F\|_{\mathcal{H}^*} = \|\psi\|_{\mathcal{H}}$.

Differentiability and functional derivatives back

The following definition of differentiability is standard in mathematics:

- **Definition 2 (Fréchet differentiability):** *Let* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be two Banach spaces with norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *respectively. Let* $A \subset \mathcal{B}_1$ *be open and* $x \in A$ *and consider a function* $f : A \to \mathcal{B}_2$. *We say that the function* f *is Fréchet differentiable at* x *if there exists a linear and continuous map* $\mathbb{d}_x f : \mathcal{B}_1 \to \mathcal{B}_2 : h \mapsto \mathbb{d}_x f(h)$, *called the differential of* f *at* x, *such that*

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - \mathbb{d}_x f(h)\|_2}{\|h\|_1} = 0\,.$$

- When the differential exists it is **unique**.
- The differential of a linear and continuous function **coincides with itself**.
- Differentiability is a useful concept that allows us to **extend many results of analysis** in $\mathbb{R}^n$ to **infinite dimensional Banach** spaces.
- This is the kind of differential used to define the **exterior derivative** in differential geometry.

Functional differentiability in gravity back

- In the Lagrangian and Hamiltonian formulations of field theories **on manifolds with boundaries**, the word *differentiability* often means that the variation of a functional $S[\phi]$ depending on fields ϕ has the form

$$\delta S = \int_M \frac{\delta S}{\delta \phi} \delta \phi \,,$$

  with **no boundary integrals in the right hand side**.
- This *RT-differentiability condition* guarantees that the variational equations coming from the action do not have boundary contributions that **might clash with the boundary conditions** imposed on the fields.
- The Fréchet and RT-differentiability concepts **have little in common**.
- The mathematical consequences of Fréchet differentiability are clear, **but it is quite dangerous to export them to situations in which differentiability is understood in the second sense.**
- A situation where this applies is the **computation of Poisson brackets in field theories on manifolds with boundaries.**

Poisson brackets again

back

- The **relevant differentiability concept** to compute Poisson brackets is **Fréchet differentiability** (why use the other?) combined here with the mathematical properties of Sobolev spaces (configuration manifolds).

- The standard formula found in the physical literature in the context of field theories is

$$\{f, g\} = \int_{\mathbb{R}^n} \left(\frac{\delta f}{\delta \varphi(x)} \frac{\delta g}{\delta \pi(x)} - \frac{\delta g}{\delta \varphi(x)} \frac{\delta f}{\delta \pi(x)} \right) \mathrm{d}^n x \,.$$

- On the other hand, remember how Poisson brackets look like for field theories.

$$\Omega(\mathbb{X}, \mathbb{Y}) = \mathbf{Y}_p(X_q) - \mathbf{X}_p(Y_q)$$

- **Can we connect both expressions?**

Functional derivatives again

back

To do this the following definition of functional derivative is very useful:

- **Definition 3 (Functional derivative):** *Let us consider a differentiable function $F : H \to \mathbb{R}$ on a Sobolev Hilbert space H. The functional derivative of F at $\psi \in H$, denoted as $D_\psi F$, is the unique element of H satisfying*

$$\mathbb{d}_\psi F(h) = \langle D_\psi F, h \rangle_H \,,$$

for all $h \in H$.

- This **functional derivative** $D_\psi F$ is the **Riesz-Fréchet representative** of the Fréchet-differential $\mathbb{d}_\psi F$.
- This is well defined as a consequence of the **Riesz-Fréchet representation theorem** and the fact that **the differential is a continuous linear functional.**

Poisson brackets again

back

- By using the **Riesz-Fréchet representatives** of $\mathbf{X}_p^f$ and $\mathbf{X}_p^g$ (X_p^f and X_p^g, respectively) we find

$$\{f,g\} = \Omega(\mathbb{X}^f, \mathbb{X}^g) = \langle X_p^g, X_q^f \rangle_H - \langle X_p^f, X_q^g \rangle_H \,.$$

- In order to write the symplectic form in terms of functional derivatives we introduce **partial functional derivatives** in the obvious way. As we have two arguments (fields and momenta) I denote them as D_φ and D_π
- If the configuration space is a **Sobolev space** H we have

$$\boxed{\{f,g\} := \langle D_\varphi f, D_\pi g \rangle_H - \langle D_\varphi g, D_\pi f \rangle_H}$$

Notice that it is written **in terms of the scalar product in** H!

Examples: the evaluation

back

- Take $H^1(0,1)$ as the **configuration space**.
- Let $x \in [0,1]$. Define the **evaluation** $\mathrm{Ev}_x : H^1(0,1) \to \mathbb{R} : \varphi \mapsto \widetilde{\varphi}(x)$, where $\widetilde{\varphi}$ is the continuous representative of $\varphi \in H^1(0,1)$ whose existence is guaranteed by Theorem 2.
- For every $\varphi \in H^1(0,1)$ we can show that $\mathrm{Ev}_x(\varphi)$ can be written **as a scalar product** in $H^1(0,1)$, indeed, let

$$\mathcal{E}_x : [0,1] \to \mathbb{R} : t \mapsto \mathcal{E}_x(t) = \begin{cases} \dfrac{\cosh(1-x)\cosh t}{\sinh 1}, & t \in [0,x] \\[2ex] \dfrac{\cosh x \cosh(1-t)}{\sinh 1}, & t \in [x,1] \end{cases}$$

  then $\mathrm{Ev}_x(\varphi) = \widetilde{\varphi}(x) = \langle \mathcal{E}_x | \varphi \rangle_{H^1}$ [Exercise]. $\mathcal{E}_x \in H^1(0,1)$.
- Notice that for $x, y \in [0,1]$ we have $\mathcal{E}_x(y) = \mathcal{E}_y(x)$.

Examples: the evaluation

back

- As a consequence of the previous result we immediately see that Ev_x is **linear and continuous**, i.e. $\mathrm{Ev}_x \in H^1(0,1)^*\,, \forall x \in [0,1]$.
- This implies that Ev_x is **differentiable** and **the differential is given by** Ev_x **itself**.

$$\mathbb{d}_\varphi \mathrm{Ev}_x = \mathrm{Ev}_x \Leftrightarrow \mathbb{d}_\varphi \mathrm{Ev}_x(h) = \mathrm{Ev}_x(h) = \tilde{h}(x)\,, \forall h \in H^1(0,1)\,.$$

- According to our definition the **functional derivative** of Ev_x is then

$$D_\varphi \mathrm{Ev}_x = \mathcal{E}_x\,,$$

which is **independent** of $\varphi \in H^1(0,1)$.

- It is not possible to define the **evaluation of the derivative** of an element of $u \in H^1(0,1)$ despite the fact that the continuous representative of such an u is absolutely continuous and, hence, differentiable a.e.

Examples: a non RT-differentiable function

back

- Let us consider now the **non-linear function**

$$\mathcal{V} : H^1(0,1) \to \mathbb{R} : \varphi \mapsto \frac{1}{2}\int_{[0,1]} (\varphi')^2 \,.$$

This is well defined because $\varphi \in H^1(0,1)$.

- This function is **not differentiable in the RT sense** because

$$\delta\mathcal{V} = \int_0^1 \varphi'(\delta\varphi)' = \varphi'(1)\delta\varphi(1) - \varphi'(0)\delta\varphi(0) - \int_0^1 \varphi''\delta\varphi \,,$$

which is not of the form $\delta\mathcal{V} = \int_0^1 \frac{\delta\mathcal{V}}{\delta\varphi}\delta\varphi$, (**there are boundary terms**).

- Furthermore, notice that the preceding computation is not justified if φ'' is not defined [which it does not have to for a generic $\varphi \in H^1(0,1)$].

Examples: a non RT-differentiable function

back

- However, $\mathcal{V}$ is **Fréchet-differentiable** and $\mathbb{d}_\varphi \mathcal{V}(h) = \int_{[0,1]} \varphi' h'$.
- In order to show that this is indeed the Fréchet-differential of $\mathcal{V}$ we have to check its **linearity**, its **continuity** and also (see the definition)

$$\lim_{h\to 0} \frac{1}{\|h\|_{H^1}} \left| \mathcal{V}(\varphi + h) - \mathcal{V}(\varphi) - \int_{[0,1]} \varphi' h' \right| = \lim_{h \to 0} \frac{1}{2\|h\|_{H^1}} \left| \int_{[0,1]} (h')^2 \right| = 0\,.$$

- The **linearity** of $\mathbb{d}_\varphi \mathcal{V}$ is obvious.
- **Continuity** $[A : \mathcal{B}_1 \to \mathcal{B}_2$ linear; $\exists C > 0 : \|Ax\|_2 \leq C\|x\|_1\,, \forall x \in \mathcal{B}_1.]$

$$\left| \int_{[0,1]} \varphi' h' \right| = \left| \int_{[0,1]} (\varphi' h' + \varphi h - \varphi h) \right| = |\langle \varphi, h \rangle_{H^1} - \langle \varphi, h \rangle_{L^2}|$$
$$\leq |\langle \varphi, h \rangle_{H^1}| + |\langle \varphi, h \rangle_{L^2}| \leq \|\varphi\|_{H^1} \|h\|_{H^1} + \|\varphi\|_{L^2} \|h\|_{L^2}$$
$$\leq 2\|\varphi\|_{H^1} \|h\|_{H^1}$$

Examples: a non RT-differentiable function back

- I have used the **Cauchy-Schwarz inequality** and $\|\psi\|_{L^2} \leq \|\psi\|_{H^1}$ for every $\psi \in H^1(0,1)$.

- Finally $0 \leq \frac{1}{\|h\|_{H^1}}\left|\int_{[0,1]}(h')^2\right| \leq \frac{1}{\|h\|_{H^1}}\left|\int_{[0,1]}(h^2+(h')^2)\right| = \|h\|_{H^1}$.

The continuity of the norm immediately gives $\lim_{h\to 0}\frac{1}{2\|h\|_{H^1}}\left|\int_{[0,1]}(h')^2\right| = 0$.

- A convenient description of the **functional derivative** $D_u\mathcal{V} \in H^1(0,1)$ can be obtained by considering its evaluation for every $x \in (0,1)$

$$\mathrm{Ev}_x(D_u\mathcal{V}) = \langle \mathcal{E}_x, D_u\mathcal{V}\rangle_{H^1} = \mathbb{d}_u\mathcal{V}(\mathcal{E}_x) = \int_{[0,1]} u'\mathcal{E}'_x = \langle \mathcal{E}_x, u\rangle_{H^1} - \langle \mathcal{E}_x, u\rangle_{L^2}$$

$$= \widetilde{u}(x) - \frac{\cosh(1-x)}{\sinh 1}\int_0^x u(t)\cosh t\,\mathrm{d}t - \frac{\cosh x}{\sinh 1}\int_x^1 u(t)\cosh(1-t)\mathrm{d}t\,.$$

Poisson brackets of the canonical evaluations back

- Take the **phase space** $H^1(0,1) \times H^1(0,1)^* \cong H^1(0,1) \times H^1(0,1)$.
- Define the **projections**

$$\mathrm{proj}_1 : H^1(0,1) \times H^1(0,1) \to H^1(0,1) : (\varphi, \pi) \mapsto \varphi\,,$$
$$\mathrm{proj}_2 : H^1(0,1) \times H^1(0,1) \to H^1(0,1) : (\varphi, \pi) \mapsto \pi\,,$$

and the **partial evaluations** ($x \in [0,1]$)

$$\Phi_x := \mathrm{Ev}_x \circ \mathrm{proj}_1\,, \quad \Pi_x := \mathrm{Ev}_x \circ \mathrm{proj}_2\,,$$

- These are **real differentiable functions** in the phase space $H^1(0,1) \times H^1(0,1)$ because proj_1, proj_2 and the evaluation Ev_x are differentiable.
- Their Poisson brackets are given by

$$\{\Phi_x, \Pi_y\} = \langle D_\varphi \Phi_x, D_\pi \Pi_y \rangle_{H^1} - \langle D_\varphi \Pi_x, D_\pi \Phi_y \rangle_{H^1}$$

Poisson brackets of the canonical evaluations back

- A simple computation using the definitions gives

$$D_\varphi \Phi_x(\varphi,\pi) = \mathcal{E}_x\,,\; D_\pi \Pi_y(\varphi,\pi) = \mathcal{E}_y\,,\; D_\pi \Phi_x(\varphi,\pi) = 0\,,\; D_\varphi \Pi_x(\varphi,\pi) = 0,$$

so that

$$\{\Phi_x,\Pi_y\} = \langle D_\varphi\Phi_x, D_\pi\Pi_y\rangle_{H^1} - \langle D_\varphi\Pi_x, D_\pi\Phi_y\rangle_{H^1} = \langle\mathcal{E}_x,\mathcal{E}_y\rangle_{H^1} = \mathcal{E}_x(y)$$

- Also $\{\Phi_x,\Phi_y\} = 0$, $\{\Pi_x,\Pi_y\} = 0$.
- These Poisson brackets **are completely determined by the canonical symplectic form** in $H^1(0,1)\times H^1(0,1)^*$.
- They are the **basic Poisson brackets** in $H^1(0,1)\times H^1(0,1)^*$ and play the role of the $\{\varphi(x),\pi(y)\} = \delta(x,y)$ in the usual presentations of the Hamiltonian formulation of field theories [with $\delta(x,y) \rightsquigarrow \mathcal{E}_x(y)$].

Poisson brackets of the canonical evaluations back

- Both Φ_x and Π_x are **real functions in phase space**.
- Their Poisson bracket is **also a real function in phase space**. In this example, this function is **constant** [it does not depend on the phase space point (φ, π)].
- Notice that $\{\Phi_x, \Pi_x\} = \mathcal{E}_x(x)$ which is well defined!
- All the objects used are suitably regular, in particular $\mathcal{E}_x \in H^1(0,1)$. Notice also that, at variance with the standard interpretation of the basic Poisson brackets for the scalar field, now $\{\Phi_x, \Pi_y\}$ is **never** zero.

Poisson brackets involving $\mathcal{V}$

back

- We start with $\{\Phi_x, \mathcal{V}\} = -\mathbb{d}\mathcal{V}(\mathbb{X}_{\Phi_x})$. Now

$$\{\Phi_x, \mathcal{V}\}(\varphi, \pi) = -\mathbb{d}_{(\varphi,\pi)}\mathcal{V}(\mathbb{X}_{\Phi_x}) = -(\mathbb{d}_\varphi \mathcal{V} \circ \mathrm{proj}_1)(\mathbb{X}_{\Phi_x})$$
$$= -\mathbb{d}_\varphi \mathcal{V}(X_\varphi) = -\mathbb{d}_\varphi \mathcal{V}(0) = 0 \Rightarrow \{\Phi_x, \mathcal{V}\} = 0.$$

- The computation of $\{\Pi_y, \mathcal{V}\} = -\mathbb{d}\mathcal{V}(\mathbb{X}_{\Pi_y})$ is analogous

$$\{\Pi_y, \mathcal{V}\}(\varphi, \pi) = -\mathbb{d}_{(\varphi,\pi)}\mathcal{V}(\mathbb{X}_{\Pi_y}) = -(\mathbb{d}_\varphi \mathcal{V} \circ \mathrm{proj}_1)(\mathbb{X}_{\Pi_y})$$
$$= -\mathbb{d}_\varphi \mathcal{V}(X_\varphi) = -\mathbb{d}_\varphi \mathcal{V}(\mathcal{E}_y) = -\int_{[0,1]} \mathcal{E}'_y \varphi'.$$

- **There are no obstructions to perform the previous computations. The boundary does not introduce any complications.**

Poisson brackets involving $\mathcal{V}$

back

- The resulting Poisson brackets are Fréchet-differentiable functions. They can be plugged into the Jacobi identity.
- In particular, $\{\Pi_y, \mathcal{V}\}$ is differentiable because it can be written as $F = f \circ \mathrm{proj}_1$ with

$$f : H^1(0,1) \to \mathbb{R} : \varphi \mapsto -\int_{[0,1]} \mathcal{E}_y' \varphi' \,,$$

which is **linear and continuous.**

Final comments

back

- Despite what we have all being taught, **the transit from (Hamiltonian) mechanics to field theories is not straightforward.**
- The **configuration spaces** for field theories **carry important structures** that play relevant roles.
- **Sobolev Hilbert spaces** are nice configuration spaces because they allow us to use derivatives while keeping a scalar product.

The big question

Can we define interesting and useful field theories in these spaces?

Reference: J.F.B.G., M. Basquens, B. Díaz and E.J.S. Villaseñor. *Poisson brackets in Sobolev spaces: a mock holonomy-flux algebra.* Physica Scripta 97 (2022) 125202. ArXiv:2207.00342