Title: Quantum groups, clusters, and Hamiltonian reduction

Speakers: Alexander Shapiro

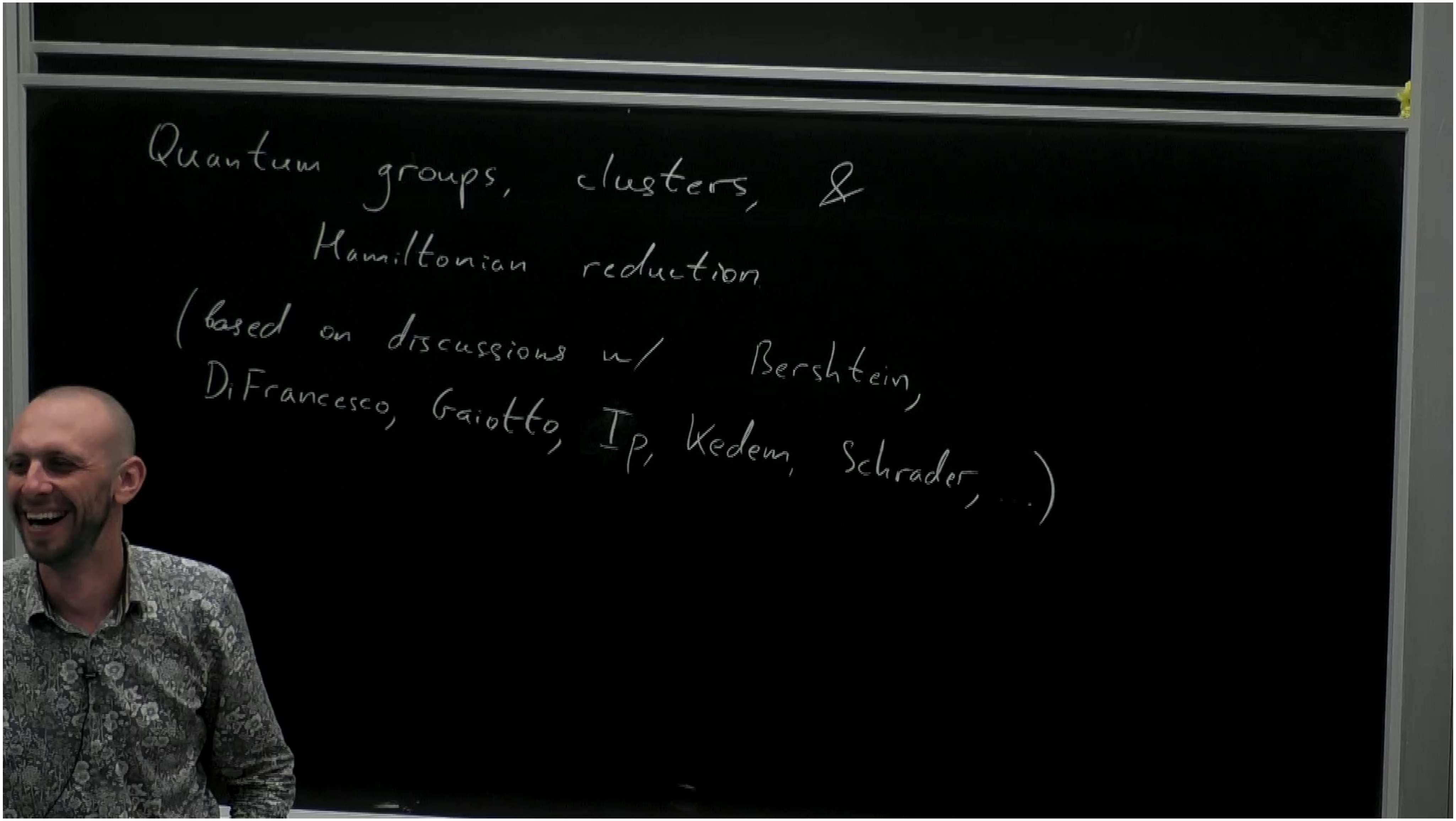
Series: Mathematical Physics

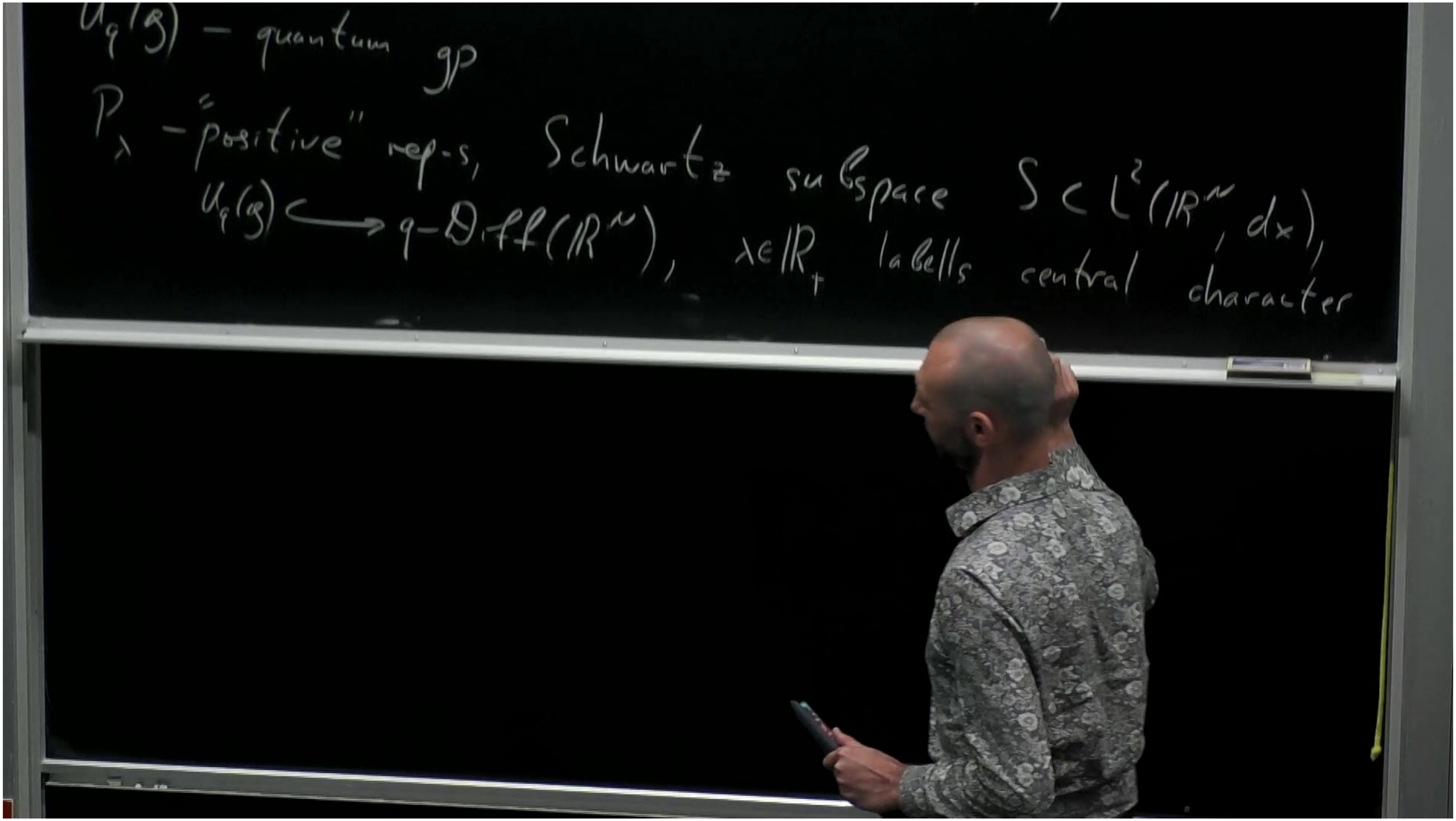
Date: April 13, 2023 - 11:00 AM

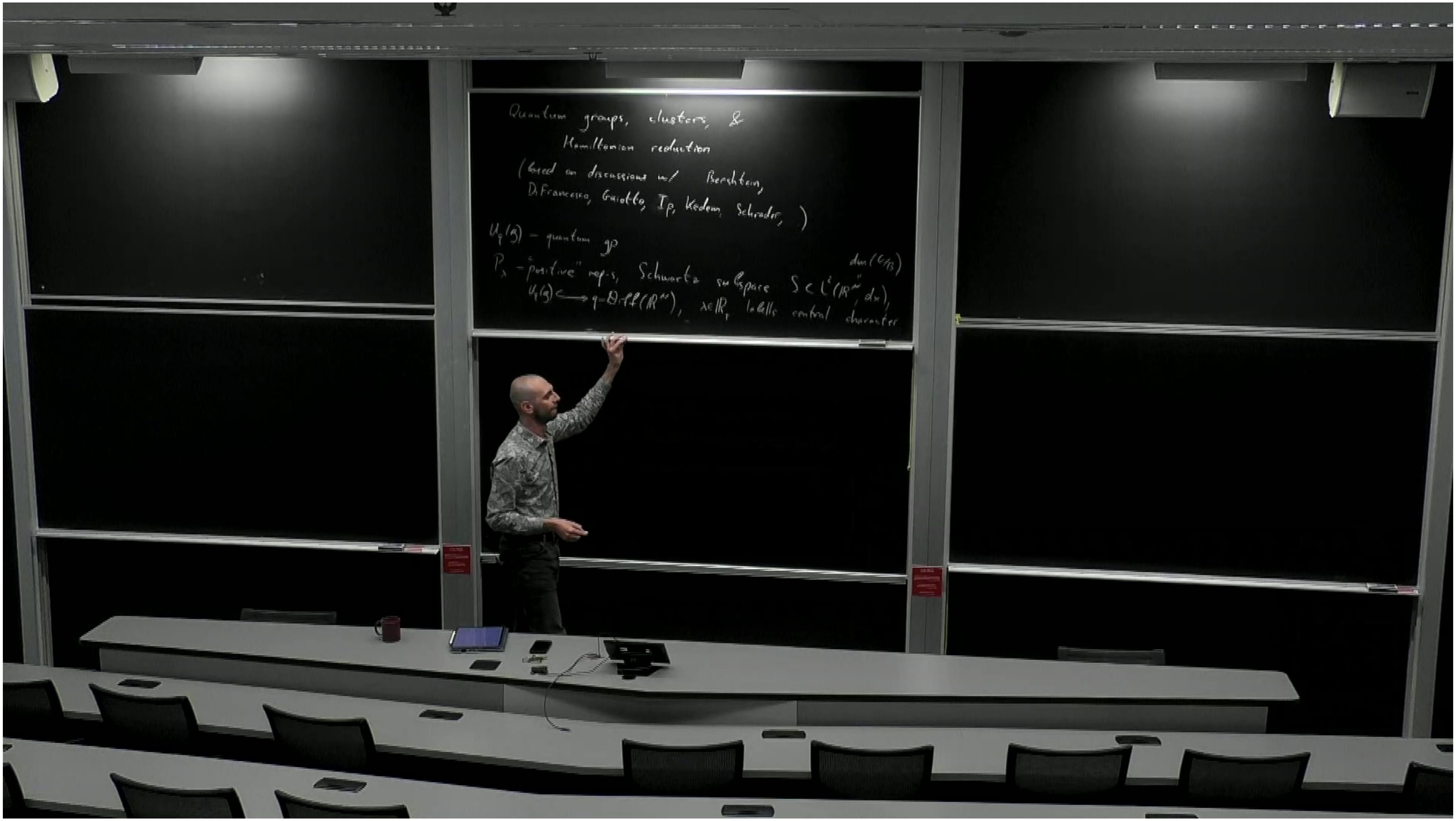
URL: https://pirsa.org/23040130

Abstract: Cluster structure on a quantum group allows one to work with its positive representations, a special class of modules similar in spirit to principal series representations but closed under tensor multiplication. On the other hand, cluster techniques proved inadequate for the study of finite-dimensional representation theory. I will discuss how one can reconcile positive and finite-dimensional representations into one theory by studying moduli spaces of local systems with non-generic monodromies.

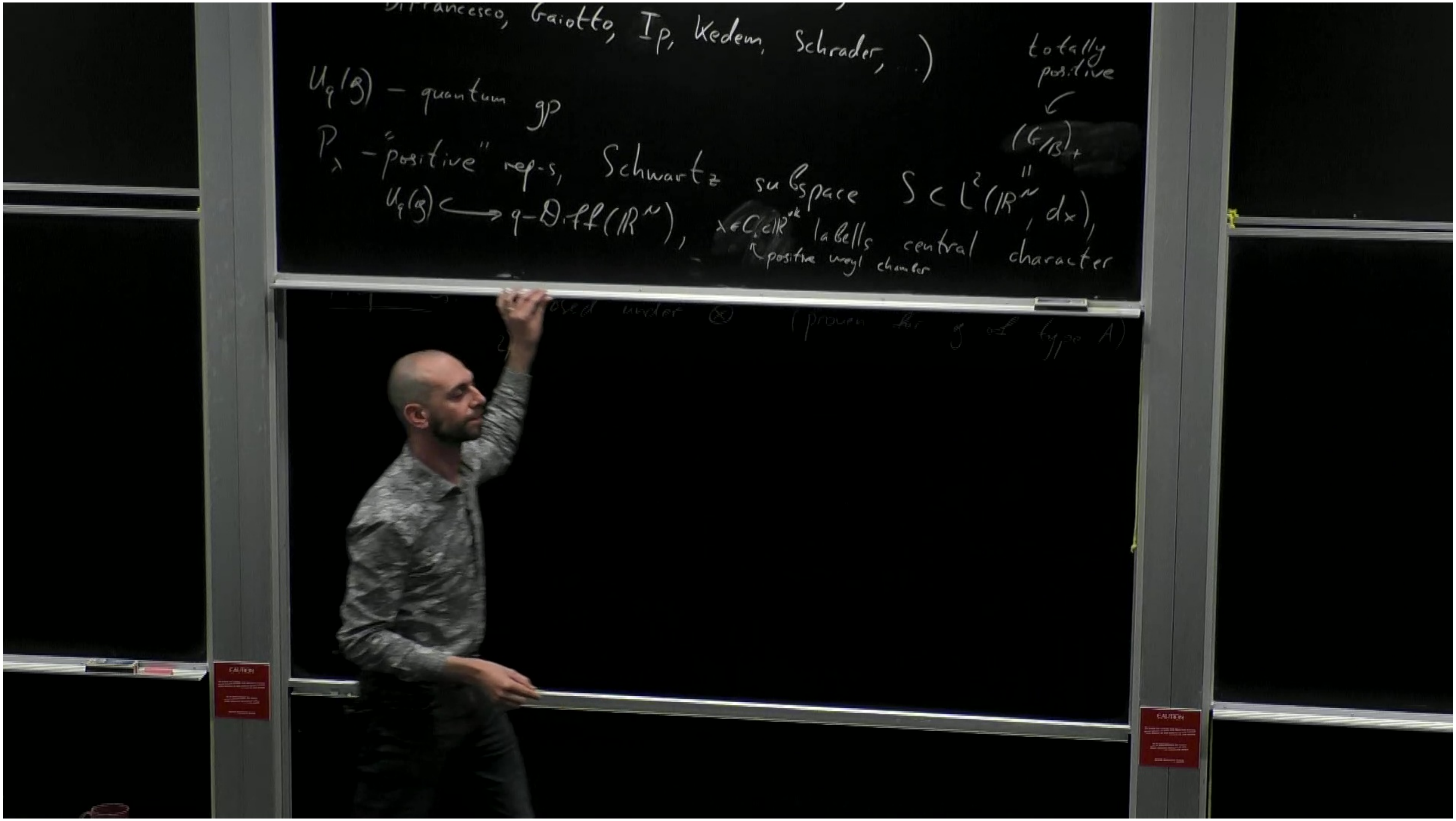
Zoom link: https://pitp.zoom.us/j/97677905324?pwd=Tkxlei9wN1llYVpncHA2cnpYKzlhUT09

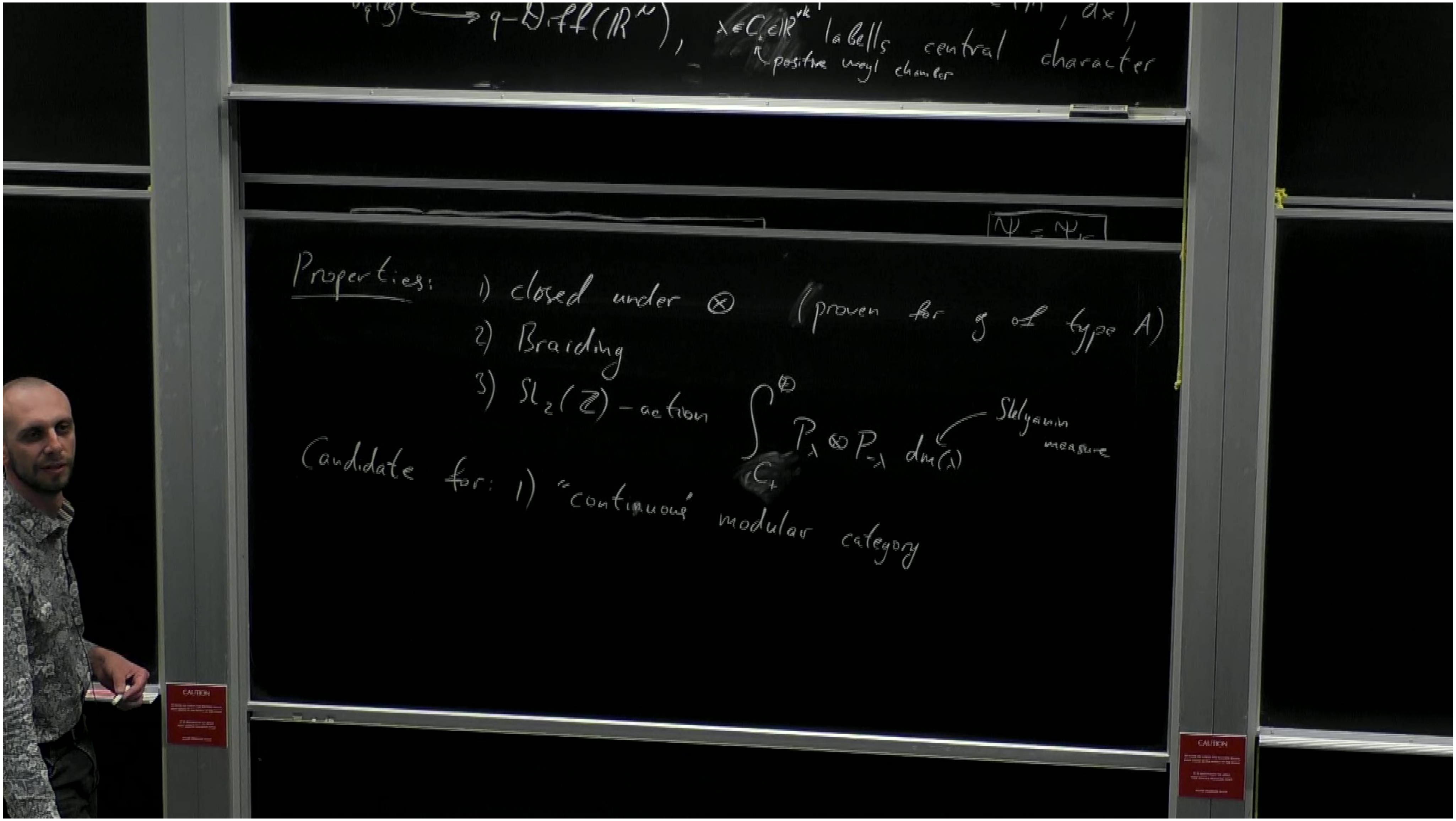

Quantum groups, clusters, &
Hamiltonian reduction
(based on discussions w/ Bershtein,
DiFrancesco, Gaiotto, Ip, Kedem, Schrader, ...)

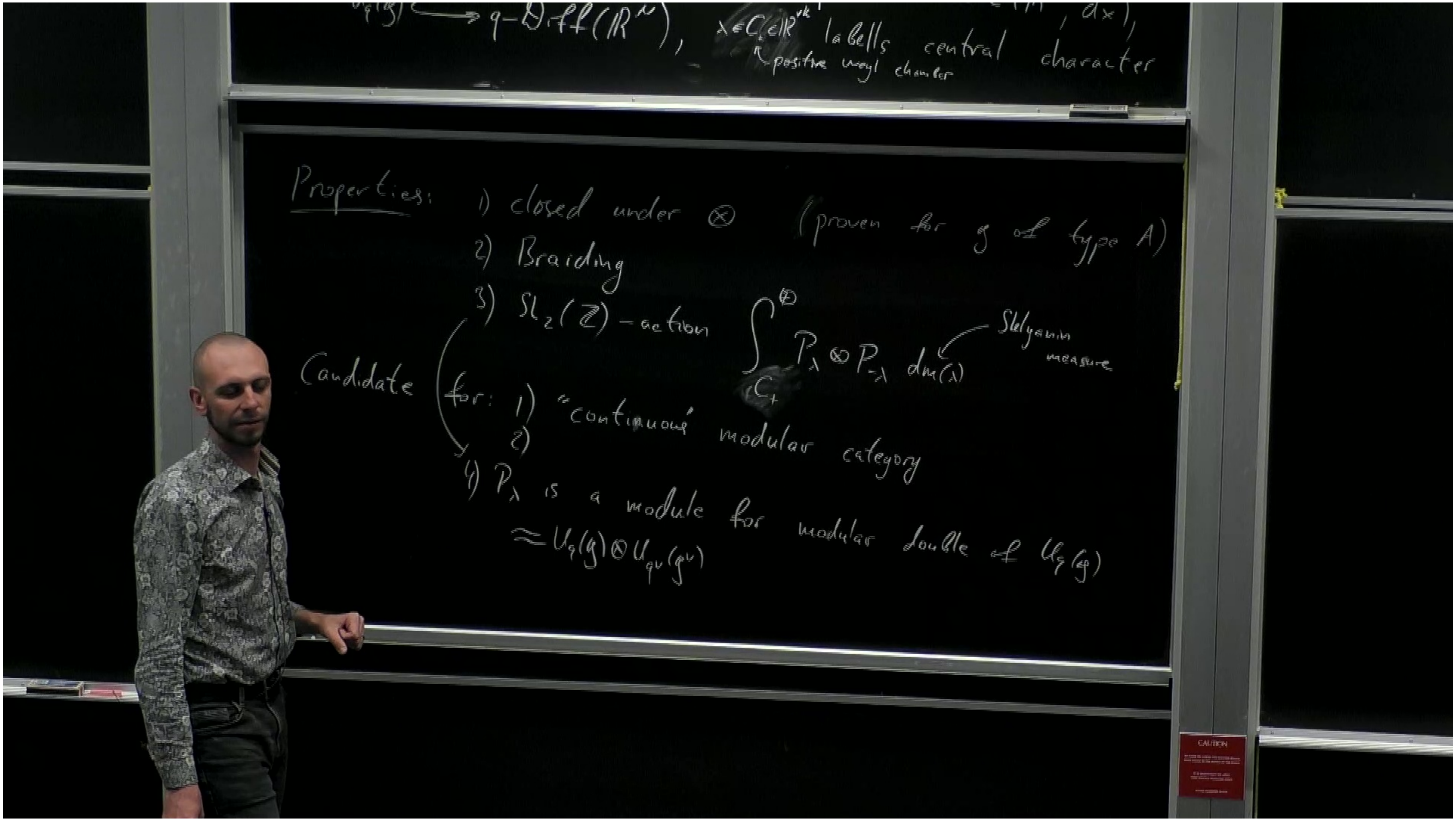

$U_q(\mathfrak{g})$ – quantum gp
P_λ – "positive" rep-s, Schwartz subspace $S \subset L^2(\mathbb{R}^n, dx)$,
$U_q(\mathfrak{g}) \hookrightarrow q\text{-}\mathcal{D}iff(\mathbb{R}^n)$, $\lambda \in \mathbb{R}_+$ labells central character

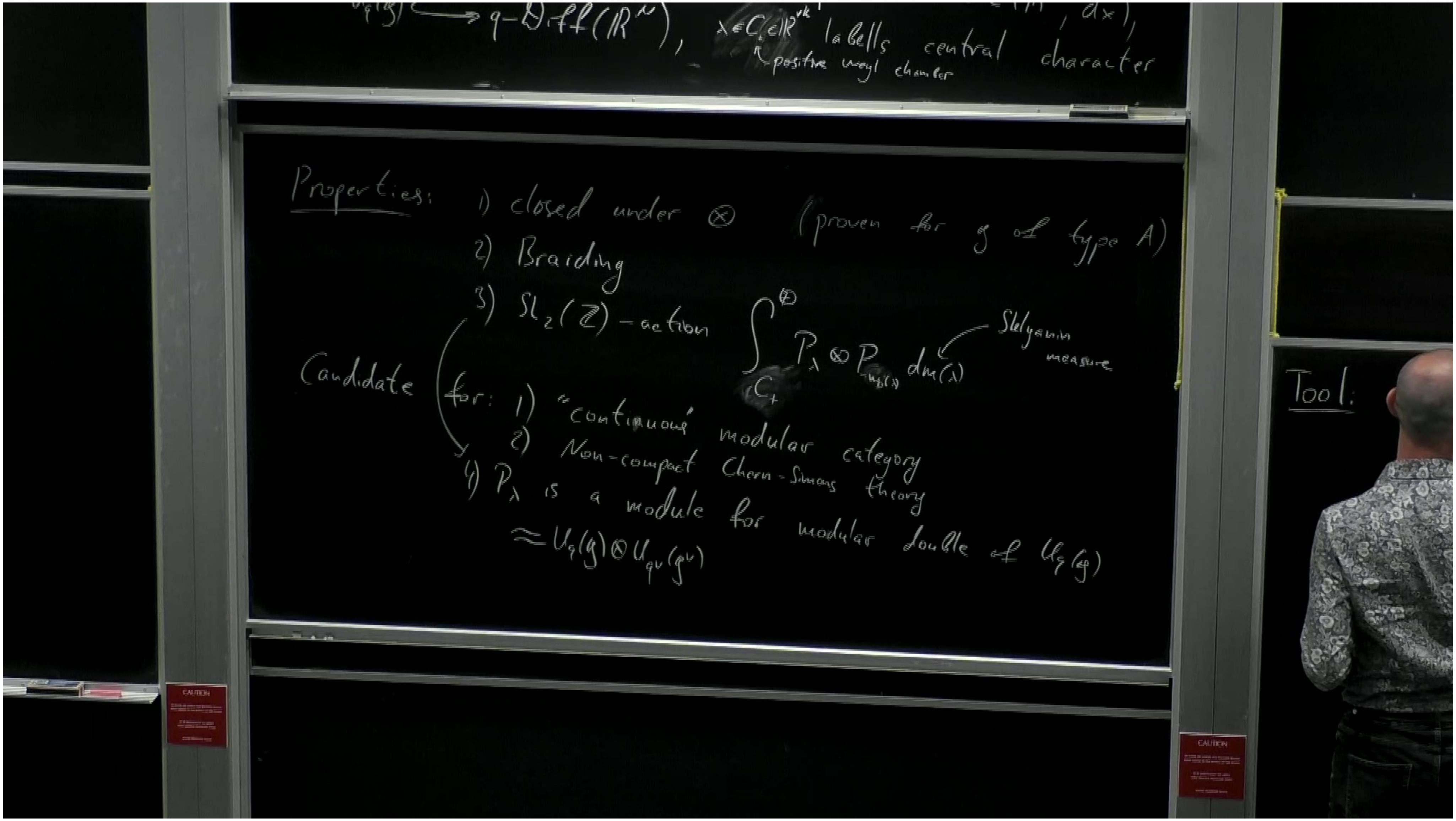


Quantum groups, clusters, &
Hamiltonian reduction
(based on discussions w/ Berenstein,
Di Francesco, Gaiotto, Ip, Kedem, Schrader,)
$U_q(\mathfrak{g})$ – quantum gp
P_λ – "positive" rep-s, Schwartz subspace $S \subset L^2(\mathbb{R}^{\dim(G/B)}, dx)$,
$U_q(\mathfrak{g}) \hookrightarrow q\text{-Diff}(\mathbb{R}^{4d})$, $\lambda \in \mathbb{R}_+$ labels central character


...rancesco, Gaiotto, Ip, Kedem, Schrader, ...)
totally positive
$U_q(\mathfrak{g})$ – quantum gp
$(G/B)_+$
P_λ – "positive" rep-s, Schwartz subspace $S \subset L^2(\mathbb{R}^n, dx)$,
$U_q(\mathfrak{g}) \hookrightarrow q\text{-}\mathcal{D}iff(\mathbb{R}^n)$, $\lambda \in C_+ \subset \mathbb{R}^{rk}$ labells central character
positive weyl chamber
...osed under $\otimes$ (proven for $\mathfrak{g}$ of type A)


q-Diff($\mathbb{R}^N$), $\lambda \in C_+ \subset \mathbb{R}^{rk}$ labells central character
positive weyl chamber
Properties: 1) closed under $\otimes$ (proven for g of type A)
2) Braiding
3) $SL_2(\mathbb{Z})$-action $\int_{C_+}^{\oplus} P_\lambda \otimes P_{-\lambda} \, dm(\lambda)$
Sklyanin measure
Candidate for: 1) "continuous" modular category


$\longrightarrow$ q-Diff($\mathbb{R}^N$), $\lambda \in C_+ \subset \mathbb{R}^{rk}$ labells central character
positive weyl chamber
Properties: 1) closed under $\otimes$ (proven for $\mathfrak{g}$ of type A)
2) Braiding
3) $SL_2(\mathbb{Z})$-action $\int_{C_+}^{\oplus} P_\lambda \otimes P_{-\lambda} \, dm(\lambda)$
Sklyanin measure
Candidate for: 1) "continuous" modular category
2)
4) P_λ is a module for modular double of $U_q(\mathfrak{g})$
$\approx U_q(\mathfrak{g}) \otimes U_{q^\vee}(\mathfrak{g}^\vee)$


$U_q(g) \hookrightarrow q\text{-Diff}(\mathbb{R}^N)$, $\lambda \in C_+ \subset \mathbb{R}^{rk}$ labells central character
positive weyl chamber
Properties: 1) closed under $\otimes$ (proven for g of type A)
2) Braiding
3) $SL_2(\mathbb{Z})$-action $\int_{C_+}^{\oplus} P_\lambda \otimes P_{-w_0(\lambda)} \, dm(\lambda)$
Sklyanin measure
Candidate for: 1) "continuous" modular category
2) Non-compact Chern-Simons theory
4) P_λ is a module for modular double of $U_q(g)$
$\approx U_q(g) \otimes U_{q^\vee}(g^\vee)$
Tool:

A)

Tool: cluster str. on $U_q(\mathfrak{g})$

Quantum cluster str: $(\mathfrak{g} = \mathfrak{sl}_2)$

$$\mathcal{D} = \mathbb{C}(q)\langle E, F, K^{\pm 1}, (K')^{\pm 1}\rangle \Big/ \begin{array}{lll} KE = q^2 EK & KF = q^{-2} FK & [E,F] = (q^{-1}-q)(K-K') \\ K'E = q^{-2} EK' & K'F = q^2 FK' & KK' = K'K \end{array}$$

$$\parallel$$

$$\mathcal{D}(U_q(\mathfrak{b}))$$

$$U_q(\mathfrak{sl}_2) = \mathcal{D}/_{KK'=1}$$

$\ldots = \mathcal{D}/_{kk'=1}$

$$Q = \begin{tikzcd} & \circ^{4} \arrow[dl] & \\ {}_{1}\square \arrow[dr] & & \square_{3} \arrow[ul] \\ & \underset{2}{\circ} \arrow[ur] & \end{tikzcd}$$

$$T_Q^q = \mathbb{C}(q)\langle Y_1^{\pm 1}, \ldots, Y_4^{\pm 1}\rangle \Big/ q^{\varepsilon_{jk}} Y_j Y_k = q^{\varepsilon_{kj}} Y_k Y_j$$

$$\varepsilon_{jk} = \#\{j \to k\} - \#\{k \to j\}$$

$$Y_j Y_{j+1} = q^{-2} Y_{j+1} Y_j \quad j \in \mathbb{Z}/4\mathbb{Z} \qquad Y_j Y_{j+2} = Y_{j+2} Y_j$$

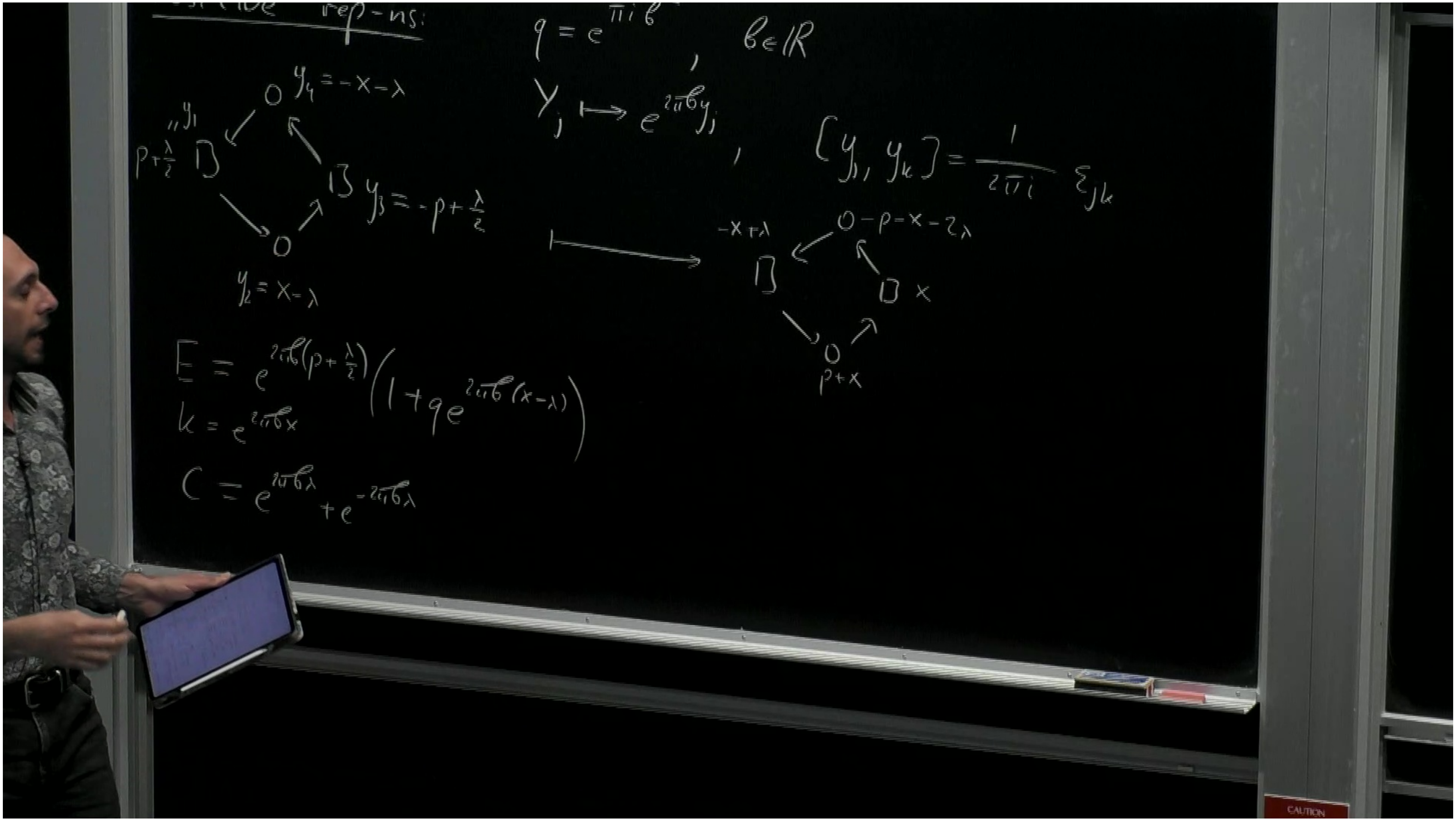
$$\exists\ \varphi_Q : \mathcal{D} \hookrightarrow T_Q^q \qquad \begin{array}{ll} E \longmapsto Y_1(1+qY_2) & K \longmapsto q^2 Y_1 Y_2 Y_3 \\ F \longmapsto Y_3(1+qY_4) & K' \longmapsto q^2 Y_3 Y_4 Y_1 \end{array}$$

Positive rep-ns:

$$q = e^{\pi i b^2}, \quad b \in \mathbb{R}$$

$$Y_j \longmapsto e^{2\pi b y_j}, \quad [y_j, y_k] = \frac{1}{2\pi i} \varepsilon_{jk}$$

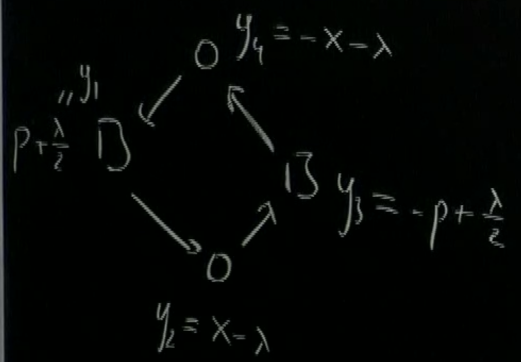
$$\begin{tikzcd}
 & \circ \, {\scriptstyle y_4 = -x-\lambda} \arrow[dl] & \\
{\scriptstyle y_1 = p+\frac{\lambda}{2}} \, \boxtimes \arrow[dr] & & \boxtimes \, {\scriptstyle y_3 = -p+\frac{\lambda}{2}} \arrow[ul] \\
 & \underset{y_2 = x-\lambda}{\circ} \arrow[ur] &
\end{tikzcd}$$


rep-ns:
$q = e^{\pi i b}$, $b \in \mathbb{R}$
$Y_j \mapsto e^{2\pi b y_j}$, $[y_j, y_k] = \frac{1}{2\pi i}\varepsilon_{jk}$
$y_4 = -x-\lambda$
y_1
$p+\frac{\lambda}{2}$
$y_3 = -p+\frac{\lambda}{2}$
$y_2 = x-\lambda$
$-x+\lambda$
$-p-x-2\lambda$
x
$p+x$
$E = e^{2\pi b(p+\frac{\lambda}{2})}\left(1+qe^{2\pi b(x-\lambda)}\right)$
$K = e^{2\pi b x}$
$C = e^{2\pi b\lambda}+e^{-2\pi b\lambda}$

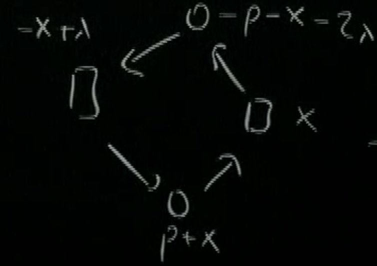
... rep-ns:

$q = e^{\pi i b^2}$, $b \in \mathbb{R}$

$Y_j \mapsto e^{2\pi b y_j}$, $[y_j, y_k] = \frac{1}{2\pi i}\varepsilon_{jk}$



$\longmapsto$



unitary equiv. rep-n

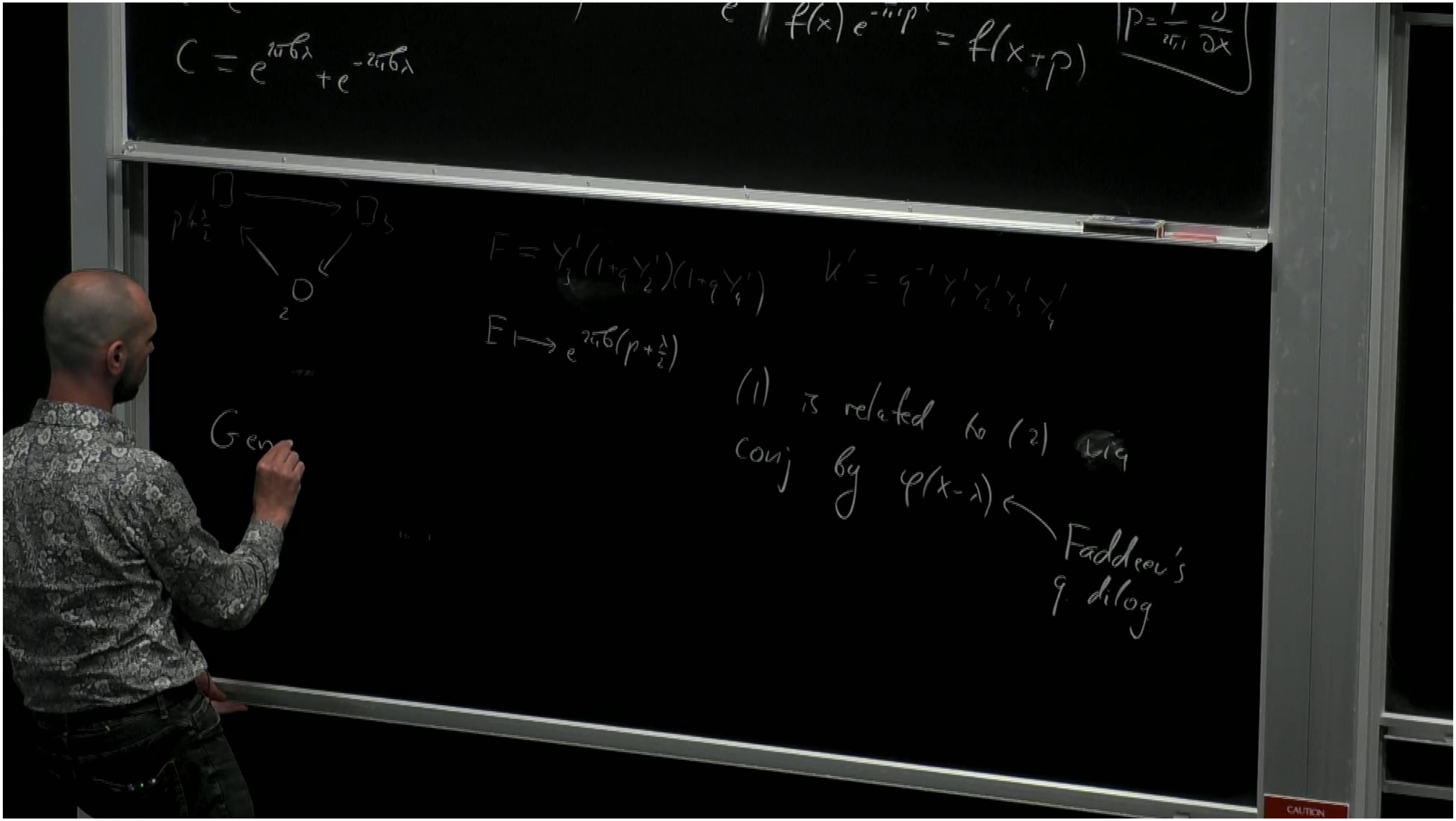
$$E = e^{2\pi b(p+\frac{\lambda}{2})}\left(1+q e^{2\pi b(x-\lambda)}\right)$$

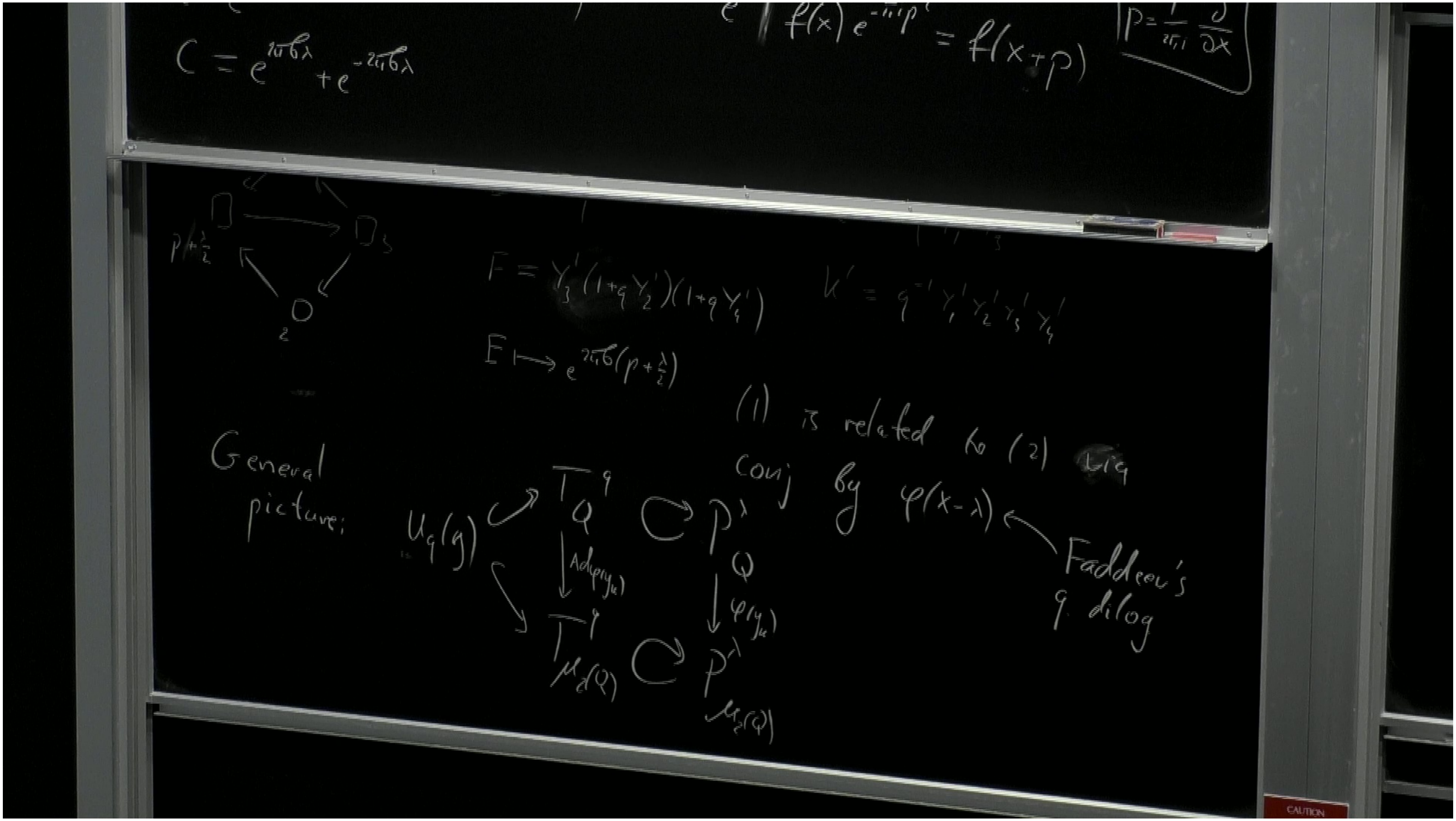
$$K = e^{2\pi b x}$$

$$C = e^{2\pi b\lambda} + e^{-2\pi b\lambda}$$

$$e^{\pi i p^2} f(x) e^{-\pi i p^2} = f(x+p)$$

$$\boxed{p = \frac{1}{2\pi i}\frac{\partial}{\partial x}}$$


C = e^{2πibλ} + e^{-2πibλ}
f(x) e^{-2πip} = f(x+p)
p = 1/2πi ∂/∂x
F = Y_3'(1+qY_2')(1+qY_4')
K' = q^{-1} Y_1' Y_2' Y_3' Y_4'
E ↦ e^{2πib(p+λ/2)}
(1) is related to (2) via
conj by φ(x−λ)
Faddeev's
q. dilog
Gen
CAUTION


C = e^{2πibλ} + e^{-2πibλ}
f(x) = f(x+p)
F = Y_3'(1+qY_2')(1+qY_4')
K' = q^{-1} Y_1' Y_2' Y_3' Y_4'
E ↦ e^{2πib(p+λ/2)}
(1) is related to (2) via
conj by φ(x-λ)
Faddeev's
q. dilog
General
picture:
U_q(g)
T_Q^q
P_Q^λ
T_{μ_2(Q)}^q
P_{μ_2(Q)}^λ

$$\boxed{\rho_{\hat{p}} = \frac{1}{\beta}\left|f\left(x+\frac{h_{\Psi}}{\beta}\right)\right|^2 e^{-\beta x}\,\Delta_{\phi|\Psi}} + O(\varepsilon \ll \beta)$$

$$\boxed{\Psi = \Psi_{dS}}$$

$$\boxed{h_{\Psi}|\Psi\rangle = 0}$$

P_λ = max. domain of $U_q(g)$ inside $L^2(\mathbb{R}, d_x)$

$$\left\{ e^{-\alpha x^2 + \beta x} P(x) \;\middle|\; \begin{array}{l} \alpha \in \mathbb{R}_{>0} \\ \beta \in \mathbb{C} \end{array} \quad P(x) \in \mathbb{C}[x] \right\}$$

$$U_q(\mathfrak{sl}_2) = \mathcal{D}/kk'=1$$

$$C = EF - q^{-1}k - qk^{-1}$$

$$Y_1Y_3 + Y_1Y_2Y_3Y_4 \quad ?$$

$Q =$

$$\mathcal{T}_Q^q = \mathbb{C}(q)\langle Y_1^{\pm1}, \dots, Y_4^{\pm1}\rangle \,/\, q^{\varepsilon_{jk}} Y_jY_k = q^{\varepsilon_{kj}} Y_kY_j$$

$$\varepsilon_{jk} = \#\{j \to k\} - \#\{k \to j\}$$

$$Y_jY_{j+1} = q^{-2}Y_{j+1}Y_j \quad j \in \mathbb{Z}/4\mathbb{Z} \qquad Y_jY_{j+2} = Y_{j+2}Y_j$$

$$\exists\ \varphi_Q : \mathcal{D} \hookrightarrow \mathcal{T}_Q^q$$

$$E \mapsto Y_1(1+qY_2) \qquad k \mapsto q^2 Y_1Y_2Y_3$$

$$F \mapsto Y_3(1+qY_4) \qquad k' \mapsto q^2 Y_3Y_4Y_1$$

P_λ = max. domain of $U_q(\mathfrak{g})$ inside $L^2(\mathbb{R}, dx)$

$$\cup$$

$$\left\{ e^{-\alpha x^2+\beta x} P(x) \;\middle|\; \begin{array}{l} \alpha\in\mathbb{R}_{>0} \\ \beta\in\mathbb{C} \end{array} \quad P(x)\in\mathbb{C}[x] \right\}$$

Why bother about clusters?

e.g. want to decompose $P_\lambda \otimes P_\mu$, need to diagonalise

$\Delta(C) \longrightarrow$ 1) Choose cluster where $\Delta(C)$ is "simple"

P_λ = max. domain of $U_q(\mathfrak{g})$ inside $L^2(\mathbb{R}, d_x)$

$$\cup$$

$$\left\{ e^{-\alpha x^2+\beta x} P(x) \;\middle|\; \begin{array}{l} \alpha \in \mathbb{R}_{>0} \\ \beta \in \mathbb{C} \quad P(x) \in \mathbb{C}[x] \end{array} \right\}$$

Why bother about clusters?

e.g. want to decompose $P_\lambda \otimes P_\mu$, need to diagonalise

$\Delta(C) \longrightarrow$ 1) Choose cluster where $\Delta(C)$ is "simple"

2) Diagonalise the latter.

P_λ = max. domain of $U_q(\mathfrak{g})$ inside $L^2(\mathbb{R}, dx)$

$$\bigcup$$

$$\left\{ e^{-\alpha x^2 + \beta x} P(x) \;\middle|\; \begin{array}{l} \alpha \in \mathbb{R}_{>0} \\ \beta \in \mathbb{C} \end{array} \quad P(x) \in \mathbb{C}[x] \right\}$$

Why bother about clusters?

Clebsch - Gordon map

$$\boxed{C : P_\lambda \otimes P_\mu \xrightarrow{\sim} \int^{\oplus} P_\nu \otimes M_{\lambda,\mu}^{\nu} \, d\nu(\nu)}$$

e.g. want to decompose $P_\lambda \otimes P_\mu$, need to diagonalise $\Delta(C)$ $\longrightarrow$

1) Choose cluster where $\Delta(C)$ is "simple"

2) Diagonalise the latter.

Finite-dim rep-s

$$c_b = \frac{i}{2}(b+b^{-1})$$

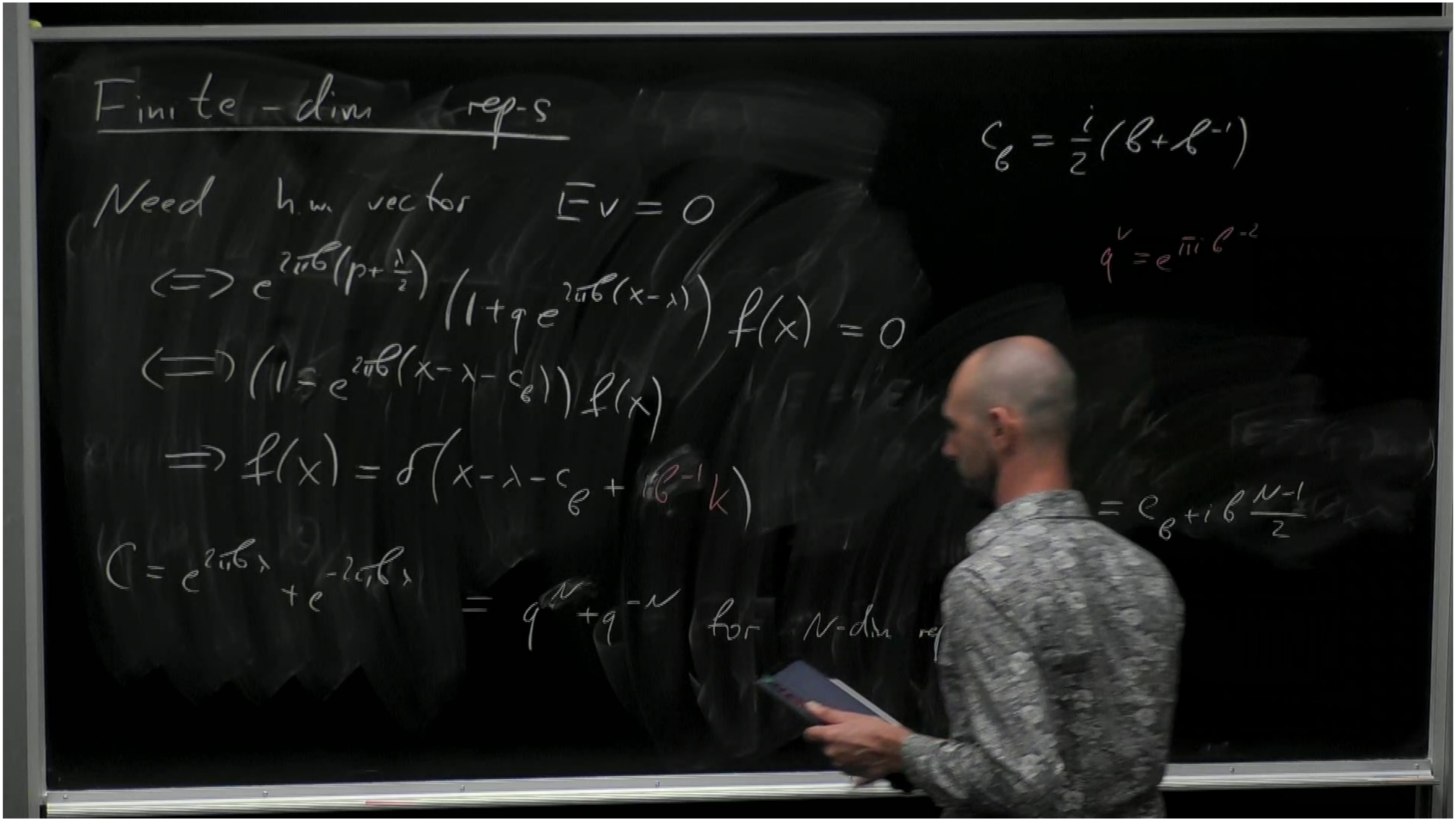
$$q^\vee = e^{\pi i b^{-2}}$$

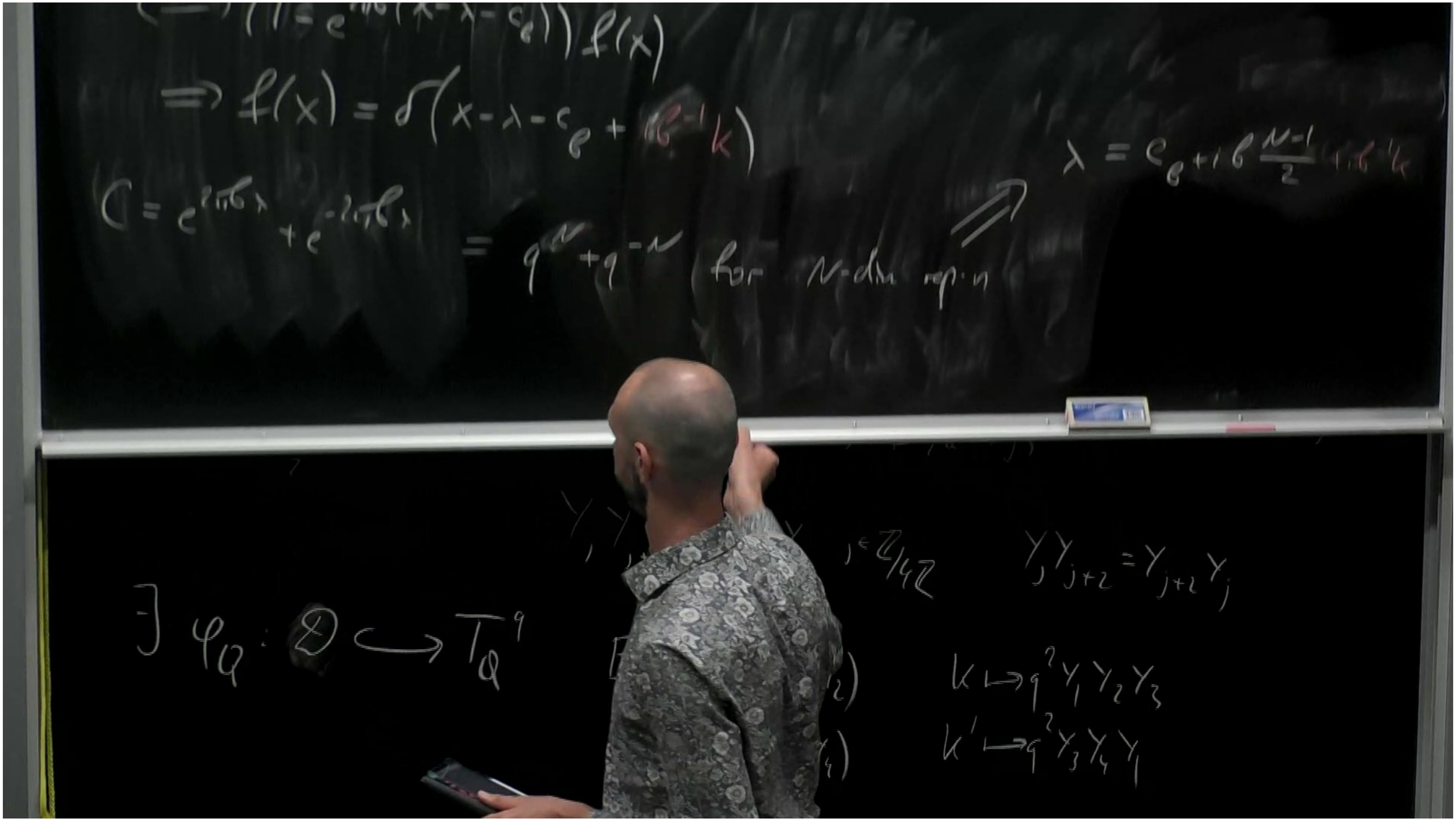
Need h.w. vector $Ev = 0$

$$\Leftrightarrow e^{2\pi b(p+\frac{\lambda}{2})}\left(1+q e^{2\pi b(x-\lambda)}\right) f(x) = 0$$

$$\Leftrightarrow \left(1 - e^{2\pi b(x-\lambda-c_b)}\right) f(x)$$

$$\Rightarrow f(x) = \delta\left(x-\lambda-c_b + ib^{-1}k\right)$$


Finite-dim rep-s
Need h.w. vector $Ev=0$
$\iff e^{2\pi b(p+\frac{\lambda}{2})}\left(1+q e^{2\pi b(x-\lambda)}\right) f(x)=0$
$\iff \left(1+e^{2\pi b(x-\lambda-c_b)}\right) f(x)$
$\implies f(x)=\delta(x-\lambda-c_b+ib^{-1}k)$
$C=e^{2\pi b\lambda}+e^{-2\pi b\lambda} = q^N+q^{-N}$ for N-dim rep
$c_b=\frac{i}{2}(b+b^{-1})$
$q^\vee=e^{\pi i b^{-2}}$
$=c_b+ib\frac{N-1}{2}$


⟹ f(x) = δ(x − λ − c_b + ib^-1 k)
λ = c_b + ib (N−1)/2 + ib^-1 k
for N-dim rep'n
Y_j Y_j+2 = Y_j+2 Y_j
∃ φ_Q : D ↪ T_Q^q
k ↦ q^2 Y_1 Y_2 Y_3
k' ↦ q^2 Y_3 Y_4 Y_1

$q \quad + q \quad$ for N-dim rep-n

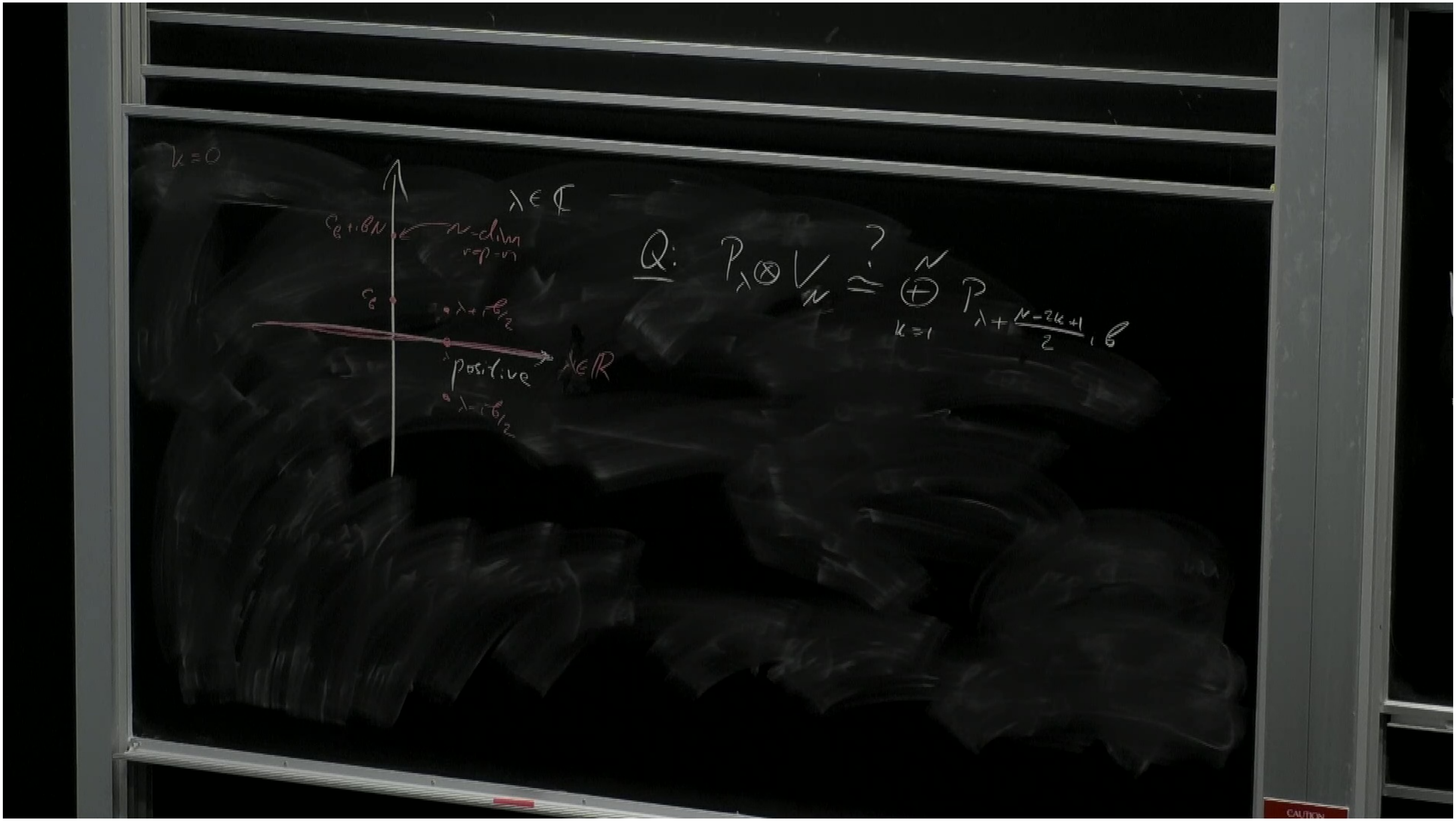
Therefore, N-dim rep-n of $U_q(sl_2)$ is supported on

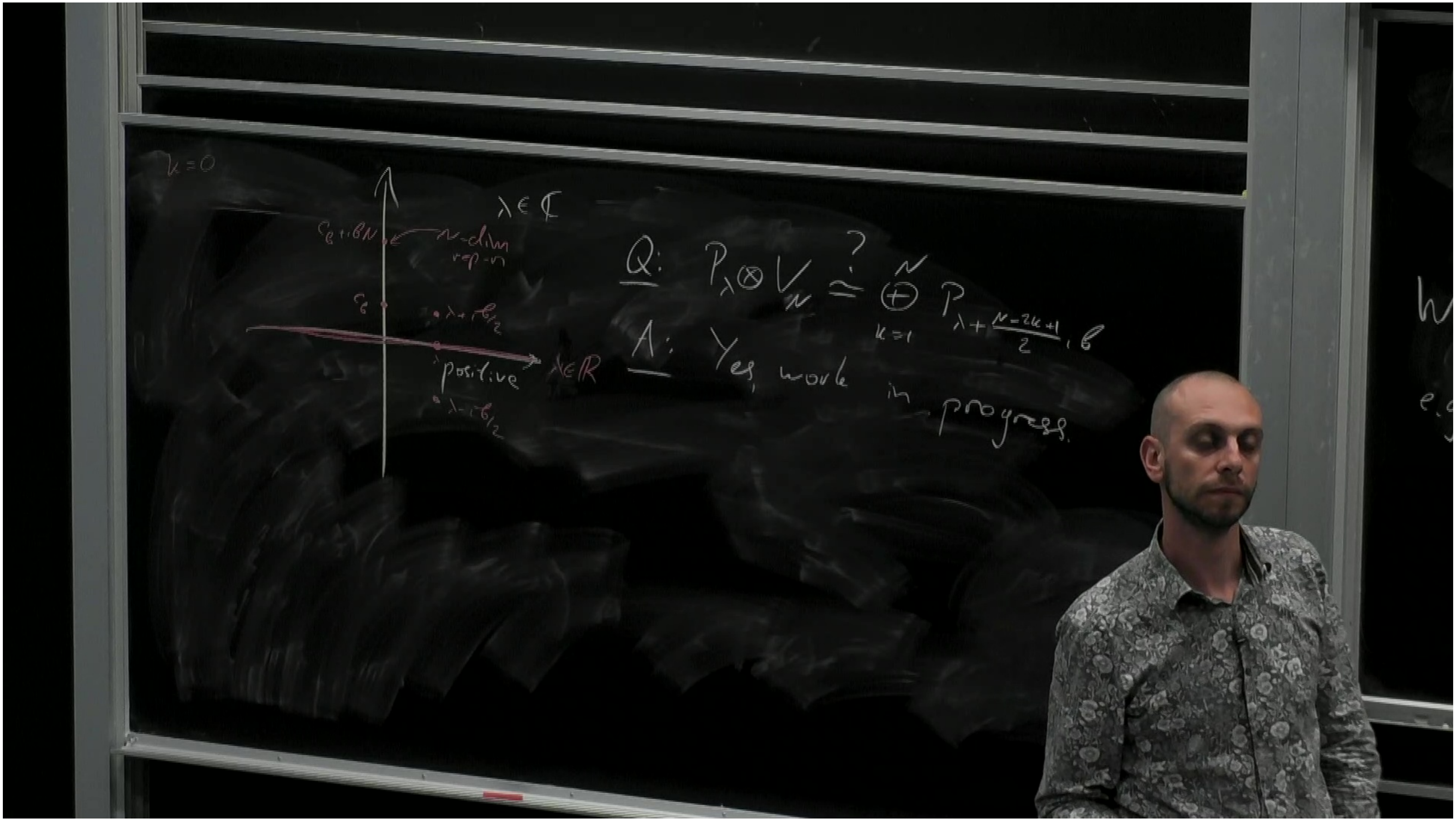
$$\left\{ \delta\left(x - \frac{N-1}{2} i b\right), \delta\left(x - \frac{N-3}{2} i b\right), \ldots, \delta\left(x + \frac{N-1}{2} i b\right) \right\}$$

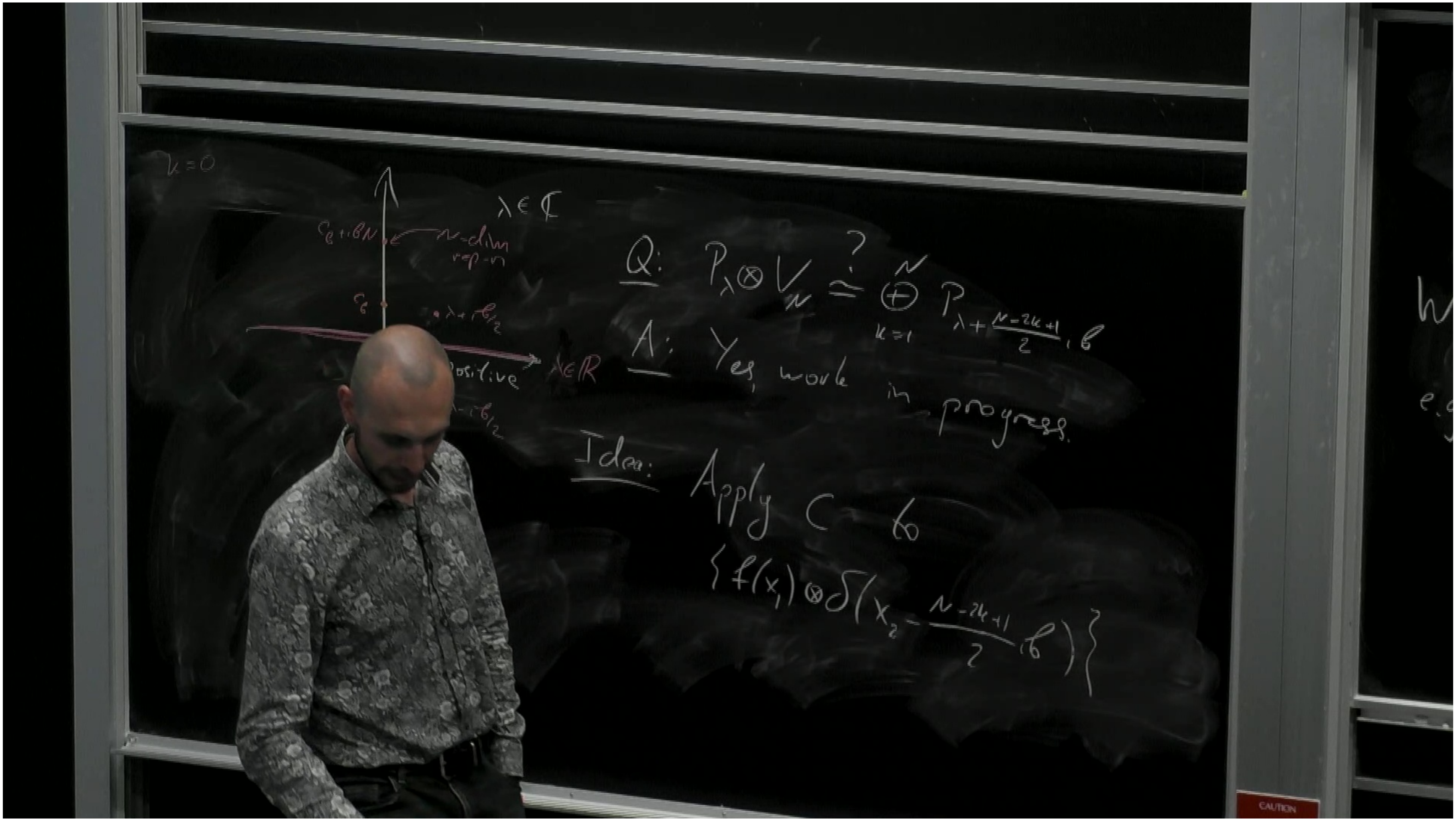
$$\lambda = c_b + i b \frac{N-1}{2}$$

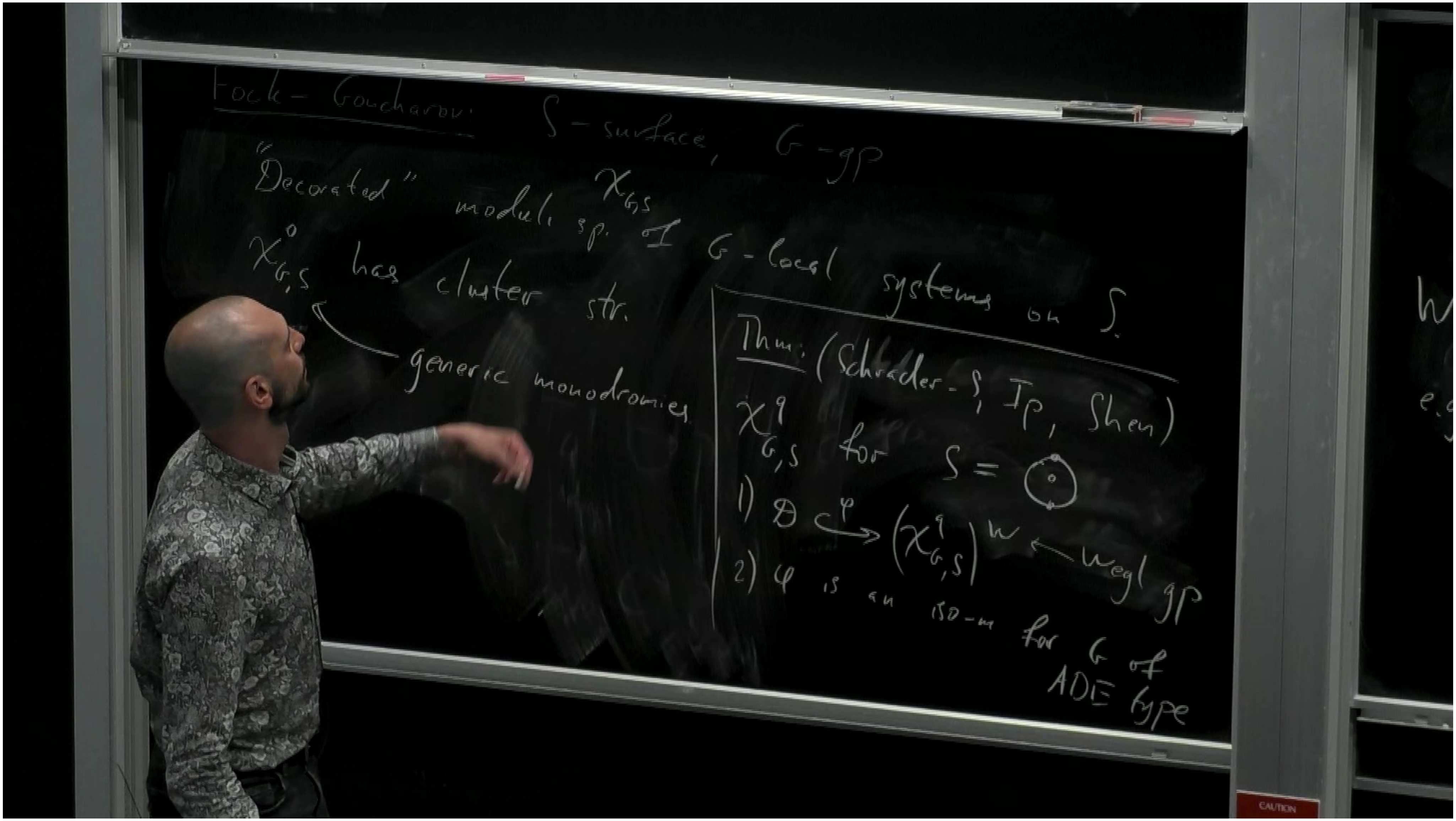
$V_N \leftarrow$ N-dim for $U_q(sl_2)$
trivial for $U_{q^\vee}(sl_2)$

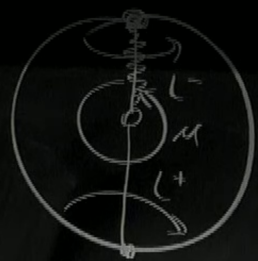
If $k \neq 0$ I get $V_N \otimes V_M^\vee$


k=0
λ ∈ ℂ
Q: P_λ ⊗ V_N ≅? ⊕_{k=1}^{N} P_{λ + (N-2k+1)/2, b}
c_b + ibN
N-dim rep-n
c_b
λ + ib/2
positive
λ ∈ ℝ
λ - ib/2


k=0
λ ∈ ℂ
N-dim rep-n
positive
λ ∈ ℝ
Q: P_λ ⊗ V_N ≅? ⊕_{k=1}^{N} P_{λ + (N-2k+1)/2, b}
A: Yes, work in progress.


k=0
λ ∈ ℂ
N-dim rep-n
positive
λ ∈ ℝ
Q: $P_\lambda \otimes V_N \overset{?}{\simeq} \bigoplus_{k=1}^{N} P_{\lambda + \frac{N-2k+1}{2} b}$
A: Yes, work in progress.
Idea: Apply C to
$\{ f(x_1) \otimes \delta(x_2 - \frac{N-2k+1}{2} b) \}$


Fock-Goncharov: S - surface, G - gp
"Decorated" moduli sp. of G-local systems on S.
X_{G,S}
X°_{G,S} has cluster str.
generic monodromies
Thm: (Schrader-Shapiro, Ip, Shen)
X^q_{G,S} for S =
1) D ↪ (X^q_{G,S})^W ← Weyl gp
2) φ is an iso-m for G of ADE type

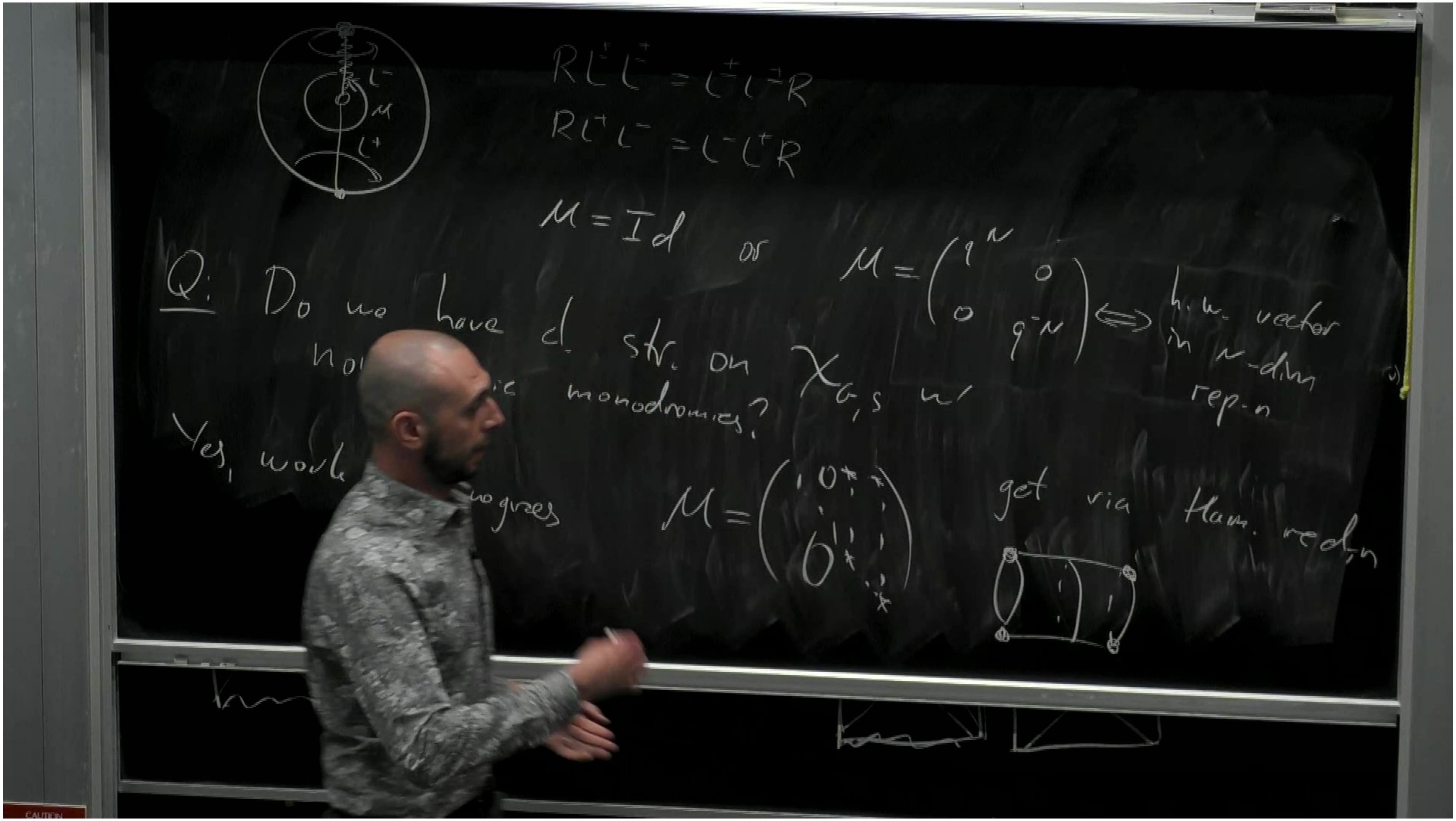


$$RL^+L^+ = L^+L^+R$$
$$RL^+L^- = L^-L^+R$$

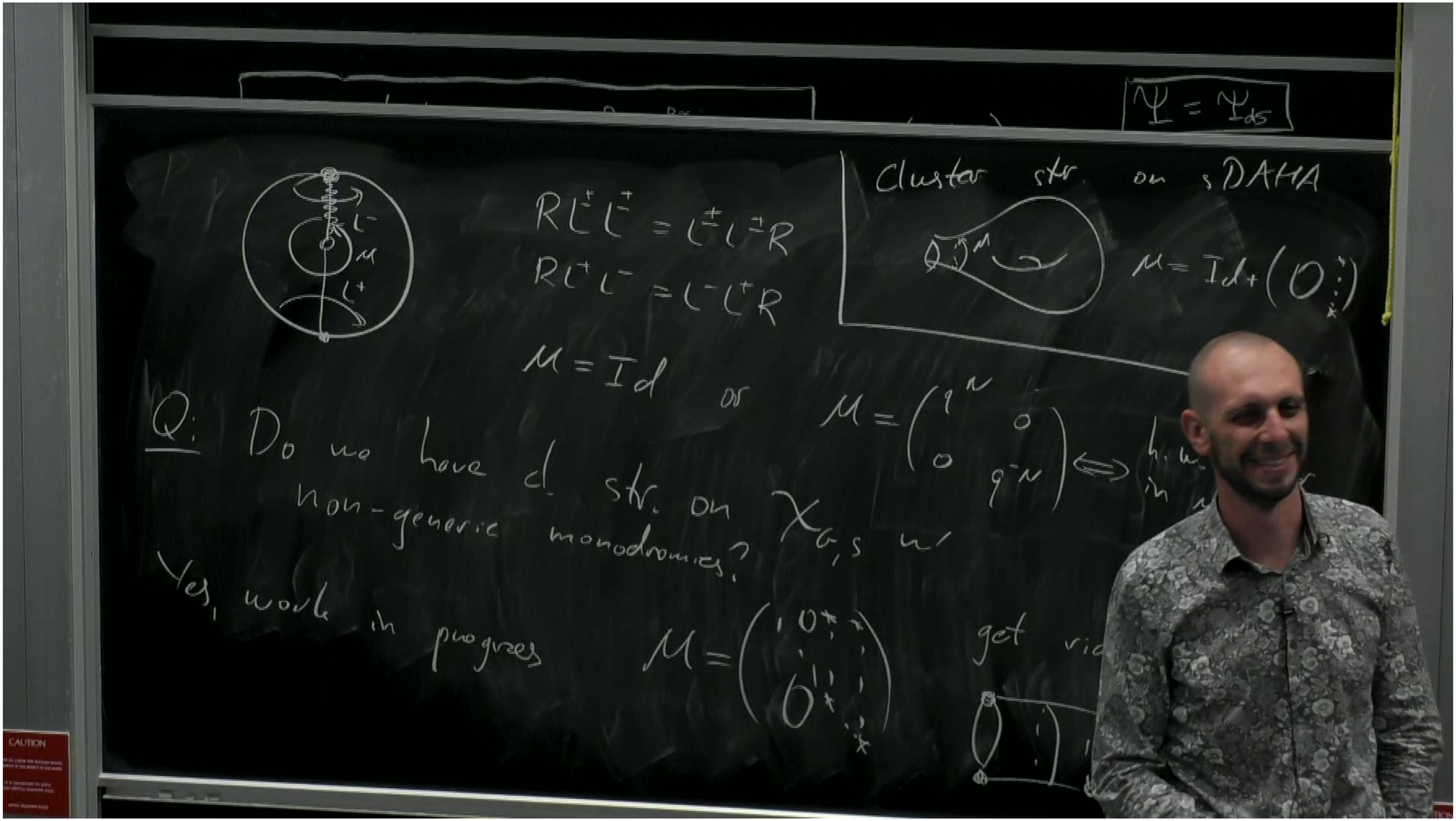
$M = Id$ or $M = \begin{pmatrix} q^N & 0 \\ 0 & q^{-N} \end{pmatrix} \Longleftrightarrow$ h.w. vector in N-dim rep-n

Q: Do we have cl. str. on $X_{0,s}$ w/ non-generic monodromies?

Yes, work in progress $M = \begin{pmatrix} 1 & * & \cdots & * \\ & \ddots & & \vdots \\ 0 & & 1 & * \\ & & & * \end{pmatrix}$



M = Id or
h.w. vector in N-dim rep-n
Q: Do we have cl. str. on X w/
monodromies?
Yes, work
get via Ham. red-n


Ψ = Ψds
Cluster str. on sDAHA
RL±L∓ = L∓L±R
RL+L− = L−L+R
M = Id or
Q: Do we have cl. str. on X_{G,S} w/ non-generic monodromies?
Yes, work in progress