Title: SDP approaches for quantum polynomial optimization

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Abstract: "Many relevant tasks in Quantum Information processing can be expressed as polynomial optimization problems over states and operators. In the earlier talk by David, we saw that this is also the case for certain (quantum) causal compatibility and causal optimization problems.
This talk will focus on several closely related semidefinite programming (SDP) hierarchies that have recently been shown to be complete for such polynomial optimization problems [arxiv:2110.14659, 2212.11299, 2301.12513]. We give a high-level overview of the techniques and mathematics that are needed for proving such statements. In particular, we will see a version of a Quantum De Finetti theorem, as well as a sketch of a constructive proof of convergence for the SDP hierarchies. Afterwards, these results are linked back to the causal compatibility problem to conclude that such SDP hierarchies are complete for a certain type of causal structures known as tree networks."

Semidefinite optimization approaches for quantum polynomial optimization

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Outline

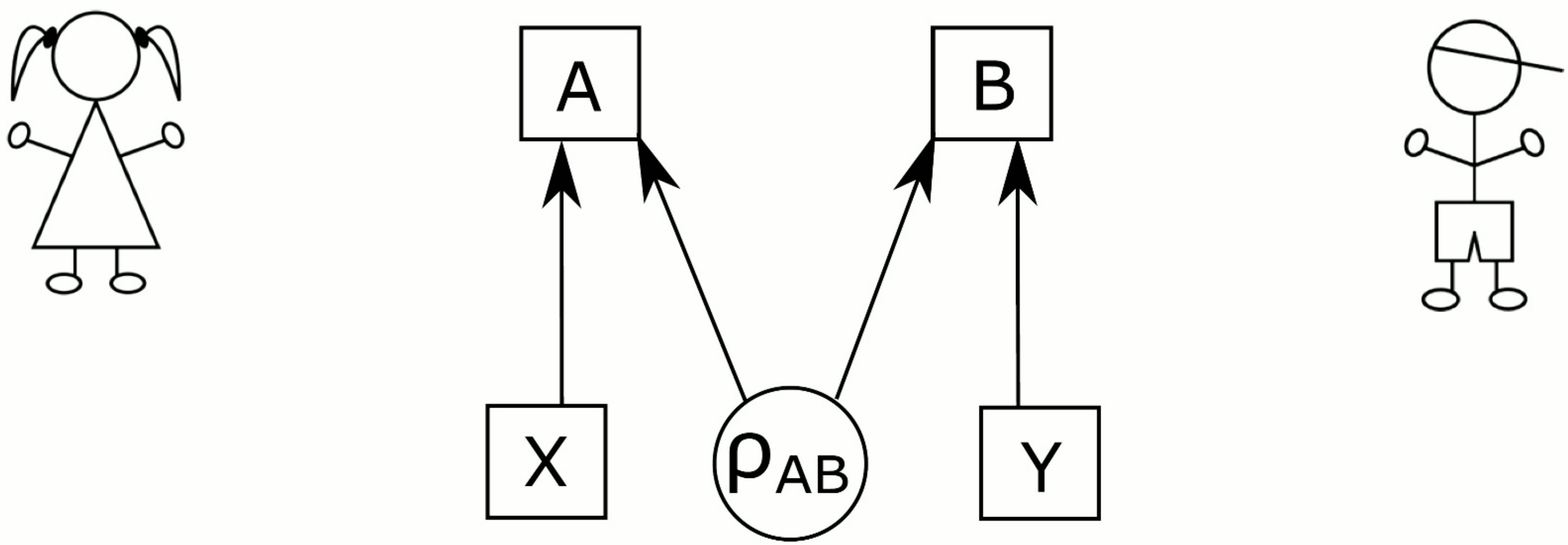
Introduction
- Causal compatibility
- (non-linear) Bell inequalities

Quantum Polynomial Optimization

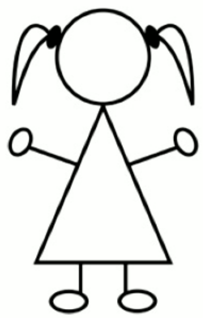
SDP approaches
- NPO
- Polarization
- Convergence

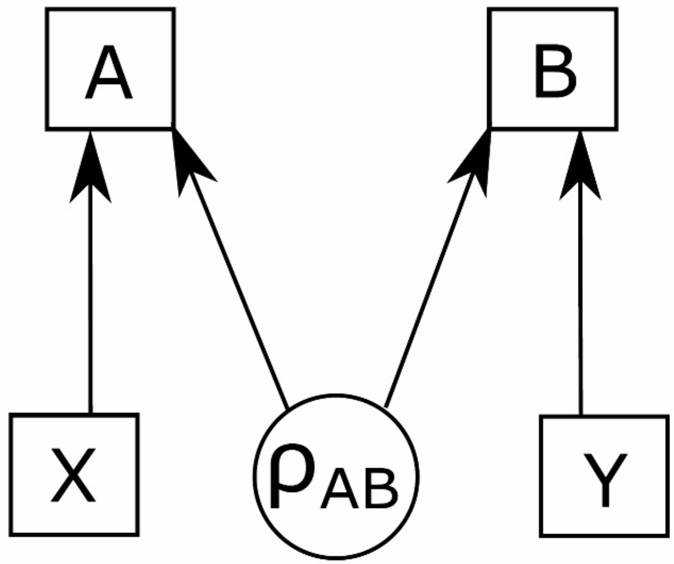
Conclusion

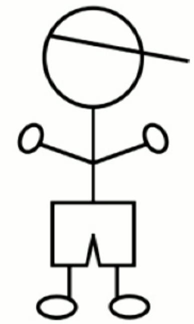
Bell scenario



Bell scenario







$$\{A_{\alpha|x}\}_{\alpha,x}$$
$$A_{\alpha|x} \succeq 0$$
$$\sum_{\alpha} A_{\alpha|x} = \mathbb{I}$$

$$P(a,b|x,y) = \rho_{AB}(A_{\alpha|x}B_{\beta|y})$$
$$\rho_{AB}(\mathbb{I}) = 1, \rho_{AB} \geq 0$$

$$\{B_{\beta|y}\}_{\beta,y}$$
$$B_{\beta|y} \succeq 0$$
$$\sum_{\beta} B_{\beta|y} = \mathbb{I}$$

The causal compatibility problem

Causal compatibility

Given a (conditional) probability distribution $P(a, b, c, \ldots | x, y, z, \ldots)$ and a causal structure (in the form of a DAG), determine whether P can be produced in this causal structure.

This is dependent on the physical model:

- Classical probability
- Quantum mechanics
- GPT

(non-linear) Bell inequalities

For binary observables we have

- The CHSH expression

$$\rho(A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2).$$

  Tsirelson bound for quantum systems is $2\sqrt{2}$.

- The expression

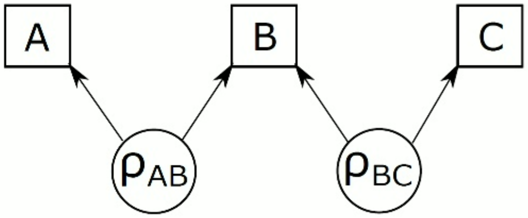
$$\rho(A_1B_2 + A_2B_1)^2 + \rho(A_1B_1 - A_2B_2)^2.$$

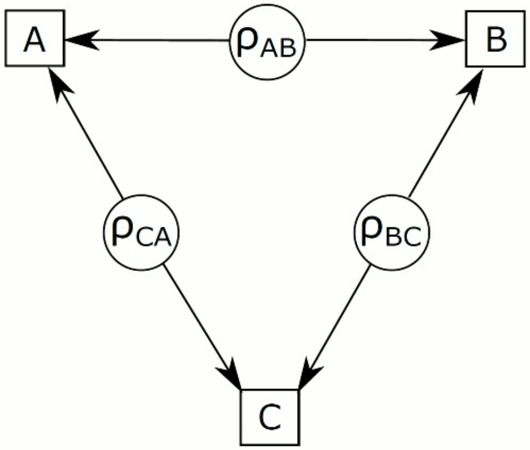
  Classical bound is 4[1]. Quantum bound is matching[2].

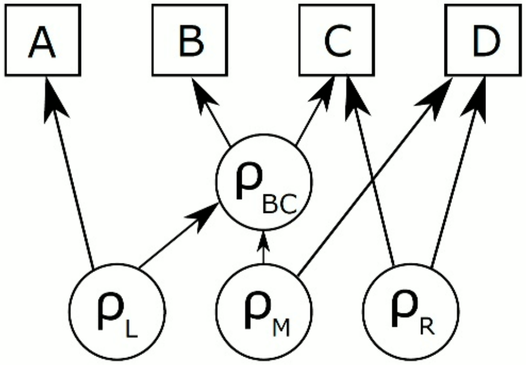
- Many more complicated expressions...

[1] Uffink (2002)
[2] Klep *et al.* (2023)

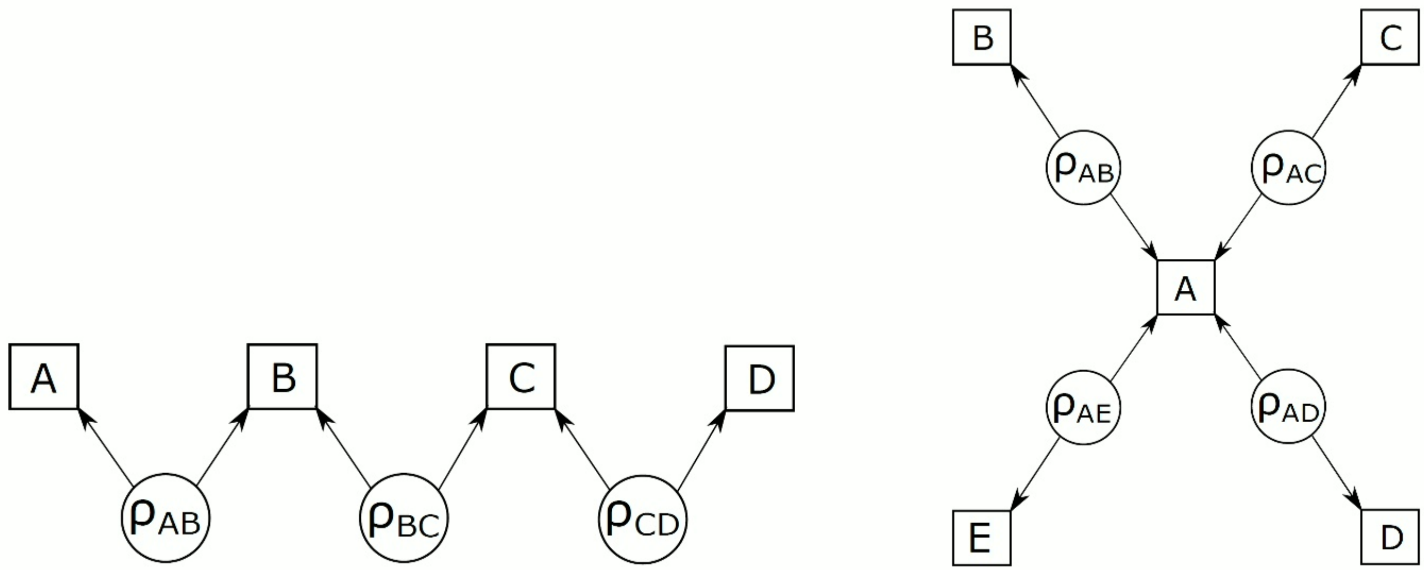
Causal structures







Tree networks



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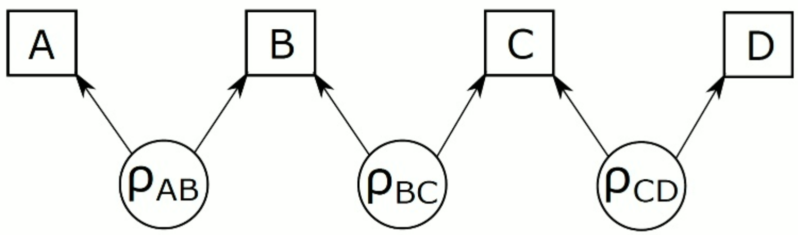
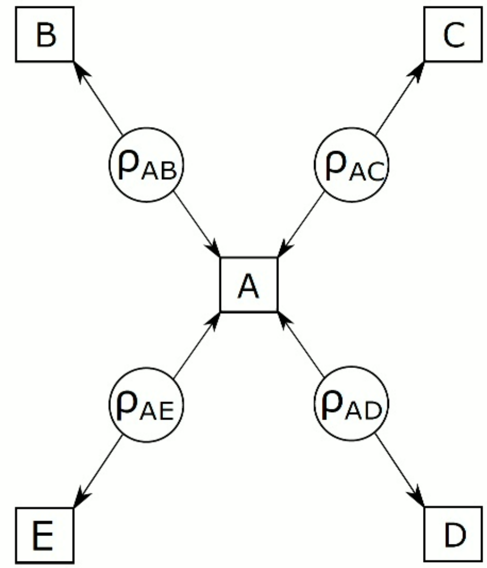
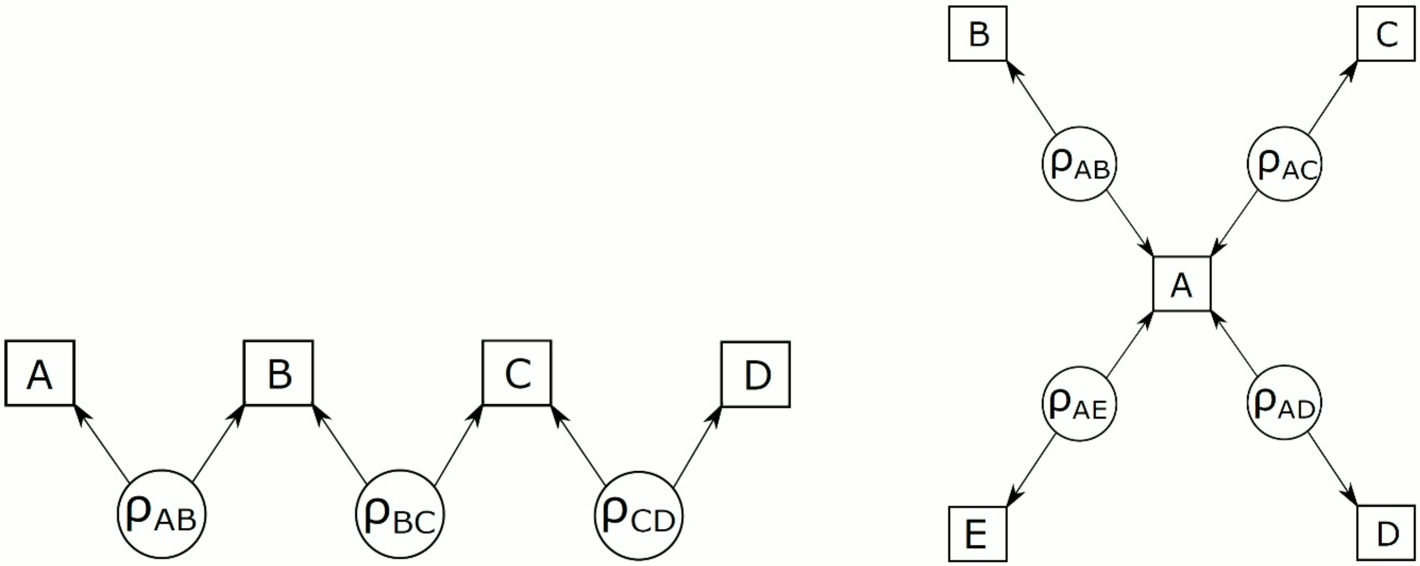




$$A \perp C, D$$
$$A, B \perp D$$

$$B \perp C, D, E$$
$$C \perp B, D, E$$
$$D \perp B, C, E$$

Tree networks



$$\rho(acd) = \rho(a)\rho(cd)$$
$$\rho(abd) = \rho(ab)\rho(d)$$

$$\rho(bcde) = \rho(b)\rho(c)\rho(d)\rho(e)$$

Polynomials

Such expressions are polynomials in two ways:

- In the operators
- But also in the state

$$\begin{aligned}\rho(abcd) &= \rho(a)\rho(b)\rho(c)\rho(d)\\ &= \rho^{\otimes 4}(a\otimes b\otimes c\otimes d)\end{aligned}$$

We call such expressions state polynomials

Optimization

Quantum Polynomial Optimization

Given a set of generators $\mathcal{G}$ and relations $\mathcal{R}$, defining a C^*-algebra $\mathcal{A} = C^*(\mathcal{G}|\mathcal{R})$, and a set of state polynomials $p_i : (\mathcal{A}, S(\mathcal{A})) \to \mathbb{C}$, solve

$$\begin{aligned} \min_{\rho \in S(\mathcal{A})} \quad & p_0(\mathcal{A}, \rho) \\ \text{s. t.} \quad & p_i(\mathcal{A}, \rho) = 0 \quad \forall i. \end{aligned}$$

For example, $\mathcal{A}$ could be the algebra generated by the measurements and the p_i could be the factorization constraints.

Difficulties for convex optimization

We would like to use convex optimization:

$$\begin{aligned} \min_{X} \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i \\ & X \succeq 0 \end{aligned}$$

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However

- The set of product states is not convex:
  For ρ, σ product states $(1-\lambda)\rho + \lambda\sigma$ is mixed unless $\lambda = 0$ or 1.

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- The dimension grows exponentially quickly

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So that seems pretty bad...

NPO

- Non-commutative Polynomial Optimization[3,4] can optimize over objective functions and constraints that are linear in the state and polynomial in the operators
- Hierarchy of relaxations that converges in the limit via a monotonically increasing sequence of lower bounds
- Can be formulated independent of the dimension!

[3] Navascués, Pironio, Acín (2008)
[4] Pironio, Navascués, Acín (2010)

NPO: How does it (roughly) work?

Construct a functional that is only positive on a subspace of the algebra $\mathcal{A}$.

- Choose a basis $\{a_i\}$ of $\mathcal{A}$, with $a_0 = \mathbb{I}$.
- Choose an integer k and construct a $k \times k$ matrix $M^{(k)}$, where

$$M^{(k)}_{i,j} = \rho(a_i^* a_j)$$

M is PSD

A matrix N is PSD iff $v^* N v \geq 0$ for all vectors v

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$$
\begin{aligned}
v^* M v &= \sum_{i,j} v_i^* M_{i,j} v_j \\
&= \sum_{i,j} v_i^* \rho(a_i^* a_j) v_j \\
&= \rho(\sum_i v_i^* a_i^* \sum_j v_j a_j)
\end{aligned}
$$

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&= \rho(X^* X) \geq 0
\end{aligned}
$$

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- Solve the SDP

$$\begin{aligned} \min_{M^{(k)}} \quad & \sum_{i,j} p^0_{ij} M^{(k)}_{i,j} \\ \text{s.t.} \quad & M^{(k)}_{0,0} = 1 \\ & \sum_{i,j} (p^\ell)_{ij} M^{(k)}_{i,j} = 0 \quad \text{when} \quad \sum_{i,j} (p^\ell)_{ij} \rho(a_i^* a_j) = 0 \\ & M^{(k)} \succeq 0 \end{aligned}$$

Difficulties for convex optimization

- The set of product states is not convex:
  For ρ, σ product states $(1-\lambda)\rho + \lambda\sigma$ is mixed unless $\lambda = 0$ or 1.
- The polynomials (in the state) can be non-convex
  NPO can handle any polynomial in operators $\frac{1}{2}\checkmark$
- The dimension grows exponentially quickly
  NPO can deal with that $\checkmark$

Independence implies symmetry

Idea:

If $\rho(AB) = \rho(A)\rho(B)$, then

$$\rho(AB) = \rho^{\otimes 2}(AB \otimes \mathbb{I}) = \rho^{\otimes 2}(A \otimes B) = \rho^{\otimes 2}(B \otimes A).$$

More generally, for n copies:

$$\rho(AB) = \rho^{\otimes n}\Big(\pi_1(A \otimes \mathbb{I}^{\otimes(n-1)})\pi_2(B \otimes \mathbb{I}^{\otimes(n-1)})\Big)$$

for all n and $\pi_1, \pi_2 \in S_n$.

[5] Wolfe, Spekkens, Fritz (2019)
[6] Wolfe *et al.* (2021)

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This is also (partly) the motivation behind the inflation technique.[5] [6]

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[5] Wolfe, Spekkens, Fritz (2019)

[6] Wolfe *et al.* (2021)

Relaxations via symmetry

Instead of solving our difficult problem

$$\min_{\rho \in S(\mathcal{A})} \quad p_0(\mathcal{A}, \rho)$$
$$s.t. \quad p_i(\mathcal{A}, \rho) = 0 \quad \forall i$$

solve the hierarchy of NPO problems

$$\min_{\omega \in S(\mathcal{A}^{\otimes n})} \quad \omega(y_0)$$
$$s.t. \quad \omega(y_i) = 0 \quad \forall i$$
$$\omega \text{ is symmetric on } \mathcal{A}^{\otimes n},$$

where y_0, y_i are the <u>polarizations</u> of p_0, p_i.

Polarization example

Suppose

$$p_i(\mathcal{A}, \rho) = \rho(AB) - \rho(A)\rho(B) = 0.$$

Then the polarization of p_i is

$$\omega(y_i) := \omega(A_1 B_1 - A_1 B_2) = 0,$$

which is now linear in the state

Relaxations via symmetry

Instead of solving our difficult problem

$$\begin{aligned} \min_{\rho \in S(\mathcal{A})} \quad & p_0(\mathcal{A}, \rho) \\ s.t. \quad & p_i(\mathcal{A}, \rho) = 0 \quad \forall i \end{aligned}$$

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where y_0, y_i are the polarizations of p_0, p_i.

Difficulties for convex optimization

- The set of product states is not convex:
  For ρ, σ product states $(1-\lambda)\rho + \lambda\sigma$ is mixed unless $\lambda = 0$ or 1.
  the set of symmetric states is convex ✓
- The polynomials (in the state) can be non-convex
  We linearized those polynomials ✓
- The dimension grows exponentially quickly
  NPO can deal with that ✓

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Quantum de Finetti Theorem

Theorem: Max tensor product Quantum de Finetti[7]

Let $\omega \in S(\mathcal{A}^\infty)$ be a symmetric state on an infinite maximal tensor product

$$\mathcal{A}^\infty = \lim_{n\to\infty} \mathcal{A}^{\otimes_{\max} n}.$$

Then there exists a unique probability measure μ over states on $\mathcal{A}$ such that for all $x \in \mathcal{A}^\infty$,

$$\omega(x) = \int_{S(\mathcal{A})} \Pi_\sigma^\infty(x)\, \mathrm{d}\mu(\sigma),$$

where Π_σ^∞ is the infinite symmetric product state on $\mathcal{A}^\infty$ associated with the state σ on $\mathcal{A}$.

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[7] **L**, Gachechiladze, Gross (2021)

Convergence proof sketch

- Each step in the hierarchy is a relaxation $\implies$ each solution is a lower bound
- NPO converges[8]
- The Quantum de Finetti Theorem shows that for $n \to \infty$ the optimal state converges to a separable state that obeys all the polarization constraints. It looks like

$$\omega(x) = \int_{S(\mathcal{A})} \Pi_\sigma^\infty(x)\,\mathrm{d}\mu(\sigma),$$

  where "each" Π_σ^∞ obeys all the polynomial constraints![9]
- Choose one such Π_σ^∞. This state gives an upper bound that matches the lower bound in the limit.

[8] Pironio, Navascués, Acín (2008)
[9] **L**, Gross (2022)

Closely related SDP hierarchy

Recent paper on state polynomial optimization[10]

- Extends the algebra $\mathcal{A}$ by including commuting generators g_i " $=$ " $\rho(a_i)$ for a basis $\{a_i\}_i$ of $\mathcal{A}$
- Now polynomials in states also become linear:

$$\begin{aligned} \rho(a_i a_j) &= \rho(a_i)\rho(a_j) \\ &= \rho(a_i \rho(a_j)) = \rho(a_i g_j) \end{aligned}$$

- They prove a Positivstellensatz to show that this converges

Resembles scalar extension[11]

[10] Klep *et al.* (2023)
[11] Pozas-Kerstjens *et al.* (2019)

Conclusion & outlook

There are several closely related converging hierarchies of SDP relaxations to the state polynomial optimization problem

$$\begin{aligned} \min_{\rho \in S(\mathcal{A})} \quad & p_0(\mathcal{A}, \rho) \\ \text{s. t.} \quad & p_i(\mathcal{A}, \rho) = 0 \quad \forall i. \end{aligned}$$

This has many applications in QIT, e.g. (non-linear) Bell inequalities and the causal compatibility problem (cf David's talk)

- Which hierarchy has faster convergence? And how are they related?
- For which causal structures is state polynomial optimization complete?

Thanks for listening!