Title: Is causal optimization polynomial optimization?

Speakers: David Gross

Collection: Causal Inference & Quantum Foundations Workshop

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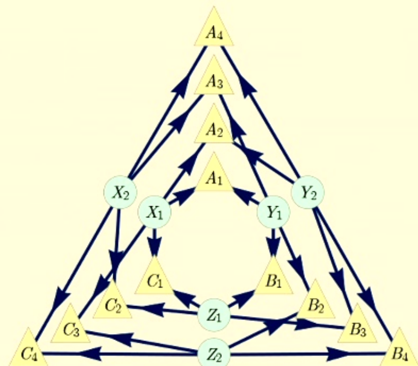
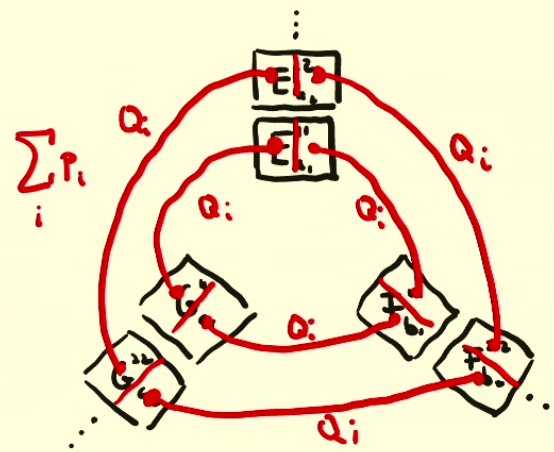
Abstract: "Is there a complete semi-definite programming hierarchy for quantum causal problems? We divide the question into two parts. First: Can quantum causal problems be expressed as polynomial optimization problems (this talk). Second: Can this class of polynomial optimizations be solved by means of SDPs (Laurens' talk). The optimizations we consider here are ""polynomial"" in two ways. They are over the unknown observable algebra of the hidden systems, which are specified by non-commutative polynomials in a set of generators. But they also involve independence constraints, which are commutative polynomials in the state. A hierarchy
of such polynomial tests is complete if one can construct a quantum model for any observed distribution that passes all of them. We've recently had some success in finding such constructions, but also ran into problems in the general case [1, 2]. I give a high-level presentation of the state of the play.

[1] https://arxiv.org/abs/2110.14659
[2] https://arxiv.org/abs/2212.11299"

Is causal optimization polynomial optimization?

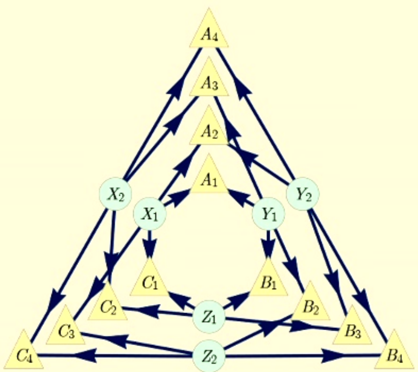




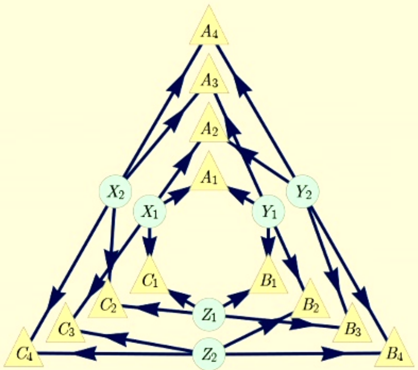
David Gross, University of Cologne
Joint work with Laurens Ligthart and Mariami Gachechiladze
arXiv:2110.14659, arXiv:2212.11299

Hierarchies of necessary conditions for causal compatibility have been proposed.

Some are asymptotically complete for classical version.



[arXiv:2110.14659, arXiv:2212.11299]



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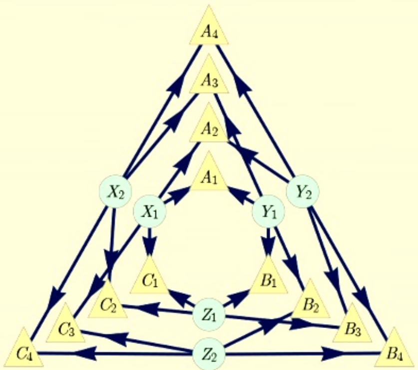
Some are asymptotically complete for classical version.

Q.: Where are we in deciding completeness for the quantum version?

[arXiv:2110.14659, arXiv:2212.11299]



Two step program:

This talk

- Can quantum causal programs be expressed as polynomial* optimization problems?

*commutative-and-non-commutative

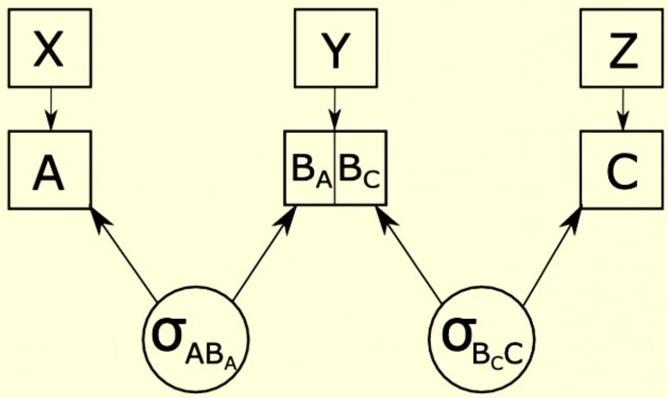
Laurens' talk

- Can these polynomial optimizations be expressed as SDP hierarchies?

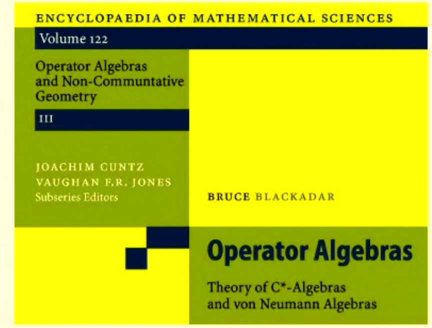
[arXiv:2212.11299]

Guiding model:

The bilocal scenario



Models of quantum systems



Textbook model of quantum system:

- Primary object is a Hilbert space H
- $\Longrightarrow$ *observables* = the *-algebra of operators on H

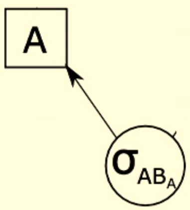
A

Algebraic model of quantum system:

- Primary object is *-algebra $\mathcal{A}$ of observables.
- [$\Longrightarrow$ may derive Hilbert space (via GNS construction)]

- *Mostly* equivalent.
- Second point of view natural for our approach.

Quantum states



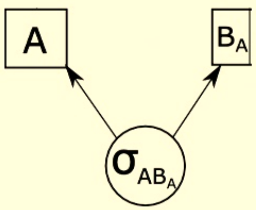
State ρ is linear function on $\mathcal{A}$ such that

- $\rho(a^* a) \geq 0$ [Positivity]
- $\rho(1) = 1$ [Normalization]

[Generalizes $\mathrm{a} \mapsto \langle\Psi|a|\Psi\rangle$ from textbook QM]

Locality

Subsystems are associated with commuting subalgebras.

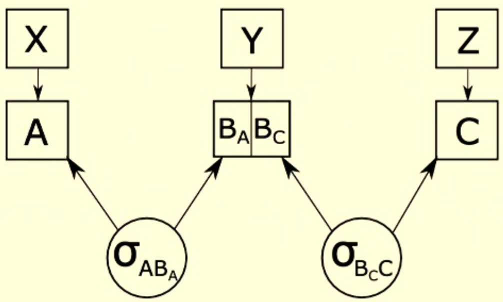


- $\mathcal{A}, \mathcal{B}_A \subset \mathfrak{D}$
- $[\mathcal{A}, \mathcal{B}_A] = 0$

- More general than textbook QM version!
  [Tsirelson's problem]

- Unclear which one is "more physical"

Quantum bilocal scenario



p$(\alpha,\beta,\gamma|x,y,z)$ is *compatible* if ∃

- Commuting observable algebras $\mathcal{A},\mathcal{B}_A,\mathcal{B}_C,\mathcal{C}\ \subset\ D$
- A joint state ρ that factorizes

$$\rho(a\ b_A\ b_C\ c)=\rho(a\ b_A)\rho(b_C\ c)$$

- POVMs $\{A_{\alpha|x}\}$, $\{B_{\beta|y}\}$, $\{C_{\gamma|z}\}$

$$A_{\alpha|x}\geq 0,\ \sum_\alpha A_{\alpha|x}=I$$

such that

$$p(\alpha,\beta,\gamma|x,y,z)=\rho\big(A_{\alpha|x}B_{\beta|y}\ C_{\gamma|z}\big).$$

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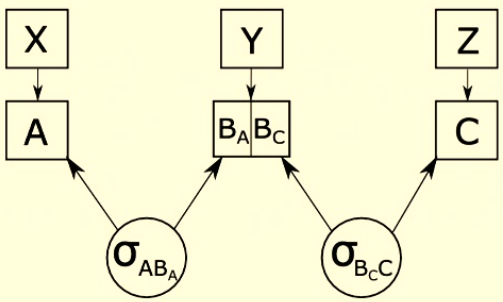

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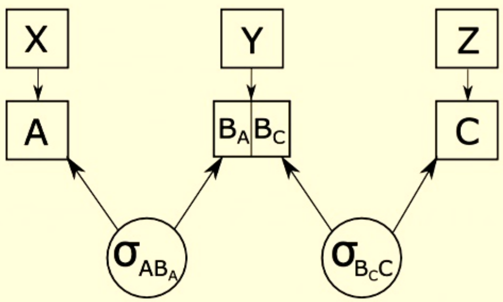

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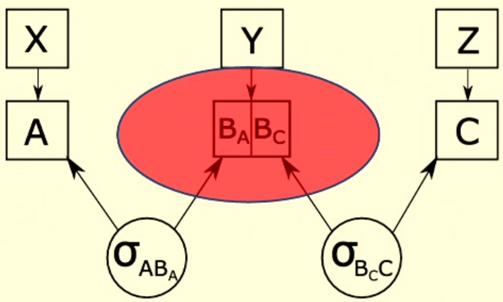

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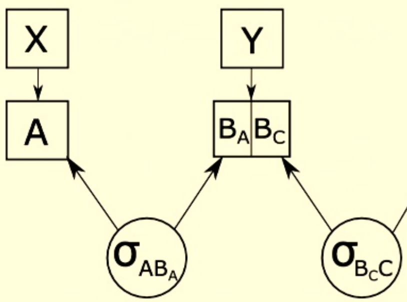
Universal algebras



Know little about observable algebra.

But it *does* contain the POVMs.

- There is a *universal* C^*-algebra with *generators*

$$\{A_{\alpha|x}\}, \{B_{\beta|y}\}, \{C_{\gamma|z}\}$$

- and *relations*

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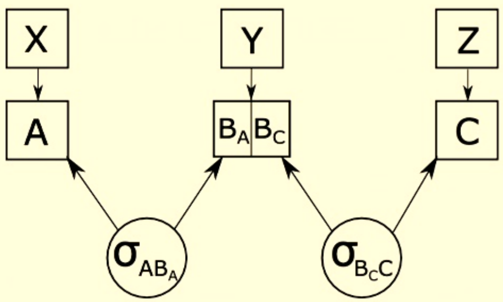
Universal algebras



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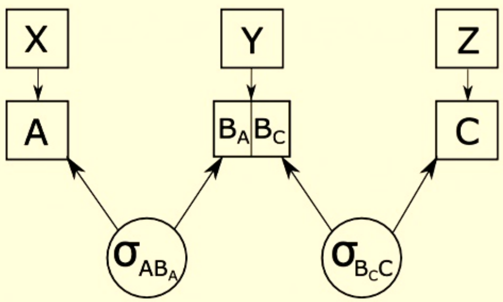
- We'll work with these kind of objects.

Universal algebras



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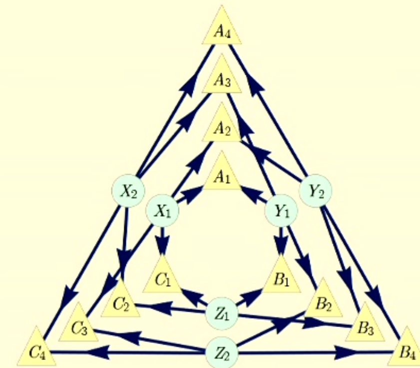
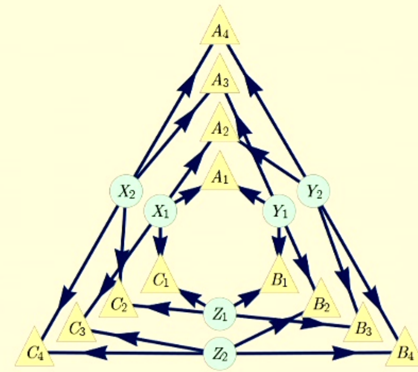
Polynomial optimization

Input
- Generators $\{g_i\}$
- Relations $\{a_j \geq 0\}$ for a_j nc-polynomial in generators.
- Polynomials p_k in states evaluated at words in generators



[Ligthart, DG 22; Ligthart, Gachechiladze, DG 20; Klep, Magron, Volcic, Wang 23]

Polynomial optimization



Input

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- Relations $\{a_j \geq 0\}$ for a_j nc-polynomial in generators.
- Polynomials p_k in states evaluated at words in generators

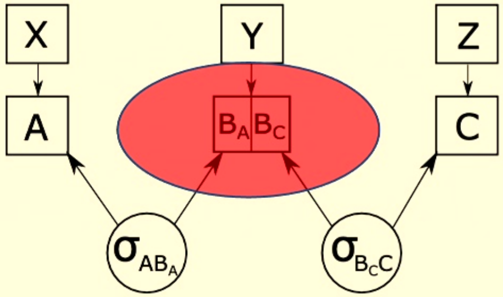
Output

$$\begin{aligned} &\min p_0(\rho) \\ &\text{s.t. } p_k(\rho) = 0 \quad \forall k \geq 1 \end{aligned}$$

With minimum over states of the of the universal C^*-algebra.

[Ligthart, DG 22; Ligthart, Gachechiladze, DG 20; Klep, Magron, Volcic, Wang 23]

Quantum bilocal scenario



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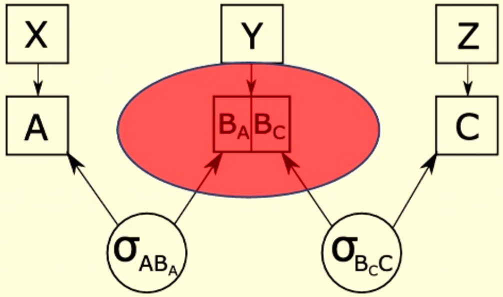
Is causal optimization polynomial optimization?

What's the issue?



Difficulty:

- Factorization constraint

$$\rho(a\, b_A\, b_C\, c) = \rho(a\, b_A)\rho(b_C\, c) \qquad (🏒)$$

involves operators b_a, b_c *that need not lie in algebra generated by the observables*.

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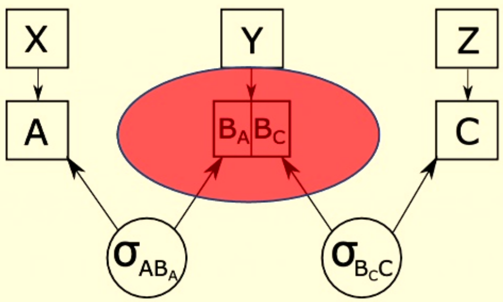

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- "Unclear how observables lies relative to the locality structure that defines causal structure."

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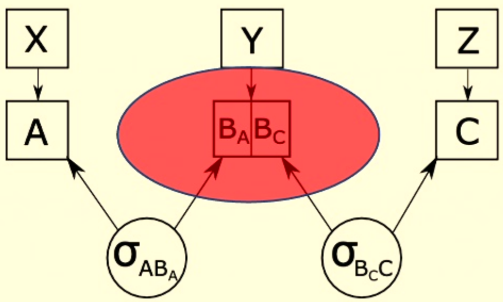

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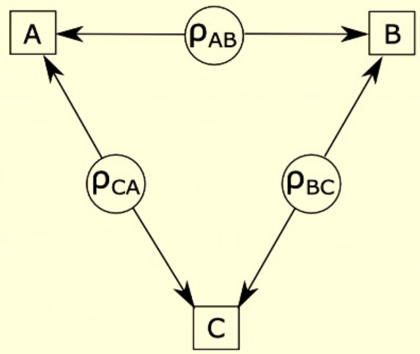
involves operators b_a, b_c *that need not lie in algebra generated by the observables*.

- "Unclear how observables lies relative to the locality structure that defines causal structure."

- Could impose (🏒) *if we knew which operators to impose it on.*

Two solutions

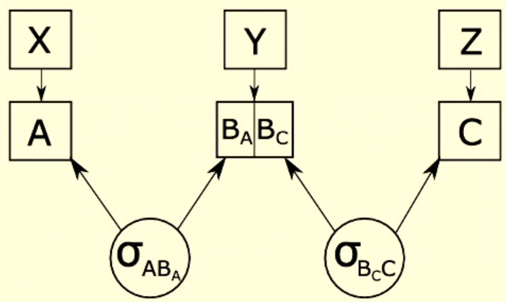


In general: A hack.

[arXiv:2110.14659]

For linear scenarios: No hack.

[arXiv:2212.11299]



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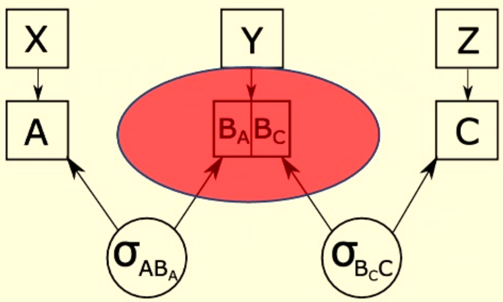


"Top-down" doesn't work:

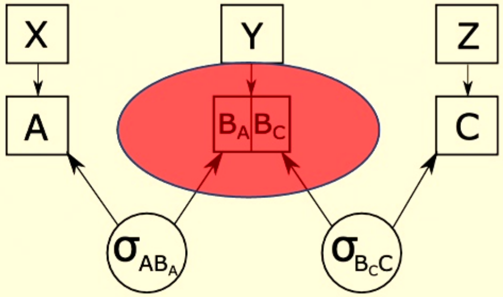
- Don't know how to write $\mathcal{B}_A, \mathcal{B}_C$ in terms of observables.

"Bottom-up" does!

- Introduce generators for $\mathcal{B}_A, \mathcal{B}_C$ and expand observables

$$B_{\beta|y} = \sum_{i=1}^{r} b_a^{(i)} \cdot b_c^{(i)}$$

A hack



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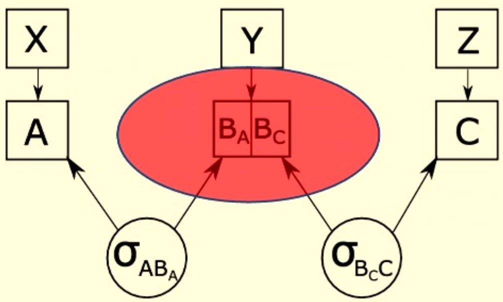
$$B_{\beta|y} = \sum_{i=1}^{r} b_a^{(i)} \cdot b_c^{(i)}$$

- Works, but rank constraint might be artifcial.
- New ideas very welcome!!

Bilocal: independences suffice



- For bilocal scenario, factorization constraint

$$\rho(a\, b_A\, b_C\, c) = \rho(a\, b_A)\rho(b_C\, c)$$

- can be replaced by

$$\rho(a\, c) = \rho(a)\rho(c)$$

- for a, c in the algebra generated by Alice's and Charlies' observables.

[Ligthart, DG 22; inspired by Renou, Xu 22]

Proof idea 1/2: Recovering $\mathcal{B}_A, \mathcal{B}_C$

Assume:

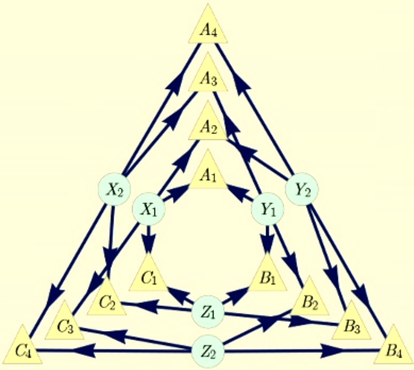
- There's a textbook quantum model given by

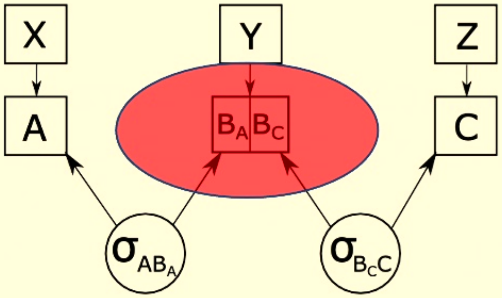
$$|\psi_{AB_A}\rangle \otimes |\psi_{B_C C}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{\mathcal{B}_\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}_\mathcal{C}} \otimes \mathcal{H}_\mathcal{C}$$

- with $|\psi_{AB_A}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{\mathcal{B}_\mathcal{A}}$ maximally entangled.

Then $B(\mathcal{H}_{\mathcal{B}_\mathcal{A}})$ is the commutant of $B(\mathcal{H}_A)$ in $B(\mathcal{H}_A \otimes \mathcal{H}_{\mathcal{B}_\mathcal{A}})$.

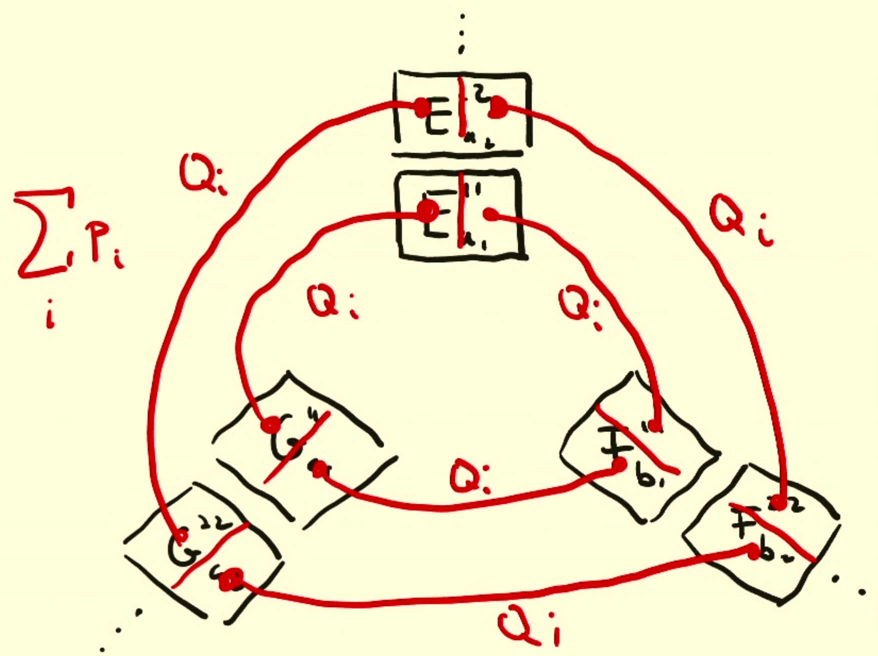
Summary





- If you write down any algebra and any set of polynomial constraints, Laurens can turn it into a convergent SDP hierarchy.
- But it's still not obvious that this captures quantum causal optimization
- That's because the independence constraints are stated with respect to observables that aren't generated by the POVMs

Thank you!



David Gross, University of Cologne

Joint work with Laurens Ligthart and Mariami Gachechiladze

Proof idea 2/2: GNS for product states

- Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be the universal algebras generated by POVMs
- Let ρ be a state such that

$$\rho(a\, c) = \rho(a)\rho(c)$$

Then:

- GNS representation for $\mathcal{A}, \mathcal{C}$ factorizes

$$\mathcal{H}_\rho = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{C}, \qquad |\rho\rangle = |\alpha\rangle \otimes |\gamma\rangle$$

- Define

$$\mathcal{B}_\mathcal{A} = \pi(\mathcal{A})', \qquad \mathcal{B}_\mathcal{C} = \pi(\mathcal{C})'$$

- There's a channel $\Lambda\colon \mathcal{B} \to \mathcal{B}_\mathcal{A} \otimes \mathcal{B}_\mathcal{C}$ such that

$$\rho(abc) = \langle\alpha\gamma|\, \pi(a)\, \Lambda(b)\, \pi(c) |\alpha\gamma\rangle$$