Title: Latent variable justifies the stronger instrumental variable bounds

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Abstract: For binary instrumental variable models, there seems to be a long-standing gap between two sets of bounds on the average treatment effect: the stronger Balke-Pearl ("sharp") bounds versus the weaker Robins-Manski ("natural") bounds. In the literature, the Balke-Pearl bounds are typically derived under stronger assumptions, i.e., either individual exclusion or joint exogeneity, which are untestable cross-world statements, while the natural bounds only require testable assumptions. In this talk, I show that the stronger bounds are justified by the existence of a latent confounder. In fact, the Balke-Pearl bounds are sharp under latent confounding and stochastic exclusion. The "secret sauce" that closes this gap is a set of CHSH-type inequalities that generalize Bell's (1964) inequality.

Latent Variable Justifies the Stronger IV Bounds

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April 19, 2023

Causal Inference & Quantum Foundations Workshop

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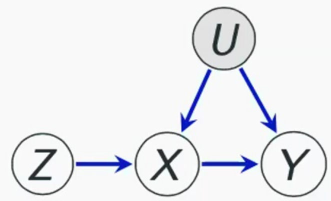
based on joint work w. Thomas Richardson

Question: *Why should statisticians care about CHSH/Bell-type inequalities?*

☞ This comes up in studying partial identification when there could be latent variables.

Instrumental variable



- Suppose $X, Y, Z \in \{0, 1\}$:

$$Z : \text{instrument}, \quad X : \text{treatment}, \quad Y : \text{outcome}.$$

$Y(x, z)$ denotes the counterfactual outcome.

- U is a latent confounder. ☞ No assumption on its state space.

☞ We want **sharp bounds** on the average treatment effect

$$\boxed{\tau \equiv \mathbb{E}\, Y(x = 1, z = 0) - \mathbb{E}\, Y(x = 0, z = 0)}$$

in terms of $P(X, Y, Z)$.

Balke–Pearl bounds

Balke and Pearl (1997) obtained bounds on τ:

$$\max\left\{\begin{array}{c}
-P(X=1,Y=0\mid Z=0)-P(X=0,Y=1\mid Z=0)\\
-P(X=1,Y=0\mid Z=1)-P(X=0,Y=1\mid Z=1)\\
P(X=1,Y=1\mid Z=0)+P(X=0,Y=0\mid Z=1)-1\\
P(X=1,Y=1\mid Z=1)+P(X=0,Y=0\mid Z=0)-1\\
P(X=1,Y=1\mid Z=0)-P(Y=1\mid Z=1)-P(X=1,Y=0\mid Z=0)-P(X=0,Y=1\mid Z=0)\\
P(X=1,Y=1\mid Z=1)-P(Y=1\mid Z=0)-P(X=1,Y=0\mid Z=1)-P(X=0,Y=1\mid Z=1)\\
P(X=0,Y=0\mid Z=1)-P(Y=0\mid Z=0)-P(X=1,Y=0\mid Z=1)-P(X=0,Y=1\mid Z=1)\\
P(X=0,Y=0\mid Z=0)-P(Y=0\mid Z=1)-P(X=1,Y=0\mid Z=0)-P(X=0,Y=1\mid Z=0)
\end{array}\right\}$$

$$\leq \tau \leq$$

$$\min\left\{\begin{array}{c}
-P(X=1,Y=1\mid Z=0)-P(X=0,Y=0\mid Z=0)\\
-P(X=1,Y=1\mid Z=1)-P(X=0,Y=0\mid Z=1)\\
1-P(X=1,Y=0\mid Z=0)+P(X=0,Y=1\mid Z=1)\\
1-P(X=1,Y=0\mid Z=1)+P(X=0,Y=1\mid Z=0)\\
-P(X=1,Y=0\mid Z=0)+P(Y=0\mid Z=1)+P(X=1,Y=1\mid Z=0)+P(X=0,Y=0\mid Z=0)\\
-P(X=1,Y=0\mid Z=1)+P(Y=0\mid Z=0)+P(X=1,Y=1\mid Z=1)+P(X=0,Y=0\mid Z=1)\\
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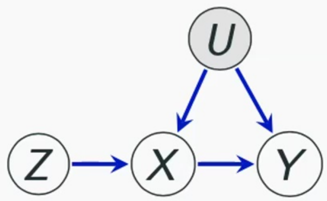
-P(X=0,Y=1\mid Z=0)+P(Y=1\mid Z=1)+P(X=1,Y=1\mid Z=0)+P(X=0,Y=0\mid Z=0)
\end{array}\right\}$$

Balke–Pearl bounds



The Balke–Pearl bounds are **sharp** under the following two **cross-world** assumptions.

- Individual-level exclusion (ind.Excl)

$$\boxed{Y(x, z=0) = Y(x, z=1), \quad x = 0, 1.}$$

(so one can write $Y(x) = Y(x, z)$) ☞ Cannot be consistently tested by randomizing (Z, X).

- Joint exogeneity (joint.Exo)

$$\boxed{Z \perp\!\!\!\perp Y(0,0), Y(0,1), Y(1,0), Y(1,1)}$$

(i.e., $Z \perp\!\!\!\perp Y(0), Y(1)$ under (ind.Excl)) ☞ Dubious in observational studies.

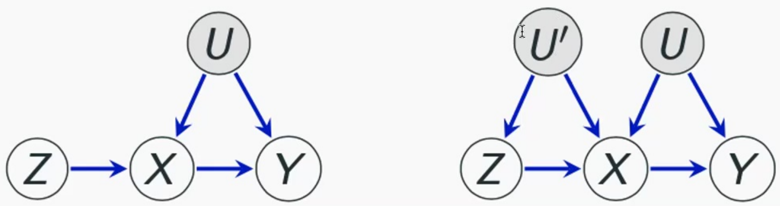
Question: *What bounds can we obtain with single-world assumptions?*

Latent variable DAG

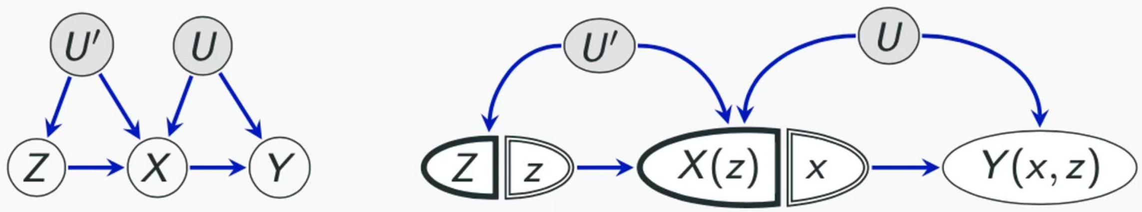
Following the convention in the literature,

- we introduce counterfactual $X(z)$,
- and allow a separate latent variable U' between Z and X for $U' \perp\!\!\!\perp U$.

☞ But these are not essential for our derivation.



SWIG interpretation of the DAG



(Population-level) Single World Intervention Graph (SWIG) (Richardson and Robins, 2013) encodes the following assumptions.

1. $Y(x,z) \perp\!\!\!\perp_d z \mid U$ on the SWIG encodes the latent-variable stochastic exclusion

$$\boxed{\mathbb{E}[Y(x,0) \mid U] = \mathbb{E}[Y(x,1) \mid U], \quad x \in \{0,1\}} \quad \texttt{(LV.sto.Excl)}$$

☞ Compare with `(ind.Excl)`: $Y(x,0) = Y(x,1)$.

2. Latent-variable marginal exogeneity

$$\boxed{Z \perp\!\!\!\perp U, \quad Y(x,z) \perp\!\!\!\perp Z, X(z) \mid U} \quad \texttt{(LV.marg.Exo)}$$

☞ Compare with `(joint.Exo)`: $Z \perp\!\!\!\perp Y(0,0), Y(0,1), Y(1,0), Y(1,1)$.

Deriving bounds

We derive the bounds on $\tau \equiv \mathbb{E}\, Y(1,0) - \mathbb{E}\, Y(0,0)$ with **linear programming**.

1. Let $\{P(X(0), X(1), Y(0,0), Y(0,1), Y(1,0), Y(1,1) \mid Z = z) : z = 0,1\}$ and $\{P(X, Y \mid Z = z) : z = 0,1\}$ be variables. ☞ U, U' are not part of the program.
2. Add constraints:
   (1) Simplex
   (2) Consistency: $P(X = x, Y = y \mid Z = z) = P(X(z) = x, Y(x,z) = y \mid Z = z)$.
   (3) Single-world assumptions

$$\boxed{\mathbb{E}[Y(x,0) \mid U] = \mathbb{E}[Y(x,1) \mid U], \quad x \in \{0,1\}} \quad \texttt{(LV.sto.Excl)}$$

$$\boxed{Z \perp\!\!\!\perp U, \quad Y(x,z) \perp\!\!\!\perp Z, X(z) \mid U} \quad \texttt{(LV.marg.Exo)}$$

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   (3) Single-world assumptions

$$\boxed{\mathbb{E}[Y(x,0) \mid U] = \mathbb{E}[Y(x,1) \mid U]}\ \texttt{(LV.sto.Excl)} \Rightarrow \boxed{\mathbb{E}\,Y(x,0) = \mathbb{E}\,Y(x,1)}\ \texttt{(sto.Excl)}$$

$$\boxed{Z \perp\!\!\!\perp U, \quad Y(x,z) \perp\!\!\!\perp Z, X(z) \mid U}\ \texttt{(LV.marg.Exo)} \Rightarrow \boxed{Z \perp\!\!\!\perp Y(x,z)}\ \texttt{(marg.Exo)}$$

Deriving bounds

We derive the bounds on $\tau \equiv \mathbb{E}\, Y(1,0) - \mathbb{E}\, Y(0,0)$ with **linear programming**.

1. Let $\{P(X(0), X(1), Y(0,0), Y(0,1), Y(1,0), Y(1,1) \mid Z = z) : z = 0, 1\}$ and $\{P(X, Y \mid Z = z) : z = 0, 1\}$ be variables. ☞ U, U' are not part of the program.
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   (2) Consistency: $P(X = x, Y = y \mid Z = z) = P(X(z) = x, Y(x,z) = y \mid Z = z)$.
   (3) Single-world assumptions

$$\boxed{\mathbb{E}\, Y(x,0) = \mathbb{E}\, Y(x,1), \quad x \in \{0,1\}} \quad \texttt{(sto.Excl)}$$

$$\boxed{Z \perp\!\!\!\perp Y(x,z), \quad x, z \in \{0,1\}} \quad \texttt{(marg.Exo)}$$

3. Also add τ as a programming variable. ☞ Degenerate convex polytope.
4. Project onto coordinates $\{P(X, Y \mid Z = z) : z = 0, 1\}$ and τ.

Robins–Manski bounds

We get the "natural" bounds (Manski, 1990; Robins, 1989):

$$\max\left\{\begin{array}{c} -P(X=1,Y=0\mid Z=0)-P(X=0,Y=1\mid Z=0) \\ -P(X=1,Y=0\mid Z=1)-P(X=0,Y=1\mid Z=1) \\ P(X=1,Y=1\mid Z=0)+P(X=0,Y=0\mid Z=1)-1 \\ P(X=1,Y=1\mid Z=1)+P(X=0,Y=0\mid Z=0)-1 \end{array}\right\}$$

$$\leq \tau \leq$$

$$\min\left\{\begin{array}{c} -P(X=1,Y=1\mid Z=0)-P(X=0,Y=0\mid Z=0) \\ -P(X=1,Y=1\mid Z=1)-P(X=0,Y=0\mid Z=1) \\ 1-P(X=1,Y=0\mid Z=0)+P(X=0,Y=1\mid Z=1) \\ 1-P(X=1,Y=0\mid Z=1)+P(X=0,Y=1\mid Z=0) \end{array}\right\}$$

Compare with the Balke–Pearl bounds

$$\max\left\{\begin{array}{c} -P(X=1,Y=0\mid Z=0)-P(X=0,Y=1\mid Z=0) \\ -P(X=1,Y=0\mid Z=1)-P(X=0,Y=1\mid Z=1) \\ P(X=1,Y=1\mid Z=0)+P(X=0,Y=0\mid Z=1)-1 \\ P(X=1,Y=1\mid Z=1)+P(X=0,Y=0\mid Z=0)-1 \\ P(X=1,Y=1\mid Z=0)-P(Y=1\mid Z=1)-P(X=1,Y=0\mid Z=0)-P(X=0,Y=1\mid Z=0) \\ P(X=1,Y=1\mid Z=1)-P(Y=1\mid Z=0)-P(X=1,Y=0\mid Z=1)-P(X=0,Y=1\mid Z=1) \\ P(X=0,Y=0\mid Z=1)-P(Y=0\mid Z=0)-P(X=1,Y=0\mid Z=1)-P(X=0,Y=1\mid Z=1) \\ P(X=0,Y=0\mid Z=0)-P(Y=0\mid Z=1)-P(X=1,Y=0\mid Z=0)-P(X=0,Y=1\mid Z=0) \end{array}\right\}$$

$$\leq \tau \leq$$

$$\min\left\{\begin{array}{c} -P(X=1,Y=1\mid Z=0)-P(X=0,Y=0\mid Z=0) \\ -P(X=1,Y=1\mid Z=1)-P(X=0,Y=0\mid Z=1) \\ 1-P(X=1,Y=0\mid Z=0)+P(X=0,Y=1\mid Z=1) \\ 1-P(X=1,Y=0\mid Z=1)+P(X=0,Y=1\mid Z=0) \\ -P(X=1,Y=0\mid Z=0)+P(Y=0\mid Z=1)+P(X=1,Y=1\mid Z=0)+P(X=0,Y=0\mid Z=0) \\ -P(X=1,Y=0\mid Z=1)+P(Y=0\mid Z=0)+P(X=1,Y=1\mid Z=1)+P(X=0,Y=0\mid Z=1) \\ -P(X=0,Y=1\mid Z=1)+P(Y=1\mid Z=0)+P(X=1,Y=1\mid Z=1)+P(X=0,Y=0\mid Z=1) \\ -P(X=0,Y=1\mid Z=0)+P(Y=1\mid Z=1)+P(X=1,Y=1\mid Z=0)+P(X=0,Y=0\mid Z=0) \end{array}\right\}$$

☞ **Question**: Is this gap fundamental to the single-world assumptions?

What caused the gap?

In terms of models, we have

$$\texttt{(ind.Excl)} \subset \boxed{\texttt{(LV.sto.Excl)} \quad \subset \quad \texttt{(sto.Excl)}}$$

and

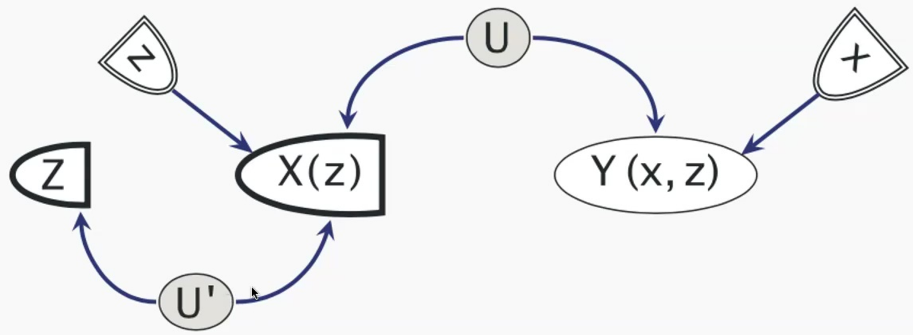
$$\texttt{(joint.Exo)} \subset \boxed{\texttt{(LV.marg.Exo)} \quad \subset \quad \texttt{(marg.Exo)}.}$$

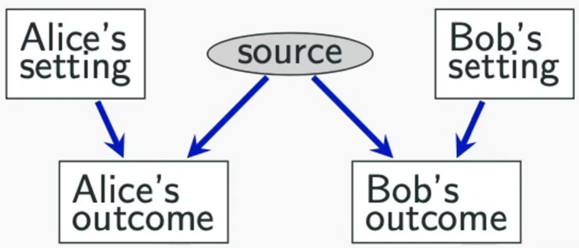
☞ The gap stems from

(LV.sto.Excl) $\Rightarrow$ (sto.Excl) and (LV.marg.Exo) $\Rightarrow$ (marg.Exo) by integrating out U,

which misses the CHSH inequality implied by the sheer **existence** of U!

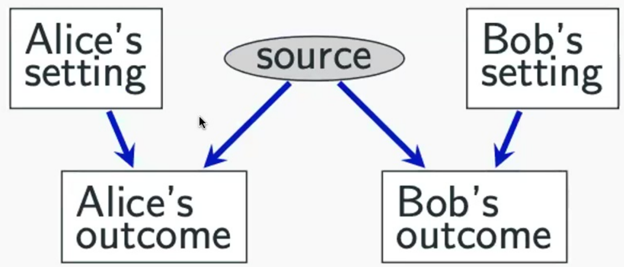
The missing piece





☞ Works similarly if $X(z)$ is not introduced.

CHSH



CHSH is postulated on the correlations:

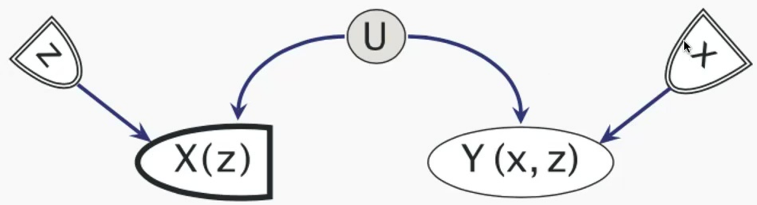
$$\langle A, B\rangle + \langle A', B\rangle + \langle A, B'\rangle - \langle A', B'\rangle \leq 2.$$

Alice either observes $A = \pm 1$ or $A' = \pm 1$; Bob observes either $B = \pm 1$ or $B' = \pm 1$.

CHSH

Lemma The latent variable U implies the Bell–CHSH inequalities

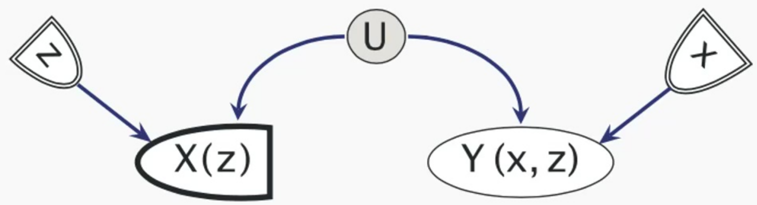
$$\begin{aligned}0 \leq P(X=1, Y(x,z)=1 \mid Z=z) + P(X=0, Y(1-x,z)=0 \mid Z=z) \\ + P(X=0, Y(x,1-z)=0 \mid Z=1-z) \\ - P(X=0, Y(1-x,1-z)=0 \mid Z=1-z) \leq 1, \quad x,z \in \{0,1\}.\end{aligned}$$



CHSH

Lemma The latent variable U implies the Bell–CHSH inequalities

$$
\begin{aligned}
0 \leq P(X=1, Y(x,z)=1 \mid Z=z) &+ P(X=0, Y(1-x,z)=0 \mid Z=z) \\
&+ P(X=0, Y(x,1-z)=0 \mid Z=1-z) \\
&- P(X=0, Y(1-x,1-z)=0 \mid Z=1-z) \leq 1, \quad x,z \in \{0,1\}.
\end{aligned}
$$



☞ Affine in $\{P(X(0), X(1), Y(0,0), Y(0,1), Y(1,0), Y(1,1) \mid Z=z) : z=0,1\}$.

Closing the gap

1. Let $\{P(X(0), X(1), Y(0,0), Y(0,1), Y(1,0), Y(1,1) \mid Z = z) : z = 0,1\}$ and $\{P(X, Y \mid Z = z) : z = 0,1\}$ be variables.
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$$\boxed{\mathbb{E}\, Y(x,0) = \mathbb{E}\, Y(x,1), \quad x \in \{0,1\}} \quad \texttt{(sto.Excl)}$$

$$\boxed{Z \perp\!\!\!\perp Y(x,z), \quad x, z \in \{0,1\}} \quad \texttt{(marg.Exo)}$$

   (4) **CHSH inequalities**

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We recover the Balke–Pearl bounds!

$$\max\left\{\begin{array}{c}
-P(X=1,Y=0\mid Z=0)-P(X=0,Y=1\mid Z=0)\\
-P(X=1,Y=0\mid Z=1)-P(X=0,Y=1\mid Z=1)\\
P(X=1,Y=1\mid Z=0)+P(X=0,Y=0\mid Z=1)-1\\
P(X=1,Y=1\mid Z=1)+P(X=0,Y=0\mid Z=0)-1\\
P(X=1,Y=1\mid Z=0)-P(Y=1\mid Z=1)-P(X=1,Y=0\mid Z=0)-P(X=0,Y=1\mid Z=0)\\
P(X=1,Y=1\mid Z=1)-P(Y=1\mid Z=0)-P(X=1,Y=0\mid Z=1)-P(X=0,Y=1\mid Z=1)\\
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P(X=0,Y=0\mid Z=0)-P(Y=0\mid Z=1)-P(X=1,Y=0\mid Z=0)-P(X=0,Y=1\mid Z=0)
\end{array}\right\}$$

$$\leq \tau \leq$$

$$\min\left\{\begin{array}{c}
-P(X=1,Y=1\mid Z=0)-P(X=0,Y=0\mid Z=0)\\
-P(X=1,Y=1\mid Z=1)-P(X=0,Y=0\mid Z=1)\\
1-P(X=1,Y=0\mid Z=0)+P(X=0,Y=1\mid Z=1)\\
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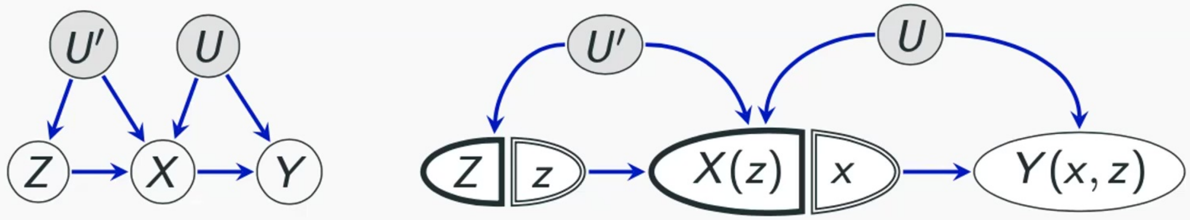
\end{array}\right\}$$

Result



Theorem Suppose $X, Y, Z \in \{0, 1\}$. Under $\texttt{(LV.sto.Excl)}+\texttt{(LV.marg.Exo)}$, which are encoded by SWIG for the latent variable DAG, Balke–Pearl bounds hold and are sharp for τ.

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Further interpretation...

Imagine you are a statistician (or econometrician) who is only willing to conceptualize $\{Z, X(z), Y(z,x) : x, z = 0, 1\}$.

Further, you only want to put down single-world assumptions

$$\boxed{\mathbb{E}\, Y(x,0) = \mathbb{E}\, Y(x,1), \quad x \in \{0,1\}} \quad \texttt{(sto.Excl)}$$

$$\boxed{Z \perp\!\!\!\perp Y(x,z), \quad x,z \in \{0,1\}} \quad \texttt{(marg.Exo)}$$

Question: What is the minimal extra condition on $\{Z, X(z), Y(z,x) : x, z = 0, 1\}$ that is necessary to yield the Balke–Pearl bounds?

Answer: CHSH!

Summary

- Balke–Pearl bounds need not rely on untestable, cross-world assumptions such as `(ind.Excl)`and `(joint.Exo)`.
    - They are justified by a SWIG with latent variable.
    - That being said, CHSH is cross-world.

      ☞ But it is a consequence of the existence of U, rather than assumptions that we impose on!
- The CHSH inequalities fill the gap between Balke–Pearl and Robins–Manski.

  ☞ Dawid (2003) showed the Balke–Pearl bounds (and their tightness) with a certain latent variable formulation that does not use counterfactuals.
- To obtain sharp bounds, we need to understand the inequality constraints in a latent variable formulation.

  ☞ Good to consult a graph over both observed and counterfactual variables (e.g., SWIGs).

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- Balke–Pearl bounds need not rely on untestable, cross-world assumptions such as (ind.Excl) and (joint.Exo).
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- The CHSH inequalities fill the gap between Balke–Pearl and Robins–Manski.
  ☞ Dawid (2003) showed the Balke–Pearl bounds (and their tightness) with a certain latent variable formulation that does not use counterfactuals.
- To obtain sharp bounds, we need to understand the inequality constraints in a latent variable formulation.
  ☞ Good to consult a graph over both observed and counterfactual variables (e.g., SWIGs).
- Linear/convex programming is a useful tool.

▶ Ref: F. R. Guo, Likelihood Analysis of Causal Models, §5.3. University of Washington, 2021.

THANKS