Title: Tutorial 1

Speakers: Robert Spekkens

Collection: Causal Inference & Quantum Foundations Workshop

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URL: https://pirsa.org/23040103

Tutorial: Causal Inference from the perspective of quantum foundations

Robert Spekkens

Causal inference and Quantum Foundations Workshop
April 17, 2023

Causarum Investigatio
“Investigate the causes”

Statistical paradigm

⇩

Causal paradigm

Operational paradigm

⇩

Realist paradigm

The formalism and conceptual scheme of causal inference resolved various puzzles of statistics (e.g., Simpson’s paradox, Berkson’s paradox)

The lesson:
Inference and causation must be clearly distinguished conceptually and in the formalism

"[...] our present Quantum Mechanical formalism [...] is a peculiar mixture describing in part realities of Nature, in part incomplete human information about Nature all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen how to unscramble."

E.T. Jaynes, 1989

But what hope do we have of succeeding if we do not even understand how to unscramble the omelette in the classical context?

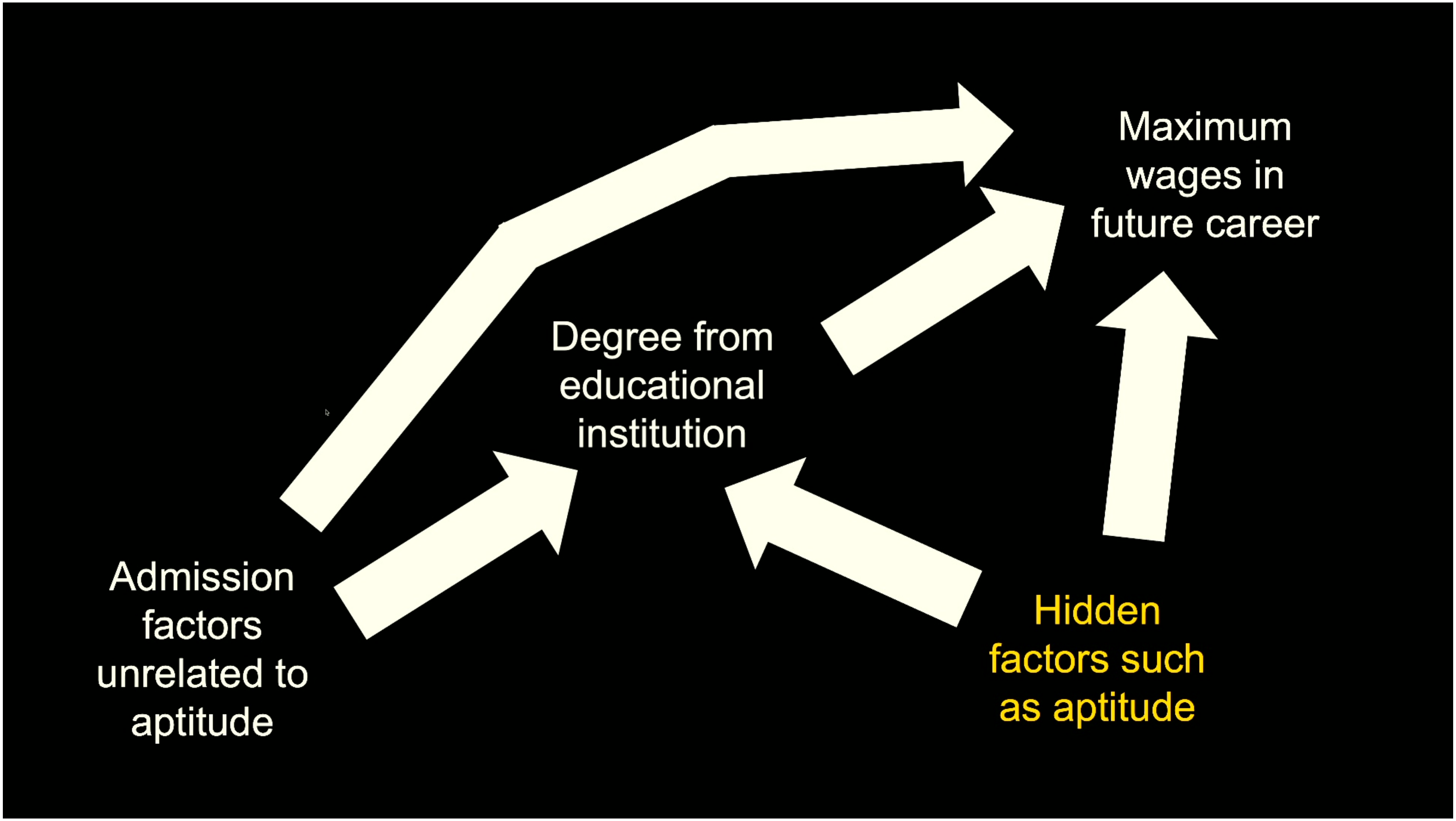
Causal Inference in the presence of hidden variables

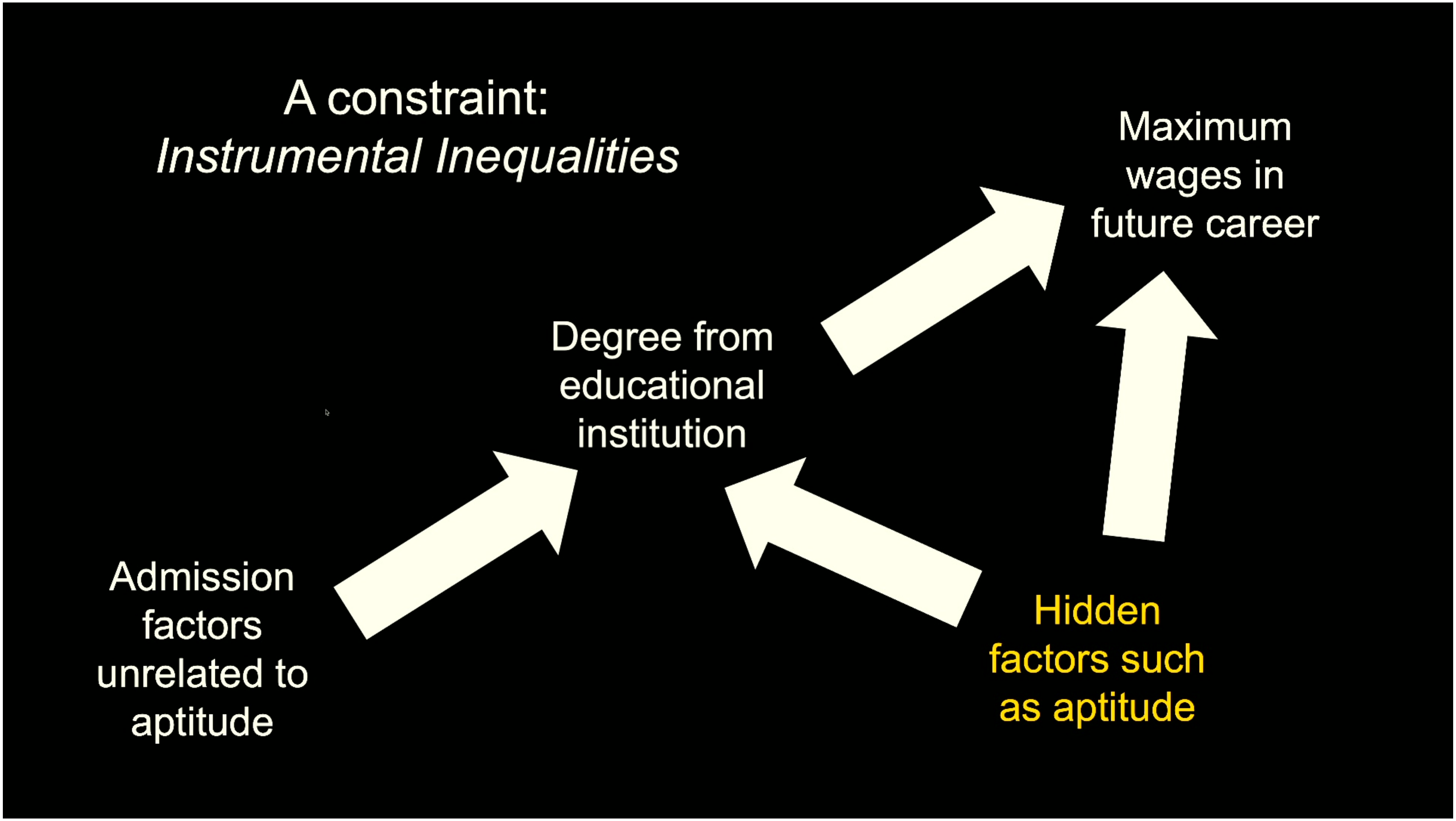
Instrumental inequalities

Maximum wages in future career above some threshold?

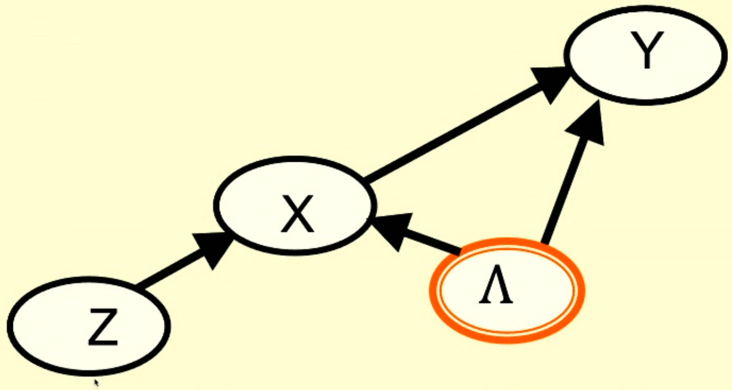
Degree from educational institution?

	Yes	No
Yes	79%	21%
No	43%	57%


Maximum
wages in
future career
Degree from
educational
institution
Admission
factors
unrelated to
aptitude
Hidden
factors such
as aptitude


A constraint:
Instrumental Inequalities
Maximum wages in future career
Degree from educational institution
Admission factors unrelated to aptitude
Hidden factors such as aptitude

Causal structure



Parameters

$P_{X|\Lambda Z}$

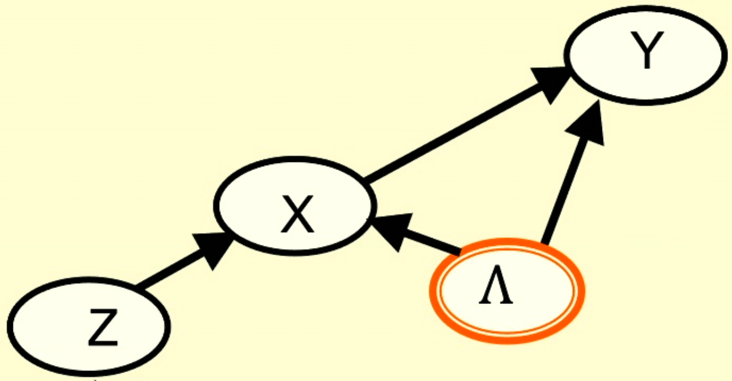
$P_{Y|\Lambda X}$

P_{Λ}

$$P_{XY|Z} = \sum_{\Lambda} P_{Y|X\Lambda} P_{X|Z\Lambda} P_{\Lambda}$$

A distribution is said to be compatible with a given causal structure if there are parameters that yield it

Causal structure



Parameters

$P_{X|\Lambda Z}$

$P_{Y|\Lambda X}$

P_{Λ}

$$P_{XY|Z} = \sum_{\Lambda} P_{Y|X\Lambda} P_{X|Z\Lambda} P_{\Lambda}$$

Example of causal compatibility constraint:

$$P_{XY|Z}(00|0) + P_{XY|Z}(01|1) \leq 1$$

Pearl, 1993

The evidence

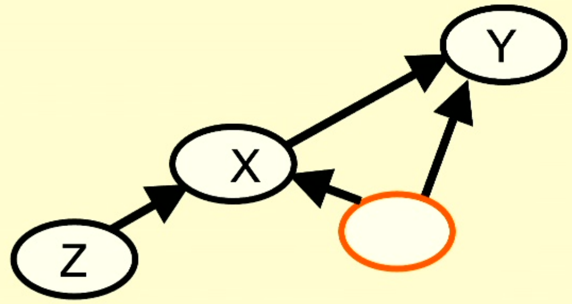
Z=0

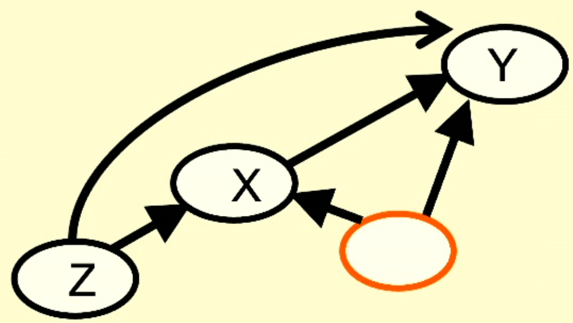
	Y=0	Y=1
X=0	0.79	0.21
X=1	0.43	0.57

Z=1

	Y=0	Y=1
X=0	0.59	0.41
X=1	0.39	0.61

The hypotheses





The evidence

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Z=1

	Y=0	Y=1
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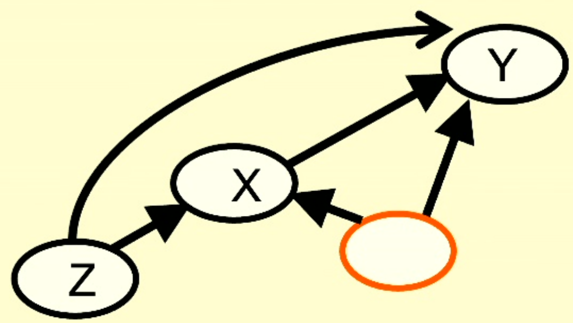
X=1	0.39	0.61

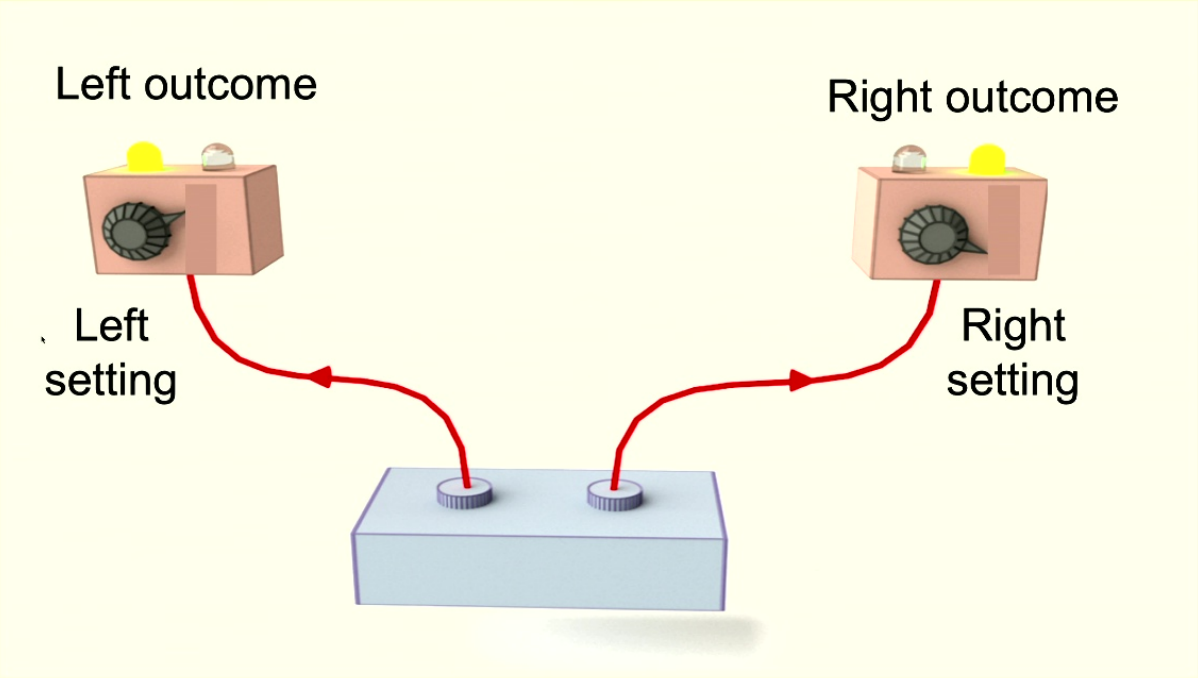
Violates Pearl's instrumental inequality!

The hypotheses

Implies a constraint:
Pearls's instrumental inequality



John Stuart Bell


Left outcome
Right outcome
Left
setting
Right
setting

		Left outcome and Right outcome			
		0 and 0	0 and 1	1 and 0	1 and 1
Left setting and Right setting	0 and 0	43%	7%	7%	43%
	0 and 1	43%	7%	7%	43%
	1 and 0	43%	7%	7%	43%
	1 and 1	7%	43%	43%	7%

The evidence

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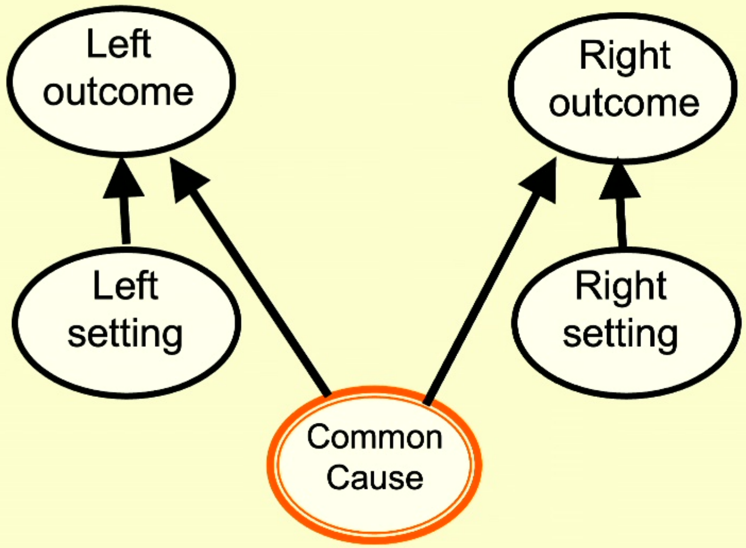
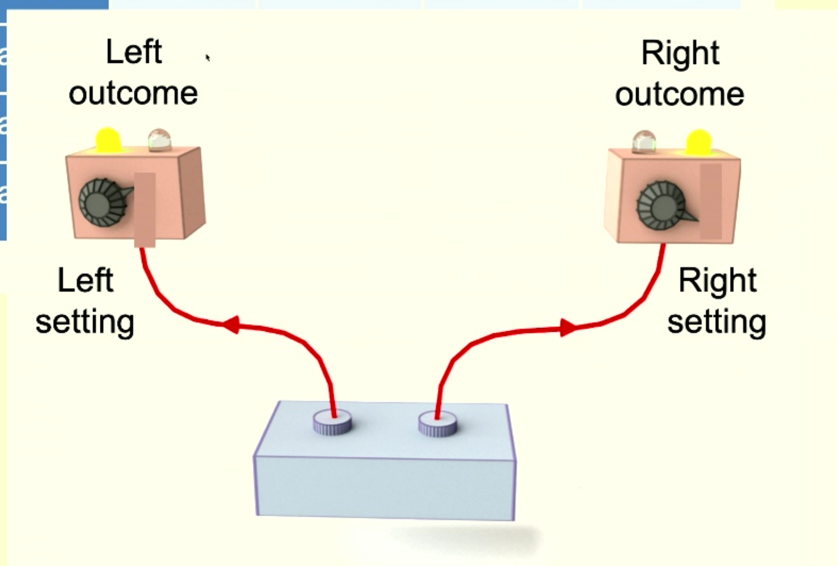
The natural hypothesis



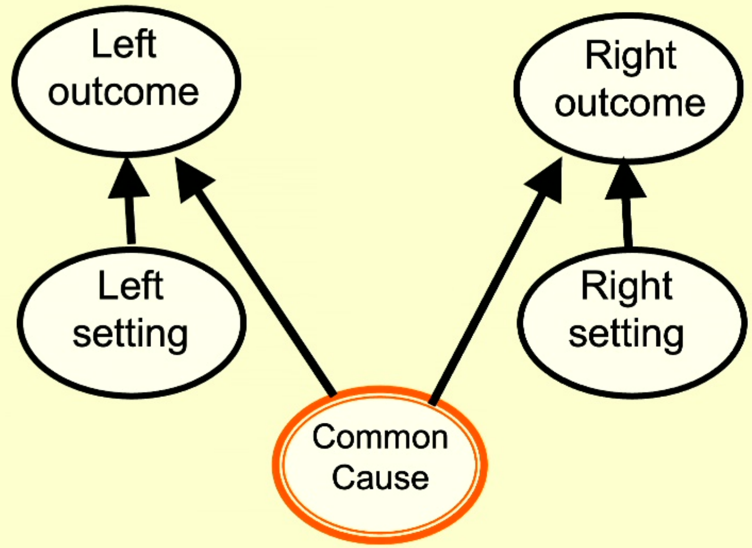
The evidence

Left outcome and Right outcome

Left setting and Right setting	0 and 0	0 and 1	1 and 0	1 and 1
0 and 0	43%	7%	7%	43%
0 a				
1 a				
1 a				



The natural hypothesis



The evidence

Left outcome and Right outcome

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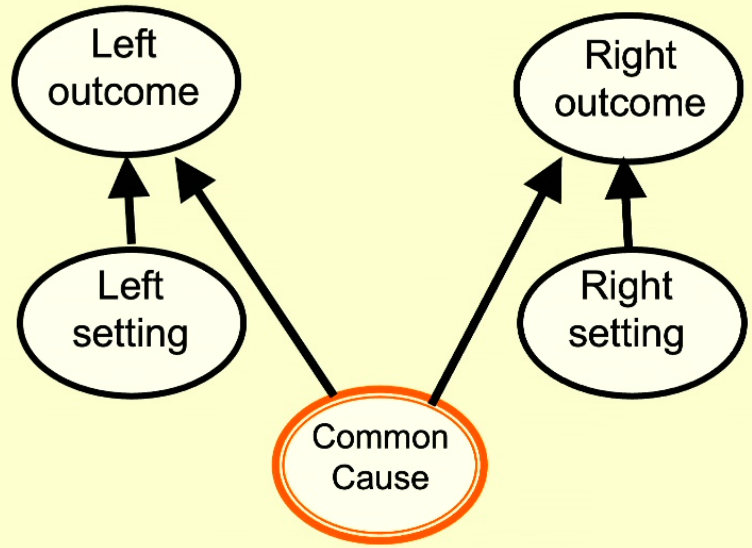
The natural hypothesis



Implies a constraint:
Bell Inequalities

The evidence

Left outcome and Right outcome

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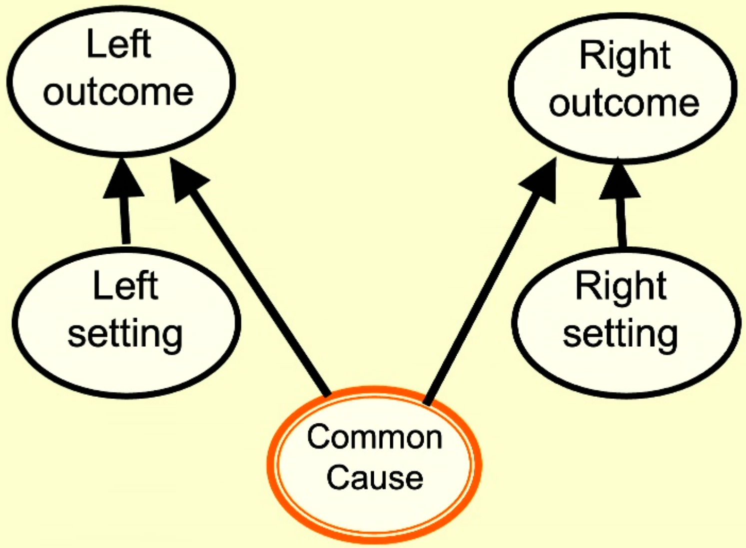
Violates Bell inequalities
(up to Tsirelson bound)

The natural hypothesis



Implies Bell inequalities

Incompatible

The evidence

		Left outcome and Right outcome			
		0 and 0	0 and 1	1 and 0	1 and 1
Left setting and Right setting	0 and 0	43%	7%	7%	43%
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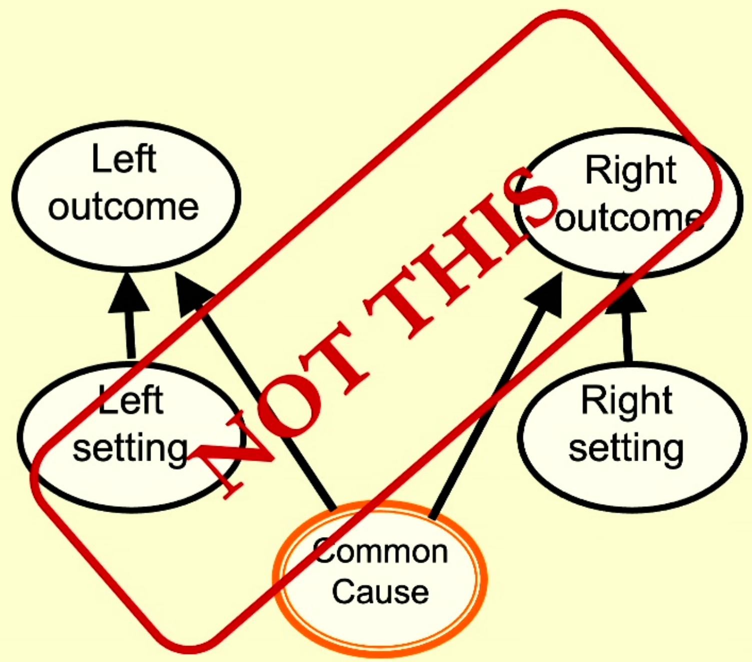
Violates Bell inequalities
(up to Tsirelson bound)

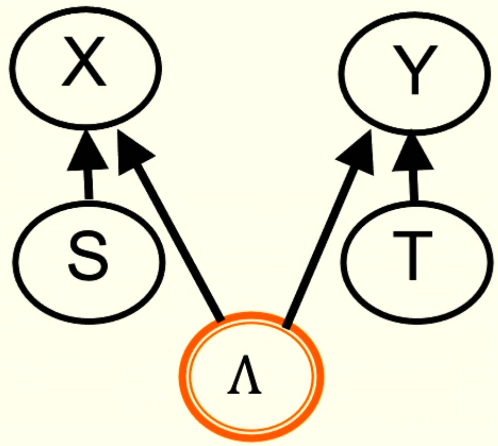
The natural hypothesis



Implies Bell inequalities

Incompatible

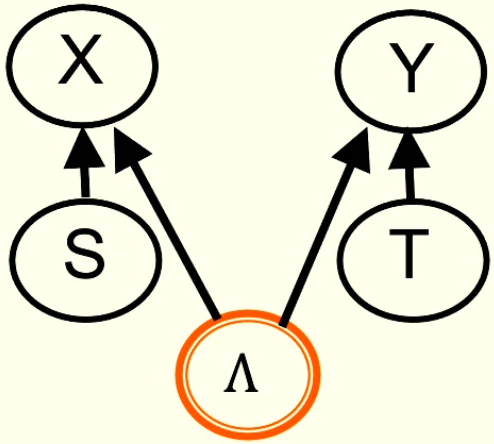
Causal structure Parameters



$P_{X|S\Lambda}$
$P_{Y|T\Lambda}$
P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{Y|T\Lambda} P_{X|S\Lambda} P_{\Lambda}$$

Causal structure Parameters



$P_{X|S\Lambda}$
$P_{Y|T\Lambda}$
P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{Y|T\Lambda} P_{X|S\Lambda} P_{\Lambda}$$

Examples of causal compatiblity constraints:

$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

Clauser, Horne, Shimony and Holte, Phys. Rev. Lett.23, 880 (1967)

The evidence

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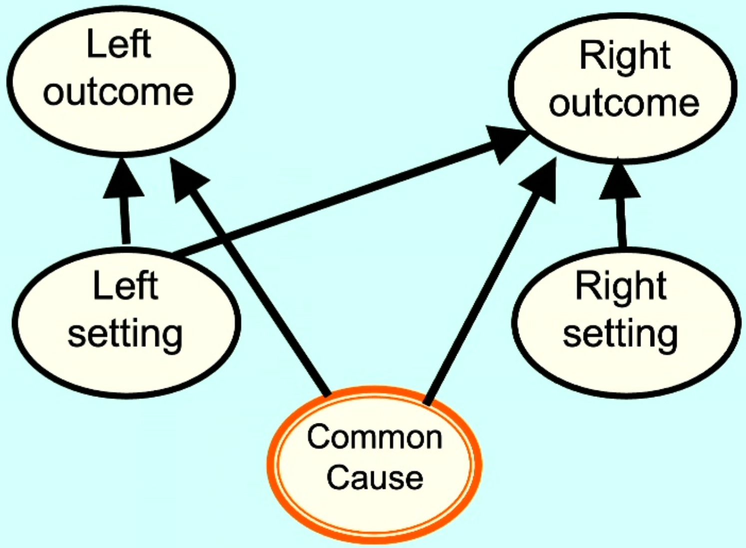
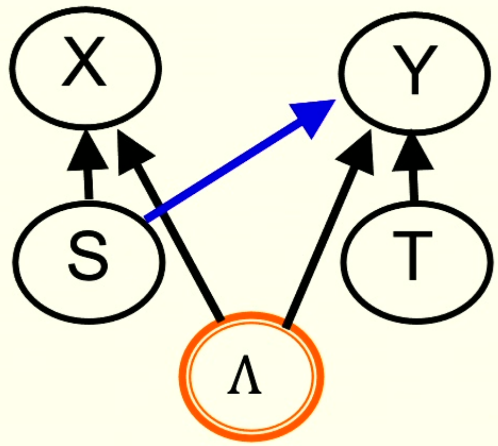
1 and 1	7%	43%	43%	7%

The 2nd possibility



Causal structure



Parameters

$P_{X|S\Lambda}$

$P_{Y|ST\Lambda}$

P_{Λ}

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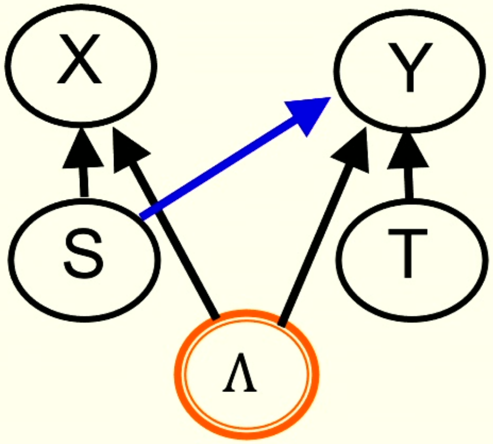
Causal structure



Parameters

$P_{X|S\Lambda}$

$P_{Y|ST\Lambda}$

P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{Y|ST\Lambda} P_{X|S\Lambda} P_{\Lambda}$$

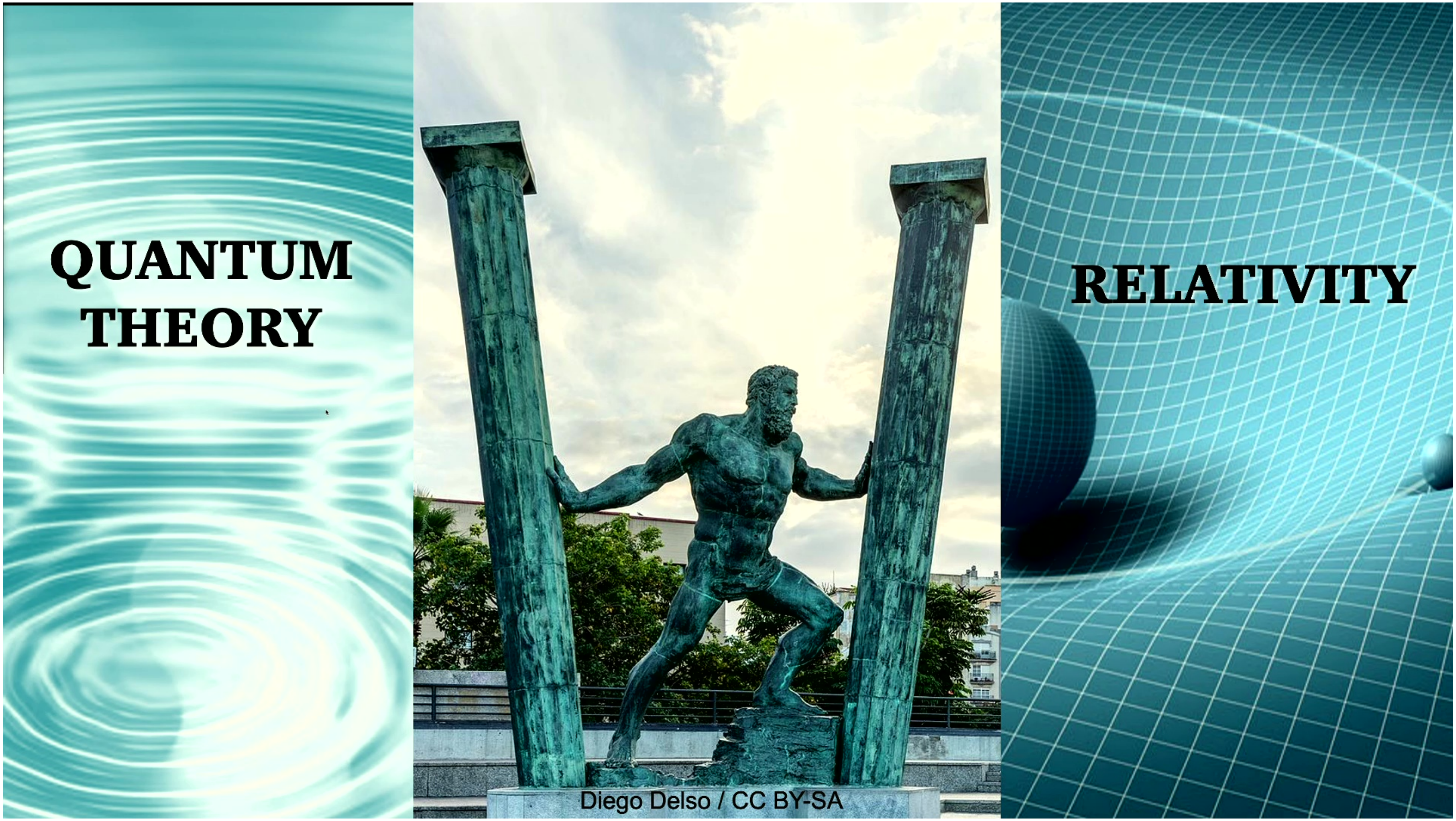
Causal compatibility constraints:

$P_{X|ST} = P_{X|S}$

But the data *also* satisfies $P_{Y|ST} = P_{Y|T}$

Reproducing this requires fine-tuning

Wood and RWS, New J. Phys. 17, 033002 (2015)


QUANTUM
THEORY
RELATIVITY
Diego Delso / CC BY-SA

The evidence

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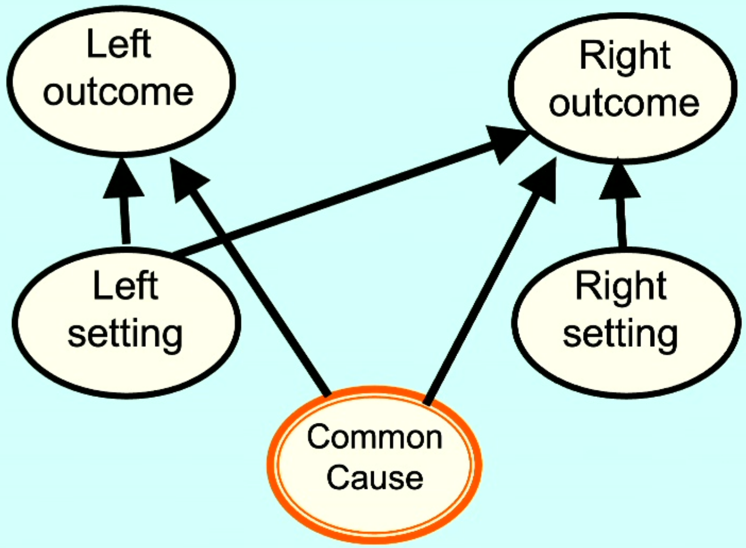

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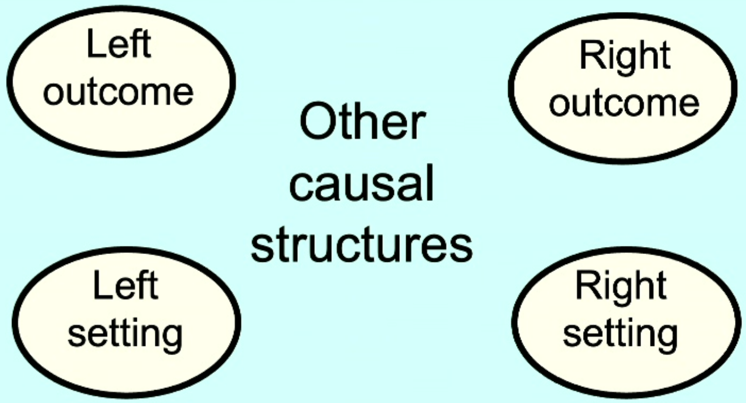
The usual suspects



Wood and RWS, New J. Phys. 17, 033002 (2015)

The evidence

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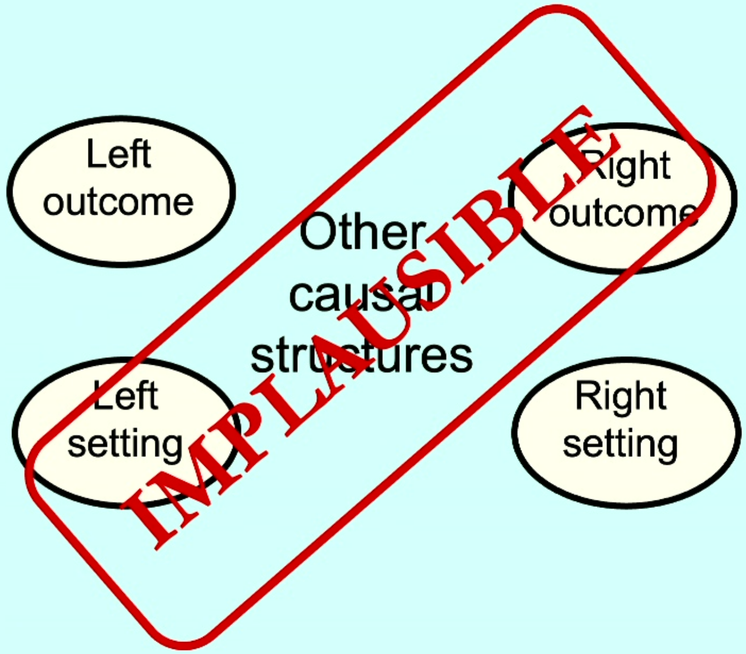

Wood and RWS, New J. Phys. 17, 033002 (2015)

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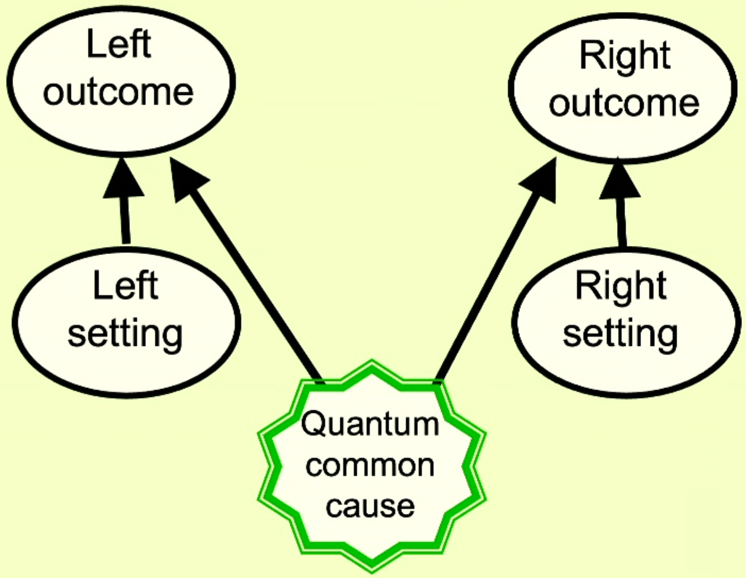
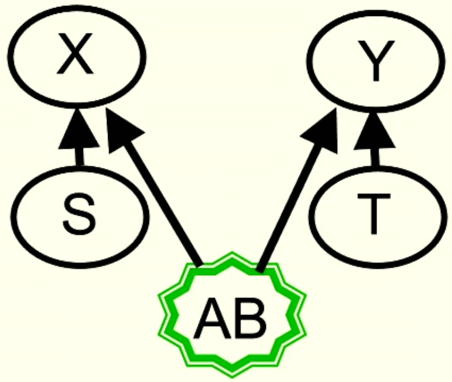
A new possibility



Quantum causal model



$\rho_{X|S\Lambda}$

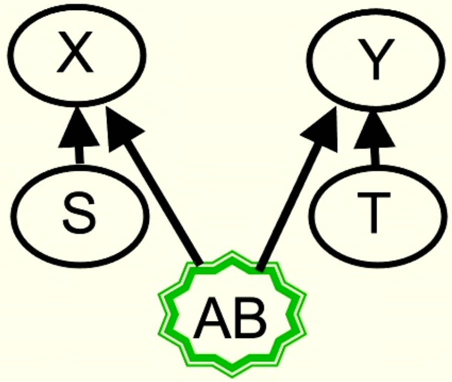
$\rho_{Y|T\Lambda}$

ρ_{Λ}

$$[\rho_{X|S\Lambda}, \rho_{Y|T\Lambda}] = 0$$

$$P_{XY|ST} = \mathrm{Tr}_{\Lambda}(\rho_{X|S\Lambda}\rho_{Y|T\Lambda}\rho_{\Lambda})$$

Quantum causal model



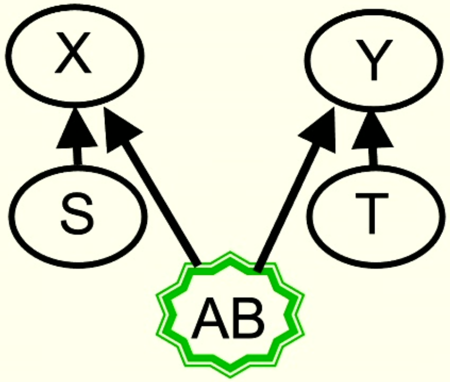
$\rho_{X|SA}$

$\rho_{Y|TB}$

ρ_{AB}

$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

Quantum causal model



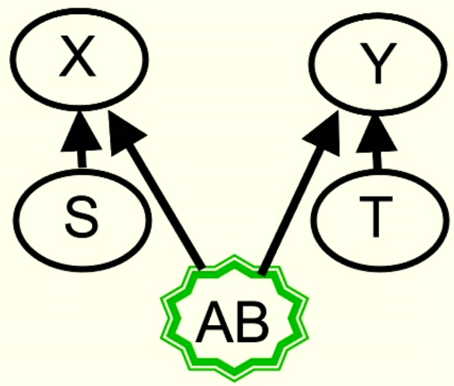
$\{E^A_{x|s}\}_x$ for each s

$\{E^B_{y|t}\}_y$ for each t

ρ_{AB}

$$P_{XY|ST}(xy|st) = \mathrm{Tr}_{AB}((E^A_{x|s} \otimes E^B_{y|t})\rho_{AB})$$

Quantum causal model



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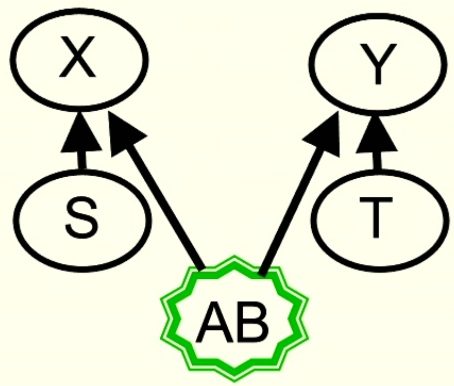
Quantum causal model



$\{E^A_{x|s}\}_x$ for each s

$\{E^B_{y|t}\}_y$ for each t

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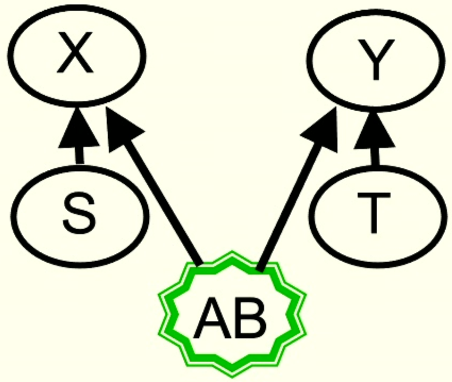
$$P_{XY|ST}(xy|st) = \mathrm{Tr}_{AB}((E^A_{x|s} \otimes E^B_{y|t})\rho_{AB})$$

Leifer and RWS, PRA 88, 052130 (2013)
Costa, Shrapnel NJP 18(6) (2016)
Allen, Barrett, Horsman, Lee & RWS, PRX 7, 031021 (2017)
Barrett, Lorenz, Oreshkov, arXiv:1906.10726

Quantum causal model



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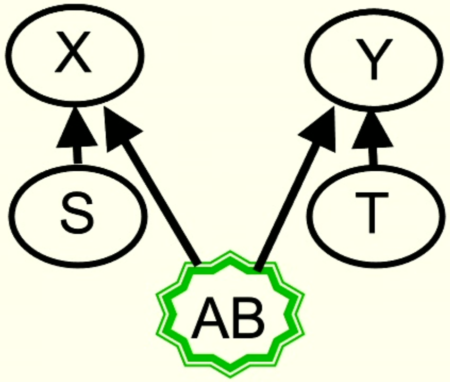
Quantum causal model



$\rho_{X|SA}$
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$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

Causal compatibility constraints:

$$P_{X|ST} = P_{X|S}$$
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$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

Tsirelson, Lett. Math. Phys. 4, 93 (1980)

The evidence

A new possibility

Left outcome and Right outcome

Left setting and Right setting	0 and 0	0 and 1	1 and 0	1 and 1
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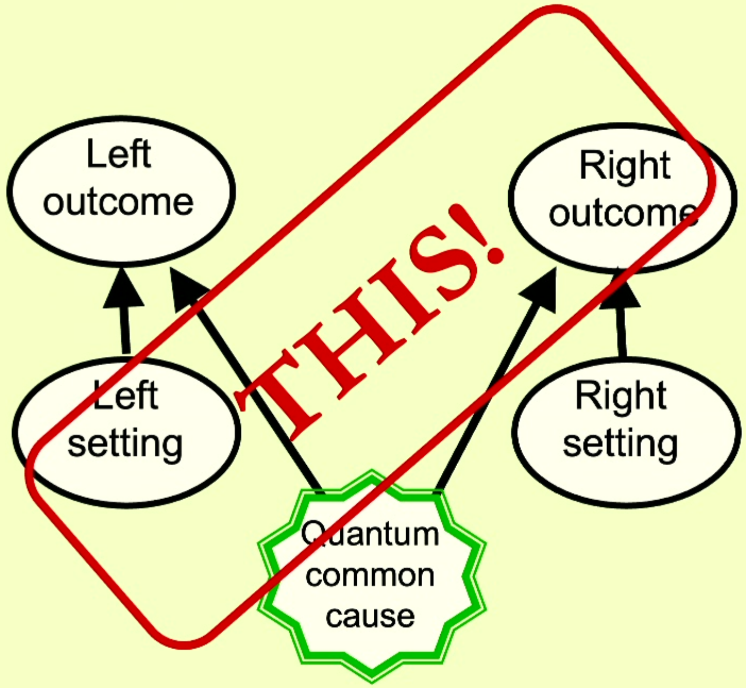
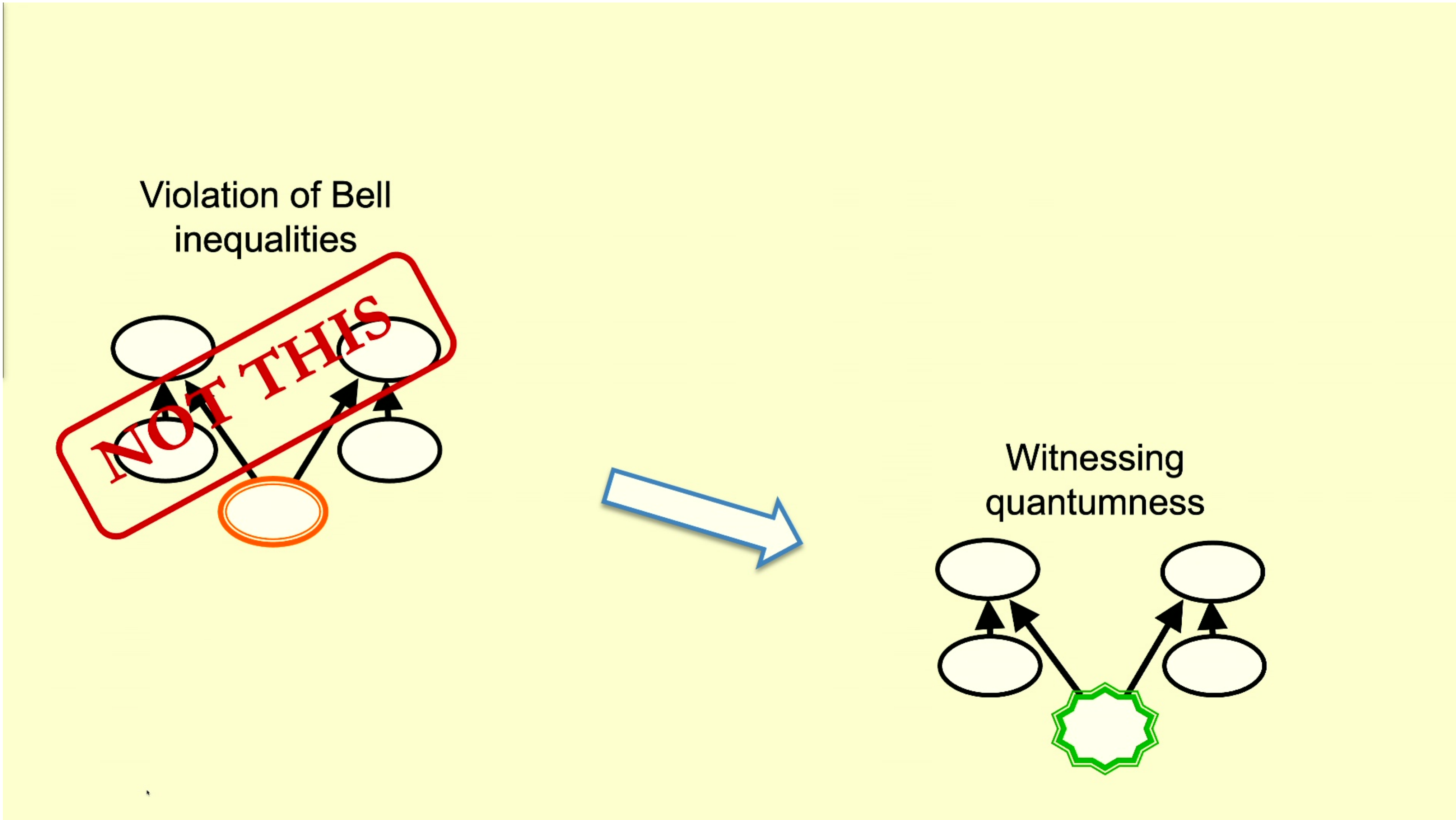
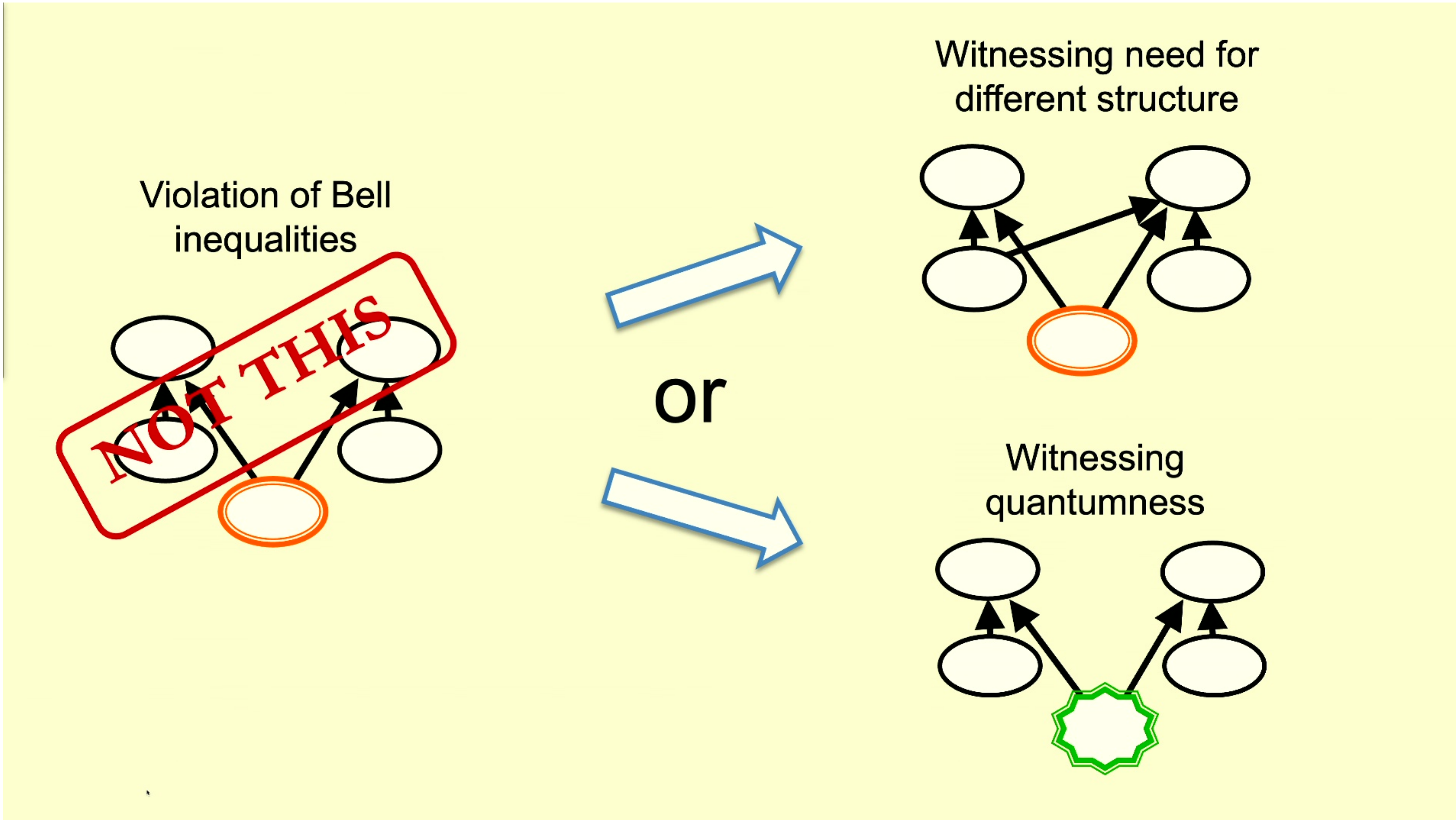


Compatible

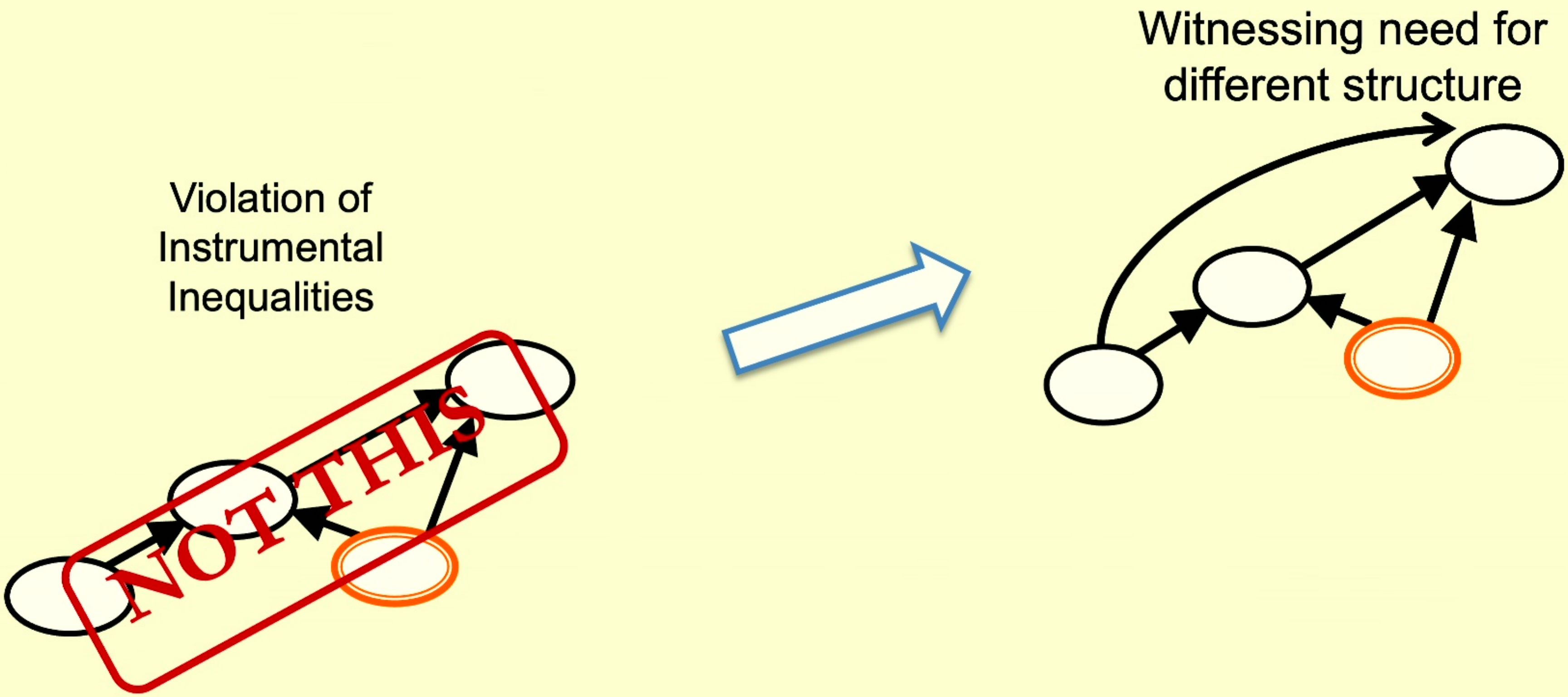
Violation of Bell inequalities

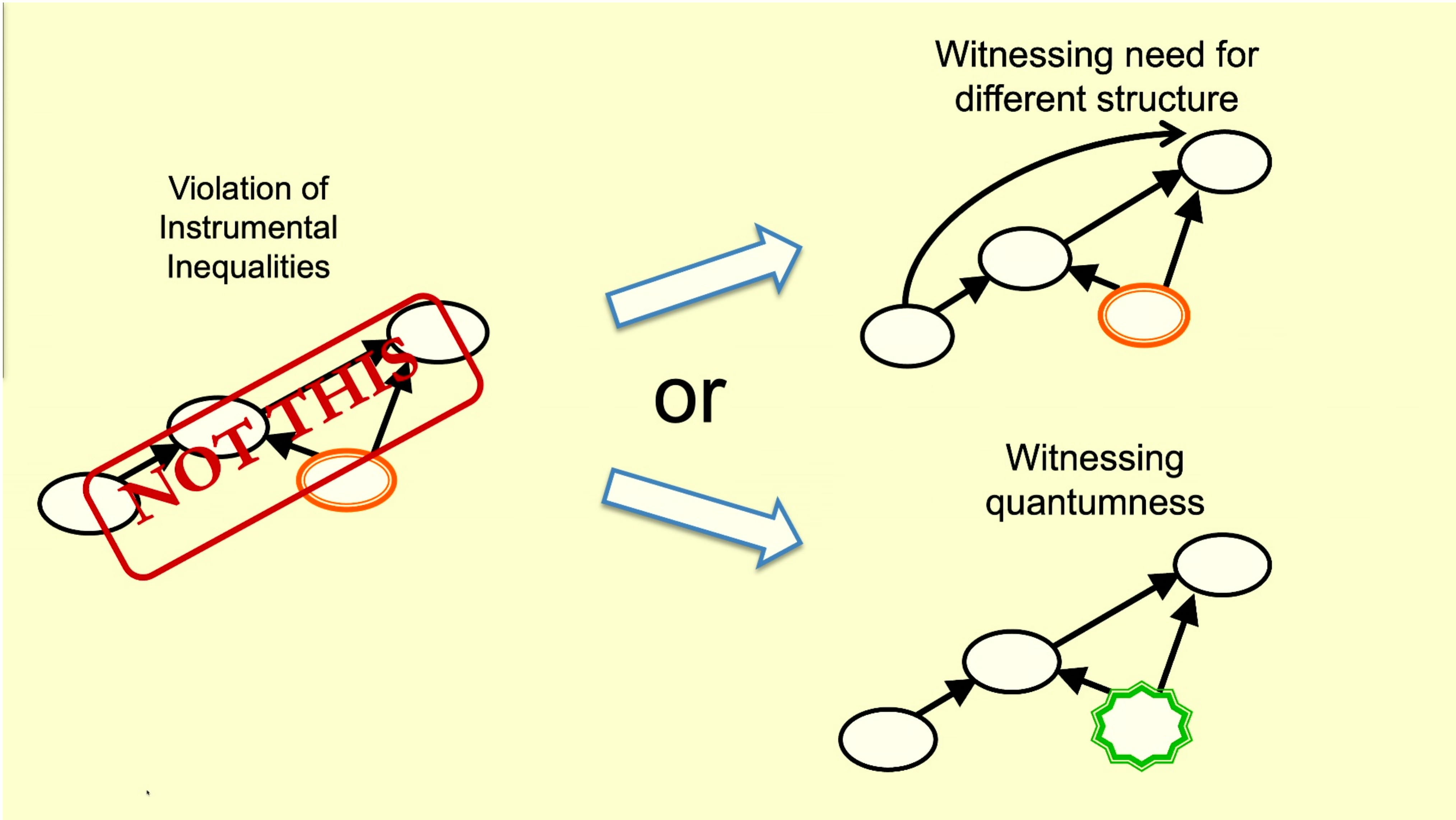


Violation of Bell inequalities
NOT THIS
Witnessing quantumness



Violation of Bell inequalities
NOT THIS
or
Witnessing need for different structure
Witnessing quantumness


Violation of
Instrumental
Inequalities
NOT THIS
Witnessing need for
different structure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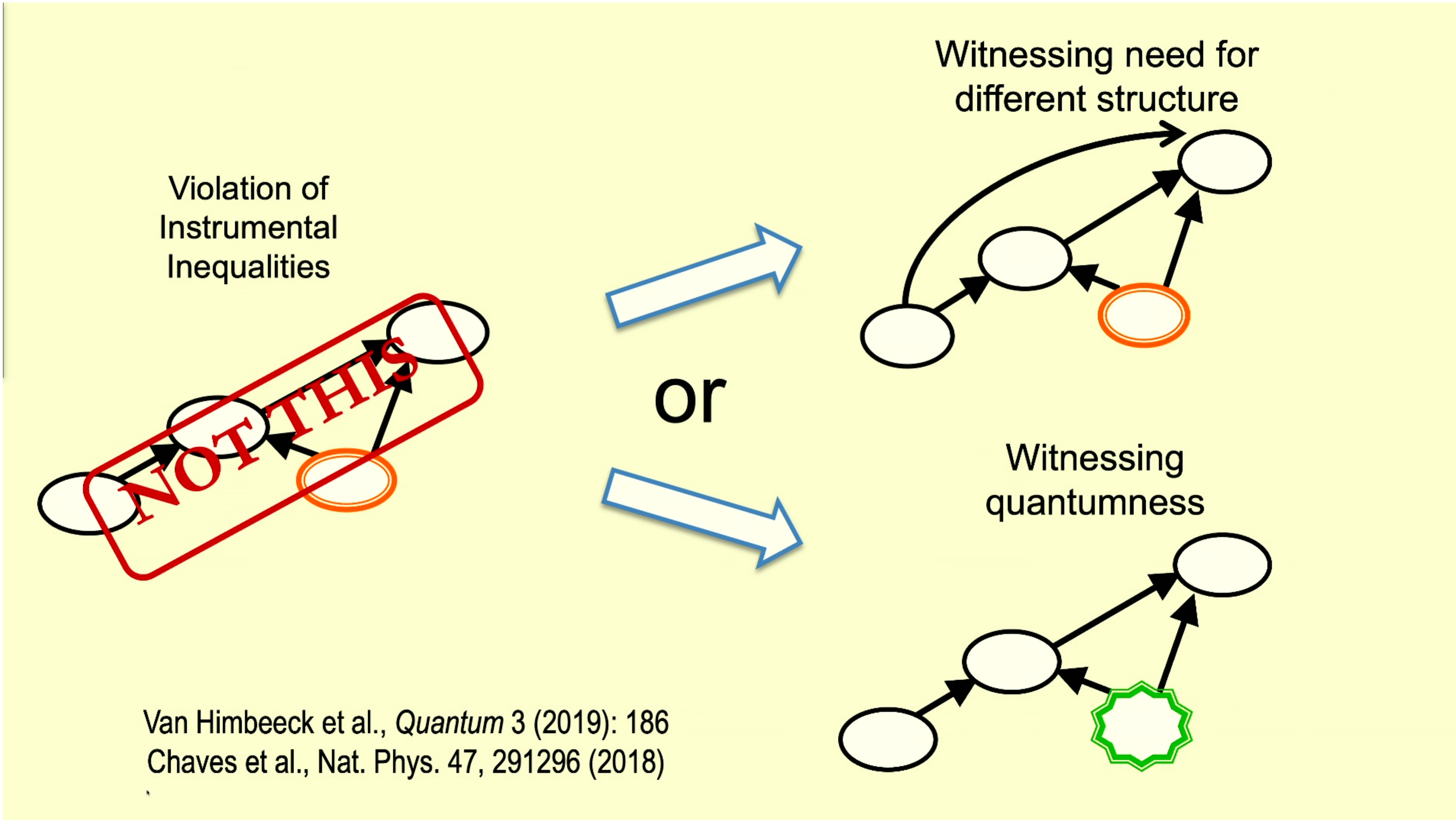
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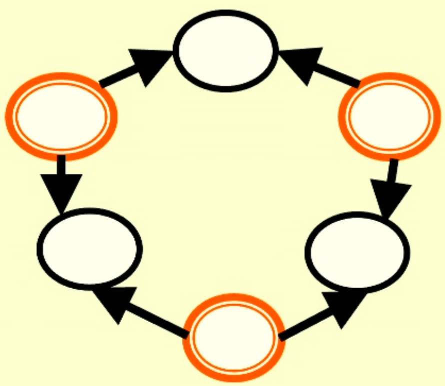
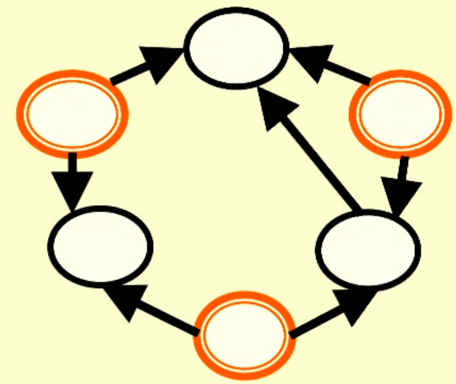
Van Himbeeck et al., *Quantum* 3 (2019): 186
Chaves et al., Nat. Phys. 47, 291296 (2018)



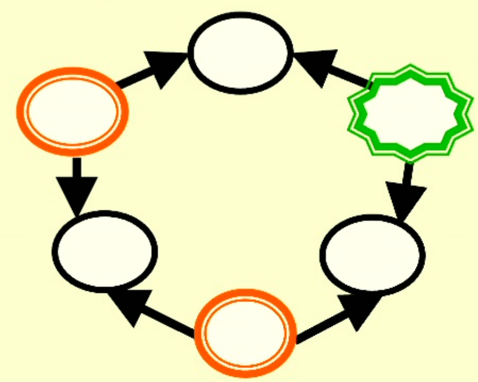
Violation of certain causal compatibility inequalities

or

Witnessing need for different structure

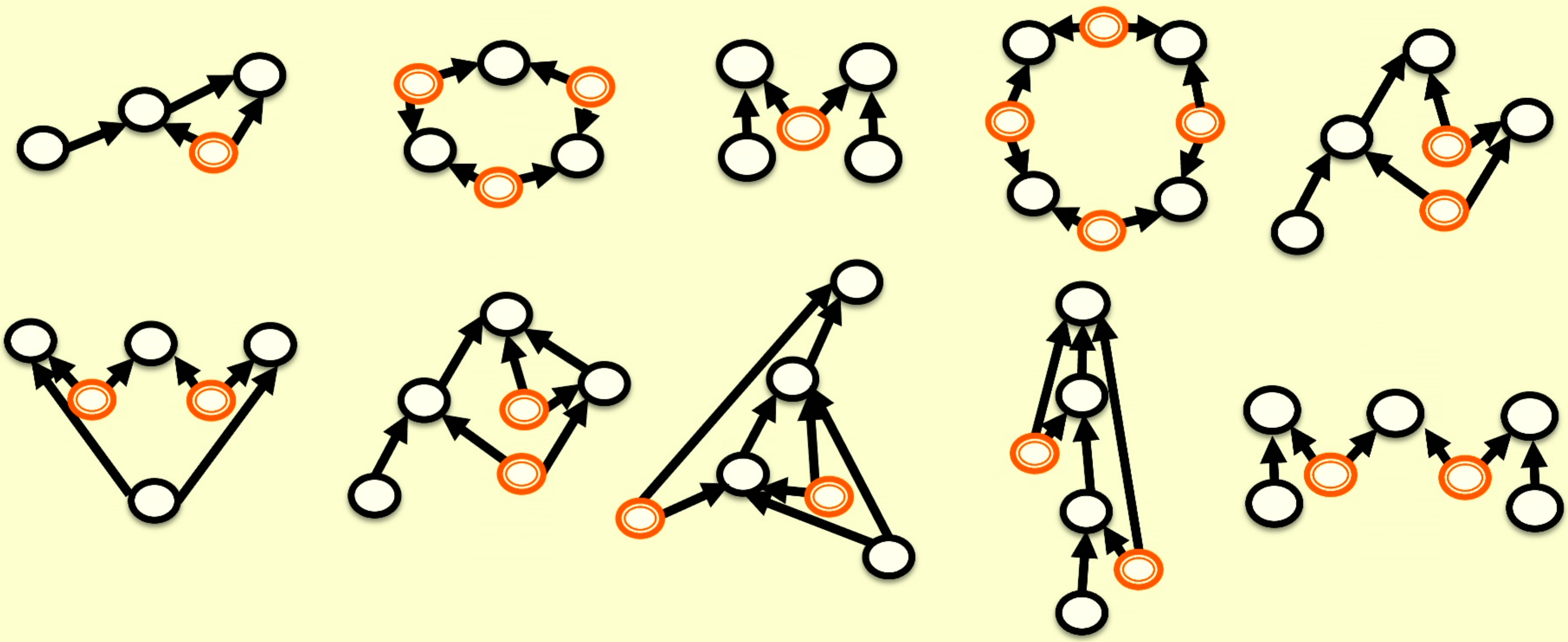


Witnessing quantumness



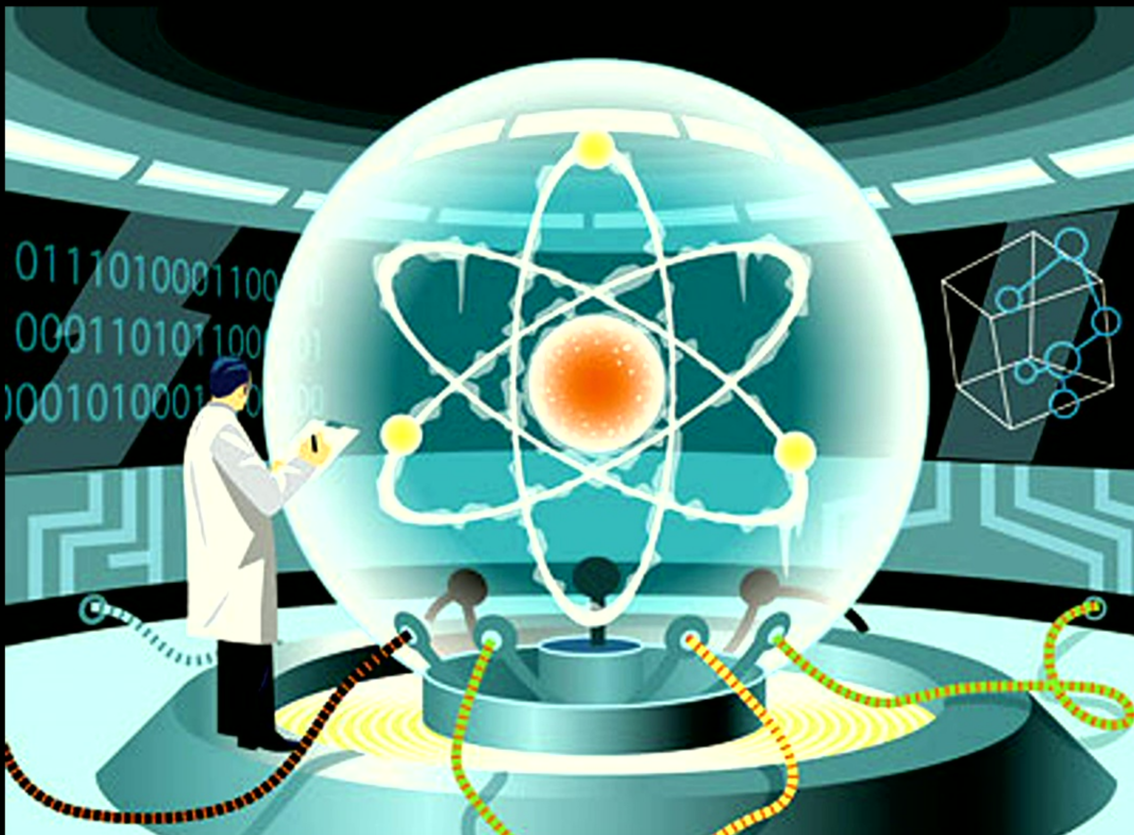
Fritz, New J. Phys. 14, 103001 (2012)
Polino et al, Nature Comm. 14, online (2023)

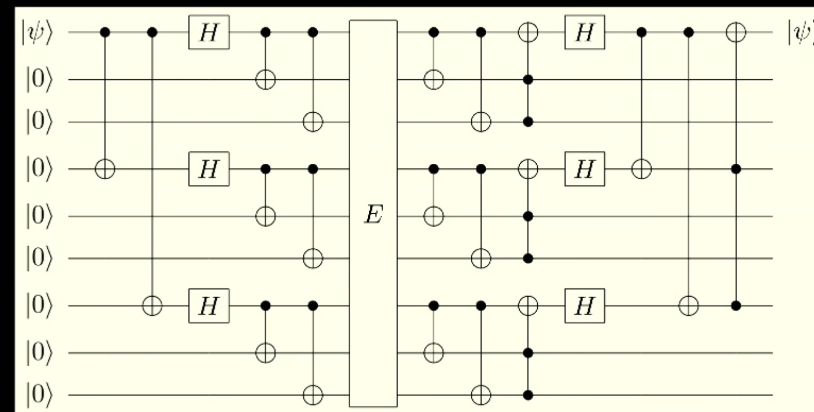
Some causal structures that admit of quantum-classical gaps:



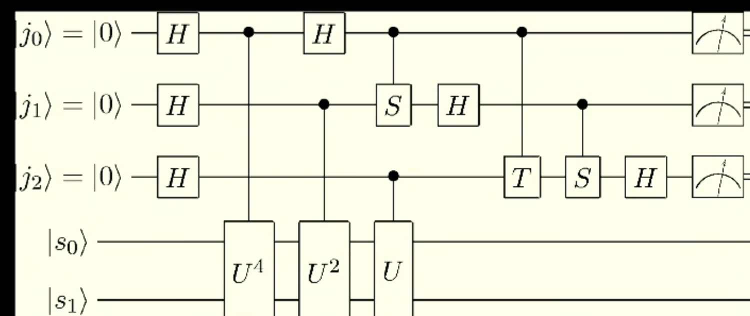
It is likely that such gaps are generic!

Applications for Quantum Technology

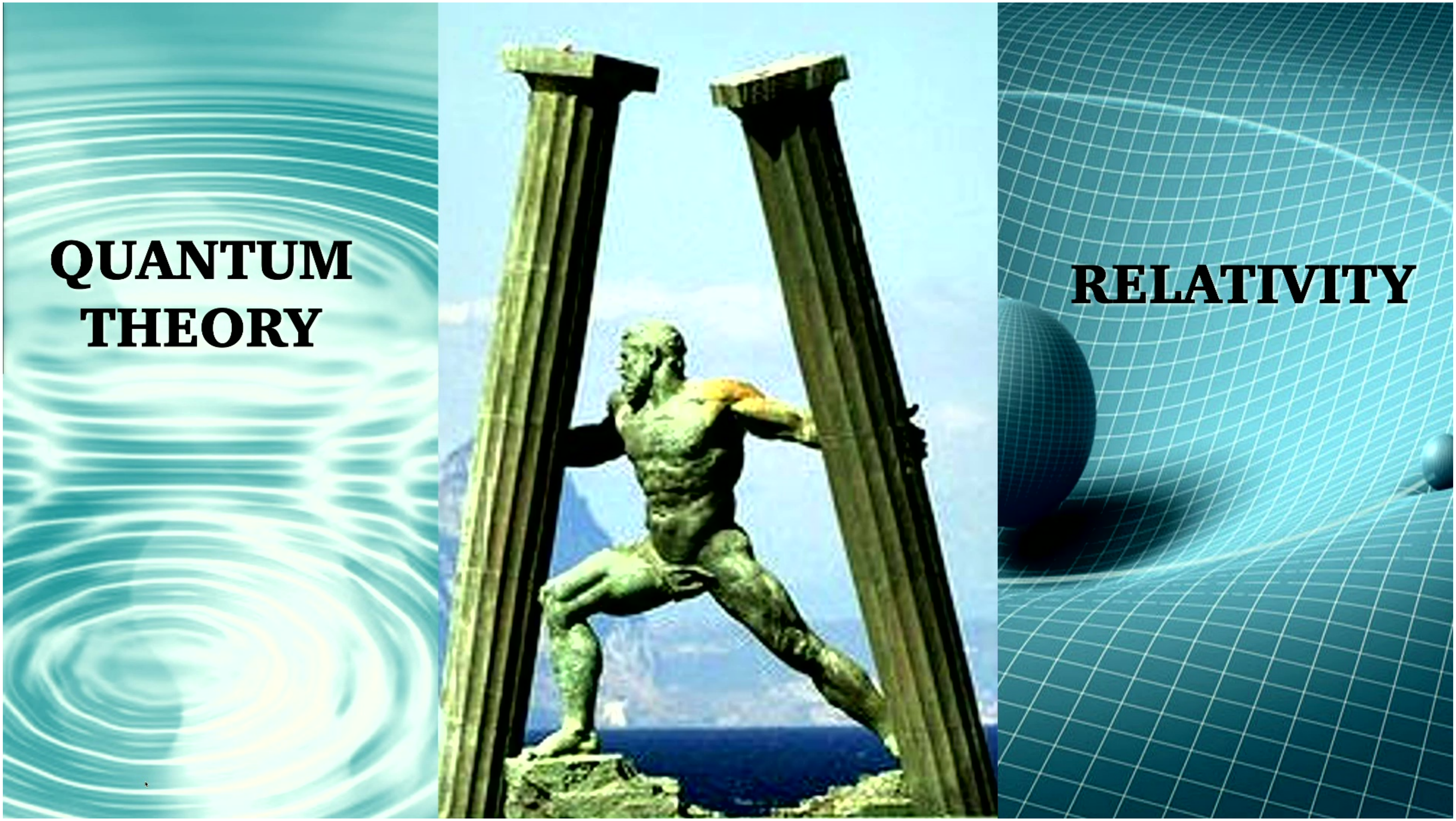




|ψ⟩
|0⟩
H
E



|j0⟩ = |0⟩
|j1⟩ = |0⟩
|j2⟩ = |0⟩
|s0⟩
|s1⟩
H
S
T
U^4
U^2
U


QUANTUM
THEORY
RELATIVITY

Quantum
Causation and Inference

Classical
Causation and Inference

≈

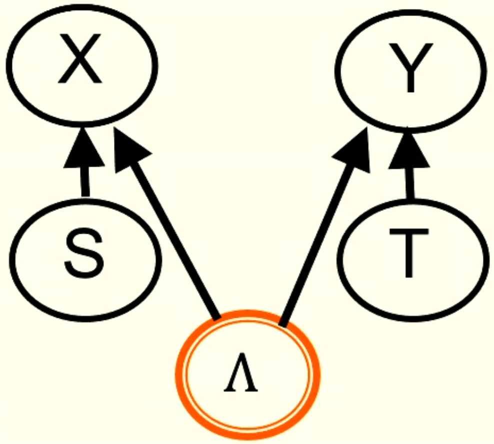
Relativistic
Notions of Space
and Time

PreRelativistic
Notions of Space
and Time

Deriving causal compatibility inequalities

Strategy 1: implicitization of parameters referring to hidden variables

Bell scenario



$P_{X|S\Lambda}$

$P_{Y|T\Lambda}$

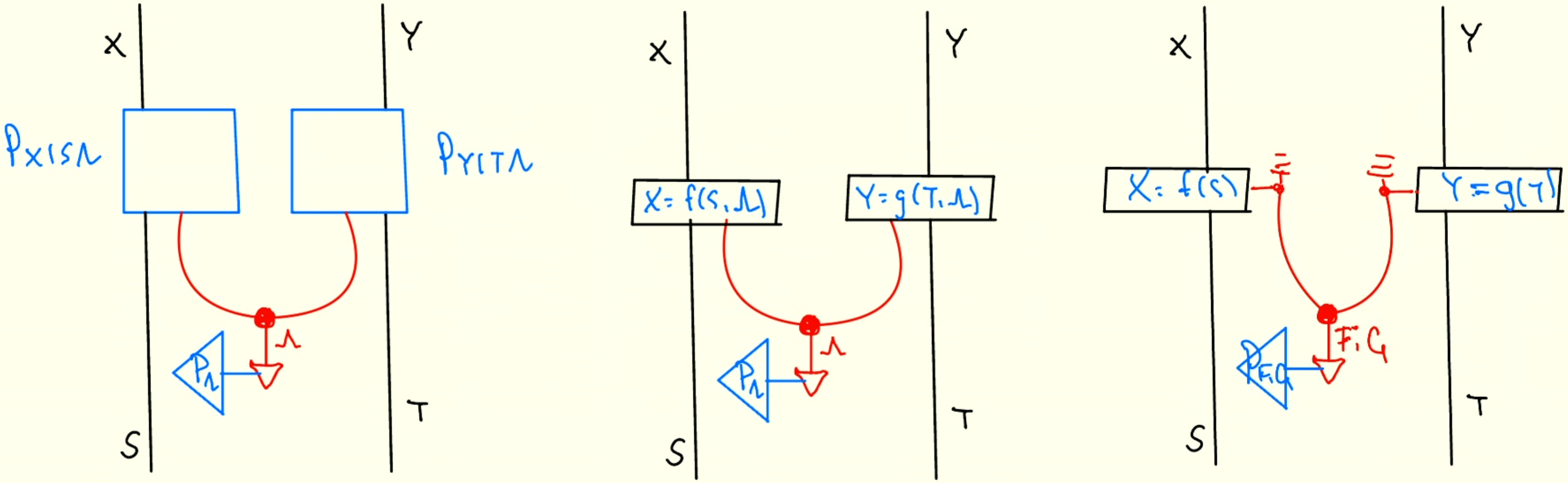
P_{Λ}

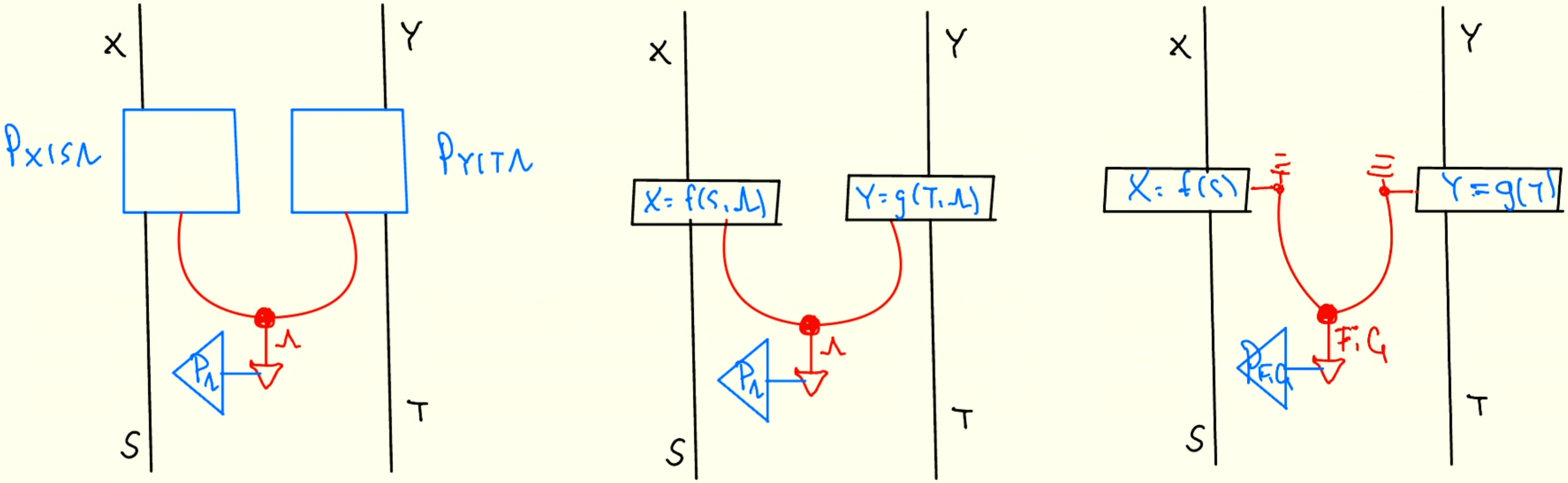
P_S

P_T

$$P_{XY|ST} = \sum_{\Lambda} P_{Y|T\Lambda} P_{X|S\Lambda} P_{\Lambda}$$

One needs to bound the cardinality of Λ


X
Y
P_{X|SΛ}
P_{Y|TΛ}
Λ
P_Λ
S
T
X
Y
X= f(S,Λ)
Y= g(T,Λ)
Λ
P_Λ
S
T
X
Y
X= f(S)
Y= g(T)
F,G
P_{F,G}
S
T



Λ can be taken to range over choices of f and g

$$P_{XY|ST} = \sum_{f,g} \delta_{X,f(S)} \delta_{Y,g(T)} P_{F,G}(f,g)$$

$$P_{XY|ST} = \sum_{f,g} \delta_{X,f(S)} \delta_{Y,g(T)} P_{F,G}(f,g)$$

If X,Y,S,T are binary, Λ can have cardinality 16

$$p_{xy|st} := P_{XY|ST}(xy|st) \qquad q_{fg} := P_{F,G}(f,g)$$

$$x, y, s, t \in \{0,1\} \qquad f, g \in \{\mathrm{id}, \mathrm{fp}, \mathrm{r}_0, \mathrm{r}_1\}$$

$$p_{00|00} = q_{\mathrm{r}_0,\mathrm{r}_0} + q_{\mathrm{r}_0,\mathrm{id}} + q_{\mathrm{id},\mathrm{r}_0} + q_{\mathrm{id},\mathrm{id}}$$

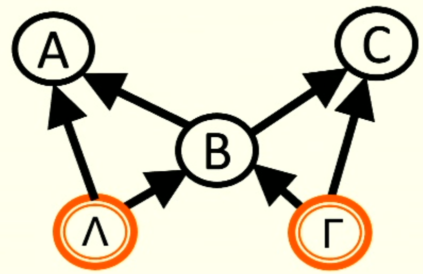
$$p_{00|01} = q_{\mathrm{r}_0,\mathrm{r}_1} + q_{\mathrm{r}_0,\mathrm{fp}} + q_{\mathrm{id},\mathrm{r}_1} + q_{\mathrm{id},\mathrm{fp}}$$

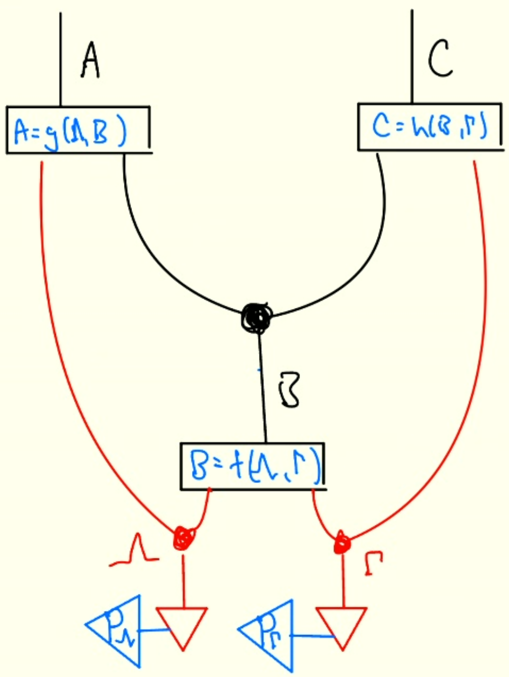
$$0 \leq q_{fg} \leq 1 \; \forall f, g$$

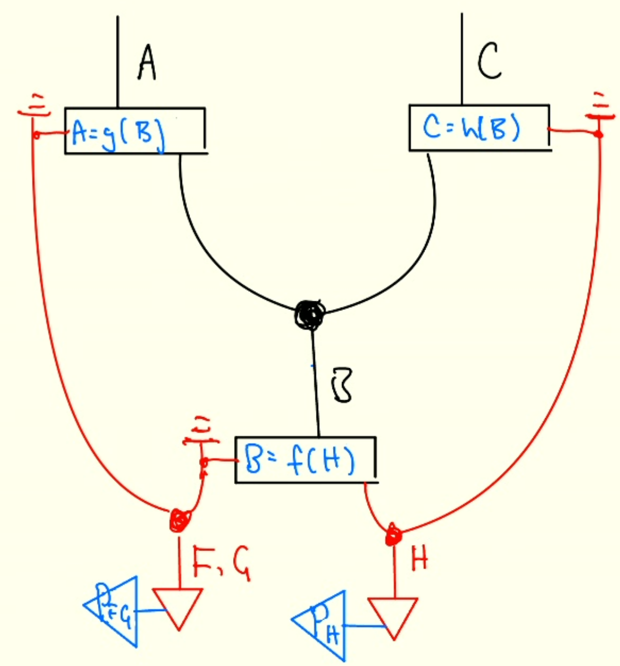
$$\vdots$$

16 linear equalities + inequalities

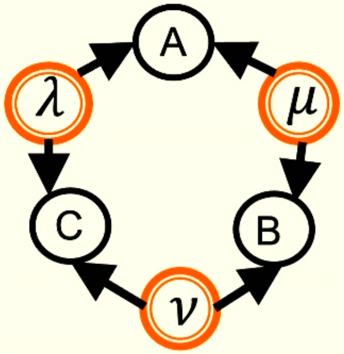
Do linear quantifier elimination on the 16 q's.


A
C
B
Λ
Γ


A
C
A=g(Λ,B)
C=h(B,Γ)
B
B=f(Λ,Γ)
Λ
Γ
$P_Λ$
$P_Γ$


A
C
A=g(B)
C=h(B)
B
B=f(H)
F, G
H
P_{FG}
P_H

DAGs for which one can deduce the cardinalities of latent variables in this manner are termed **gearable** (R. Evans, Annals of Statistics, **46**, 2623 (2018))



Triangle

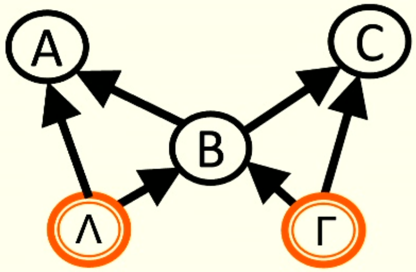
The triangle scenario is not gearable

Nonetheless, there are other techniques for determining sufficient cardinalities of the latent variables

See: D. Rosset, N. Gisin, and E. Wolfe. Quantum Inf. & Comp. **18,** 0910 (2018)

E.g. If A, B, C are binary, then it is sufficient if the the latent variables are 6-valued

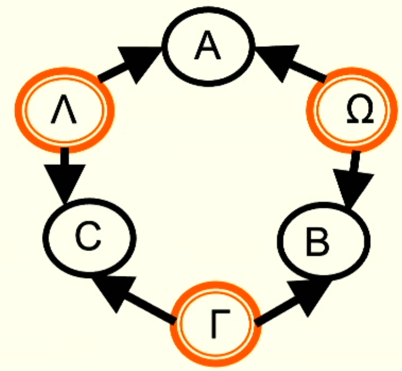
Evans' scenario



$\mathrm{P}_{A|B\Lambda}$
$\mathrm{P}_{C|B\Gamma}$
$\mathrm{P}_{B|\Lambda\Gamma}$
P_{Λ}
P_{Γ}

$$P_{ABC} = \sum_{\Lambda\Gamma} \mathrm{P}_{A|B\Lambda}\mathrm{P}_{C|B\Gamma}\mathrm{P}_{B|\Lambda\Gamma}\mathrm{P}_{\Lambda}\mathrm{P}_{\Gamma}$$

Triangle scenario



$\mathrm{P}_{A|\Lambda\Omega}$
$\mathrm{P}_{B|\Omega\Gamma}$
$\mathrm{P}_{C|\Gamma\Lambda}$
P_{Λ}
P_{Ω}
P_{Γ}

$$P_{ABC} = \sum_{\Lambda\Omega\Gamma} \mathrm{P}_{A|\Lambda\Omega}\mathrm{P}_{B|\Omega\Gamma}\mathrm{P}_{C|\Gamma\Lambda}\mathrm{P}_{\Lambda}\mathrm{P}_{\Omega}\mathrm{P}_{\Gamma}$$

With more than one latent variable,
we require **nonlinear** quantifier elimination
which scales badly

Entropy cone technique

R. Chaves and T. Fritz, Phys. Rev. A 85 (2012)
T. Fritz, New J. Phys. 14 103001 (2012)
R. Chaves, L. Luft, D. Gross, New J. Phys. 16, 043001 (2014)
R. Chaves, L. Luft, T. O. Maciel, D. Gross, D. Janzing, B. Schölkopf, Proceedings of UAI 2014
M. Weilenmann and R. Colbeck,Proc. Roy. Soc. A 473.2207 (2017): 20170483.

Entropy vector

For the joint distribution of the random variables $X_1, \dots X_n$, the components of the entropy vector are the entropies of the marginals for all possible subsets of variables:

$$(H(X_1), H(X_2), \dots, H(X_n), H(X_1 X_2), H(X_1 X_3), \dots, H(X_1, X_2, \cdots, X_n))$$

Shannon entropy

$$H(X) := -\sum_x P_X(x) \log P_X(x)$$

Entropy cone

The closure of this set of vectors is called the
entropy cone

It is a convex cone
therefore characterized by **linear** inequalities

An outer approximation to the entropy cone is the Shannon cone

Monotonicity

$$H(\mathbf{X}A) \geq H(\mathbf{X})$$

for every variable A and set of variables **X**

Submodularity

$$H(\mathbf{X}) + H(\mathbf{X}AB) \leq H(\mathbf{X}A) + H(\mathbf{X}B)$$

where A and B are variables not in the set **X**

Inequalities describing the Shannon cone are termed **Shannon-type**

Valid inequalities for the entropy cone that are not Shannon-type are termed **non-Shannon-type** (R. Yeung, IEEE Trans. Inf. Th., 43, 1997)

Example: for distributions on X, Y, Z, the linear equalities defining the Shannon cone are

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} H(X) \\ H(Y) \\ H(Z) \\ H(XY) \\ H(XZ) \\ H(YZ) \\ H(XYZ) \end{pmatrix} \geq 0$$

Weilenmann and Colbeck,Proc. Roy. Soc. A 473.2207 (2017): 20170483.

The entropic technique for deriving inequality constraints:

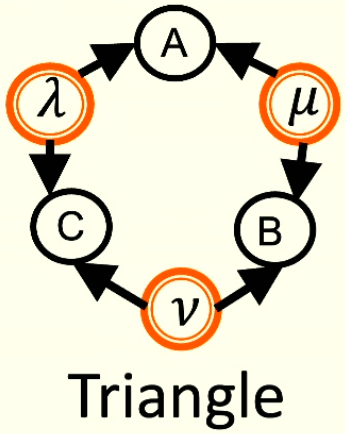
The set of all constraints on observed and latent variables are the conditional independence relations among these

These imply linear equalities on the components of the entropy vector for observed and latent variables

Add the linear inequalities of Shannon-type

Implicitize all entropic quantities that refer to latent variables

The result is the **marginal Shannon cone,** described by linear inequalities on entropies over observed variables only



Entropic constraint for the triangle scenario

T. Fritz, 2012

$$A \perp B|\mu \quad \lambda \perp \mu\nu$$
$$A \perp C|\lambda \quad \mu \perp \lambda\nu$$
$$B \perp C|\nu \quad \nu \perp \lambda\mu$$

$$A \perp B|\mu \implies I(A:B|\mu) = 0 \implies I(A:B) \leq I(A:\mu)$$
$$A \perp C|\lambda \implies I(A:C|\lambda) = 0 \implies I(A:C) \leq I(A:\lambda)$$

By Shannon-type inequalities

$$I(A:B) + I(A:C) \leq I(A:\mu) + I(A:\lambda)$$
$$\leq H(A) + I(\mu:\lambda)$$

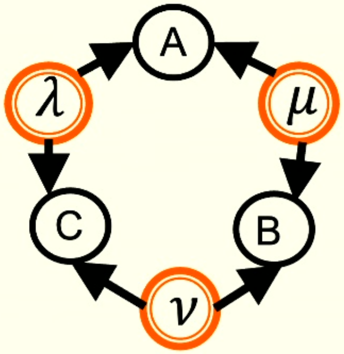
By Shannon-type inequalities

$$\mu \perp \lambda \implies I(\mu:\lambda) = 0$$

$$I(A:B) + I(A:C) \leq H(A)$$

Entropic constraint for the triangle scenario

T. Fritz, 2012



Triangle

$$I(A:B) + I(A:C) \leq H(A)$$

However, the constraints obtained by entropy cone techniques generally do not succeed at distinguishing quantum and classical models

Strategy 2: Implicitization of parameters referring to counterfactual possibilities

Example 1:

$$P_{XY}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$

Question: $\exists P_{XYZ}$

such that

$$P_{XY} = P_{XY}^{\text{target}}$$
$$P_{YZ} = P_{YZ}^{\text{target}}$$
$$P_{XZ} = P_{XZ}^{\text{target}}$$

where

$$P_{XY} := \sum_Z P_{XYZ}$$
$$P_{YZ} := \sum_X P_{XYZ}$$
$$P_{XZ} := \sum_Y P_{XYZ}$$

?

Example 1:

$$P_{XY}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$

Question: $\exists P_{XYZ}$

such that
$$P_{XY} = P_{XY}^{\text{target}}$$
$$P_{YZ} = P_{YZ}^{\text{target}}$$
$$P_{XZ} = P_{XZ}^{\text{target}}$$

where
$$P_{XY} := \sum_Z P_{XYZ}$$
$$P_{YZ} := \sum_X P_{XYZ}$$
$$P_{XZ} := \sum_Y P_{XYZ}$$

?

Answer: yes!

$$P_{XYZ} = \tfrac{1}{2}[001] + \tfrac{1}{2}[110]$$

Example 2:

$$P_{XY}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11] = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$

Question: $\exists P_{XYZ}$

such that $P_{XY} = P_{XY}^{\text{target}}$, $P_{YZ} = P_{YZ}^{\text{target}}$, $P_{XZ} = P_{XZ}^{\text{target}}$

where $P_{XY} := \sum_Z P_{XYZ}$, $P_{YZ} := \sum_X P_{XYZ}$, $P_{XZ} := \sum_Y P_{XYZ}$?

Answer: no!

consider $[000], [001], [010], [011], [100], [101], [110], [111]$

Consider binary X, Y and Z

$$\exists P_{XYZ} \text{ with } \quad P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ} \text{ as marginals}$$

$$\Downarrow$$

Marginal compatibility constraints

$$P_X = \sum_Y P_{XY} = \sum_Z P_{XZ}$$
$$P_Y = \sum_X P_{XY} = \sum_Z P_{YZ}$$
$$P_Z = \sum_X P_{XZ} = \sum_Y P_{YZ}$$
$$P_X + P_Y + P_Z - P_{XY} - P_{YZ} - P_{XZ} \leq 1$$

How this is proven:

$$\exists Q_{XYZ} \text{ with } \quad P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ} \text{ as marginals}$$

$$\forall x,y : P_{XY}(xy) = \sum_z Q_{XYZ}(xyz)$$
$$\forall y,z : P_{YZ}(yz) = \sum_x Q_{XYZ}(xyz)$$
$$\forall x,z : P_{XZ}(xz) = \sum_y Q_{XYZ}(xyz)$$
$$\forall x : P_X(x) = \sum_{y,z} Q_{XYZ}(xyz)$$
$$\forall y : P_Y(y) = \sum_{x,z} Q_{XYZ}(xyz)$$
$$\forall z : P_Z(z) = \sum_{x,y} Q_{XYZ}(xyz)$$
$$\forall x,y,z : 0 \leq Q_{XYZ}(xyz) \leq 1$$

Linear quantifier elimination

$$\forall x : P_X(x) = \sum_y P_{XY}(xy) = \sum_z P_{XZ}(xz)$$
$$\forall y : P_Y(y) = \sum_x P_{XY}(xy) = \sum_z P_{YZ}(yz)$$
$$\forall z : P_Z(z) = \sum_x P_{XZ}(xz) = \sum_y P_{YZ}(yz)$$
$$\forall x,y,z : P_X(x) + P_Y(y) + P_Z(z) - P_{XY}(xy) - P_{YZ}(yz) - P_{XZ}(xz) \leq 1$$

How this is proven:

$$\exists Q_{XYZ} \text{ with } \quad P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ} \text{ as marginals}$$

$$\forall x,y : P_{XY}(xy) = \sum_z Q_{XYZ}(xyz)$$

$$\forall y,z : P_{YZ}(yz) = \sum_x Q_{XYZ}(xyz)$$

$$\forall x,z : P_{XZ}(xz) = \sum_y Q_{XYZ}(xyz)$$

$$\forall x : P_X(x) = \sum_{y,z} Q_{XYZ}(xyz)$$

$$\forall y : P_Y(y) = \sum_{x,z} Q_{XYZ}(xyz)$$

$$\forall z : P_Z(z) = \sum_{x,y} Q_{XYZ}(xyz)$$

$$\forall x,y,z : 0 \leq Q_{XYZ}(xyz) \leq 1$$

Linear quantifier elimination

$$\forall x : P_X(x) = \sum_y P_{XY}(xy) = \sum_z P_{XZ}(xz)$$

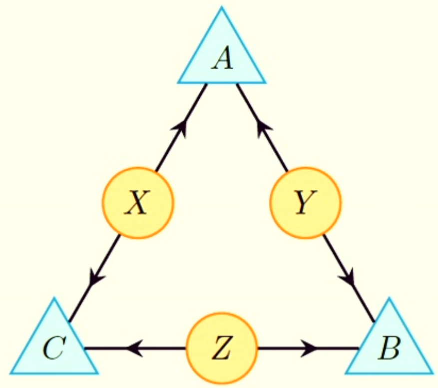
$$\forall y : P_Y(y) = \sum_x P_{XY}(xy) = \sum_z P_{YZ}(yz)$$

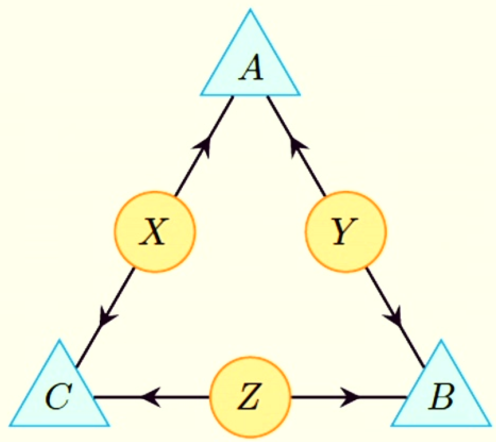
$$\forall z : P_Z(z) = \sum_x P_{XZ}(xz) = \sum_y P_{YZ}(yz)$$

$$\forall x,y,z : P_X(x) + P_Y(y) + P_Z(z) - P_{XY}(xy) - P_{YZ}(yz) - P_{XZ}(xz) \leq 1$$

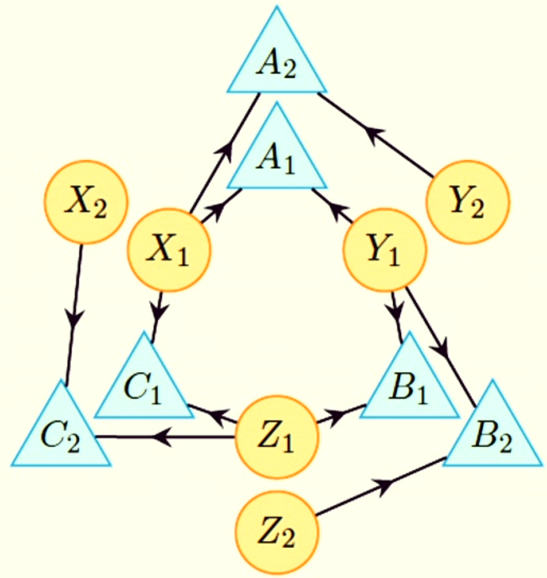
The inflation technique

Wolfe, RWS, Fritz, J. Causal Inference 7(2), (2019)

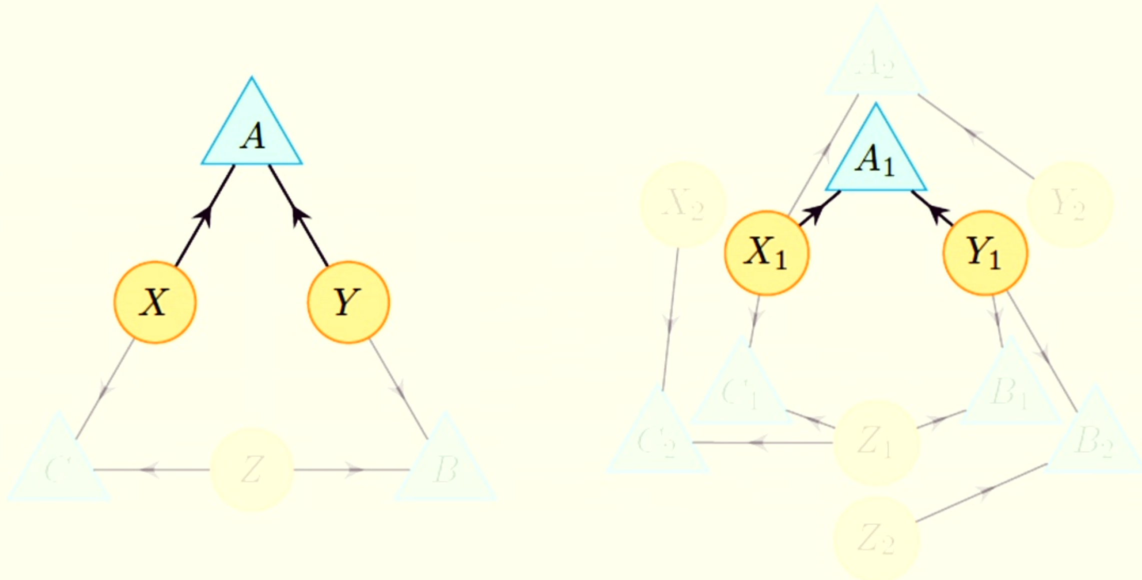

A
X
Y
C
Z
B

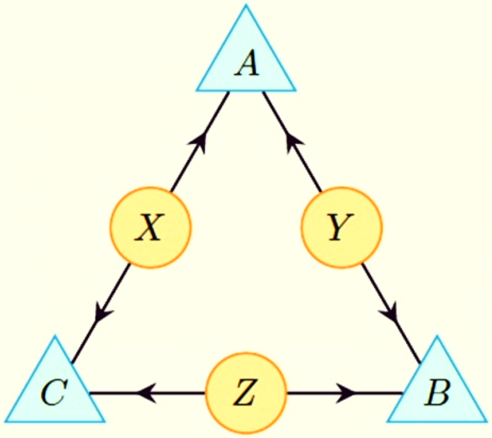


A
X
Y
C
Z
B

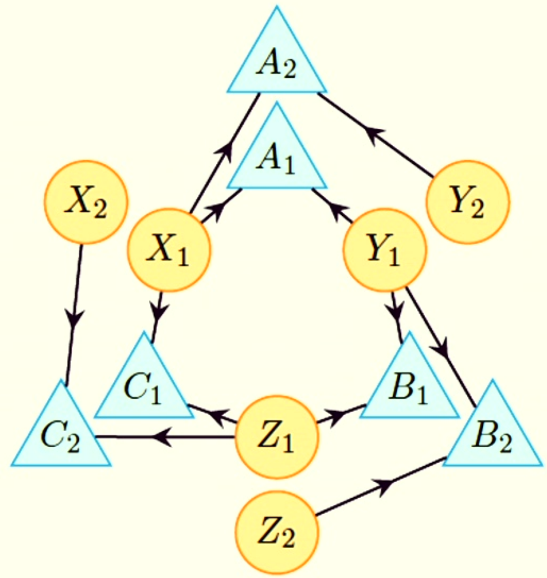


A_2
A_1
X_2
X_1
Y_1
Y_2
C_1
B_1
C_2
Z_1
B_2
Z_2


A
X
Y
C
Z
B
A_2
A_1
X_2
X_1
Y_1
Y_2
C_1
B_1
C_2
Z_1
B_2
Z_2



A
X
Y
C
Z
B

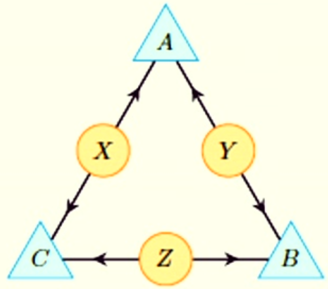


A_2
A_1
X_2
X_1
Y_1
Y_2
C_1
B_1
C_2
Z_1
B_2
Z_2

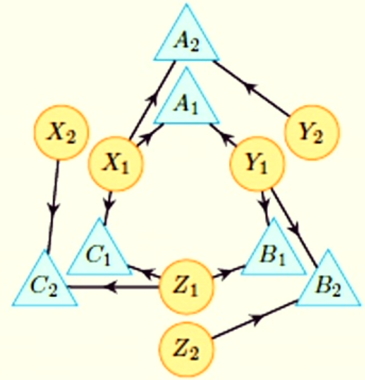
$$G' \in \mathrm{inflations}(G)$$

if and only if

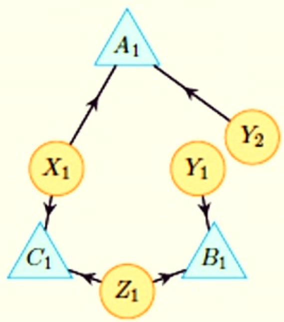
$$\forall A_i \in \mathrm{nodes}(G') : \ \mathrm{ansubgraph}_{G'}(A_i) \sim \mathrm{ansubgraph}_G(A).$$



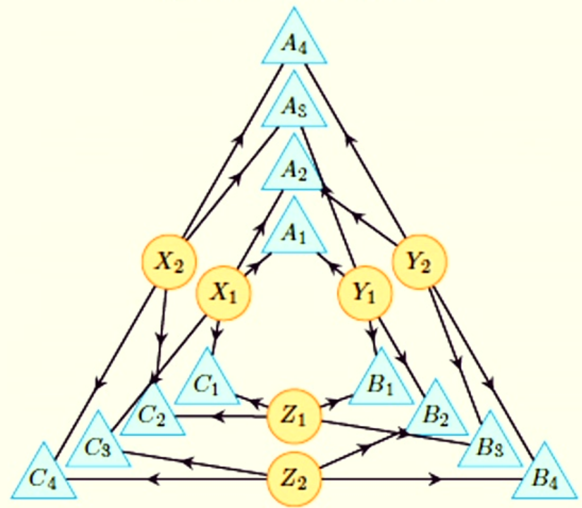
The Triangle Scenario



Spiral Inflation

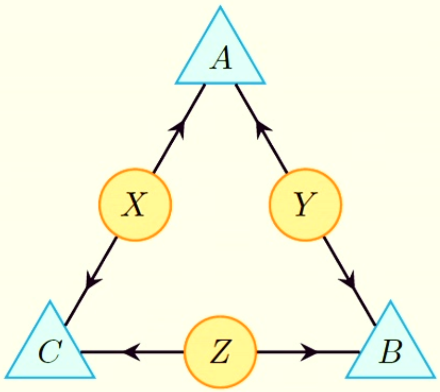


Cut Inflation



Large Inflation

model M on DAG G



$\mathrm{P}_{A|XY}$

$\mathrm{P}_{B|YZ}$

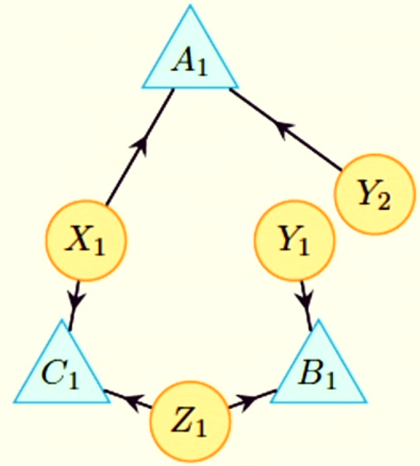
$\mathrm{P}_{C|XZ}$

P_{X}

P_{Y}

P_{Z}

M' = G → G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|Y_1Z_1}$

$\mathrm{P}_{C_1|X_1Z_1}$

P_{X_1}

P_{Y_1}

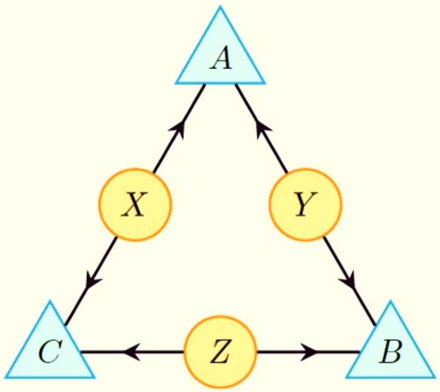
P_{Y_2}

P_{Z_1}

with symmetry constraint:

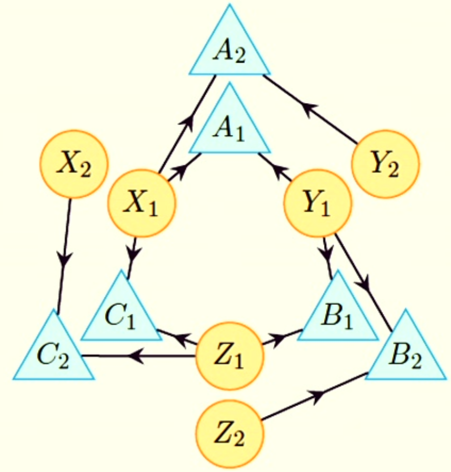
$$\mathrm{P}_{Y_1} = \mathrm{P}_{Y_2}$$

model M on DAG G



$P_{A|XY}$
$P_{B|YZ}$
$P_{C|XZ}$
P_X
P_Y
P_Z

M' = G → G' Inflation of M



$P_{A_1|X_1Y_1}$	$P_{A_2|X_1Y_2}$
$P_{B_1|Y_1Z_1}$	$P_{B_2|Y_1Z_2}$
$P_{C_1|X_1Z_1}$	$P_{C_2|X_2Z_1}$
P_{X_1}	P_{X_2}
P_{Y_1}	P_{Y_2}
P_{Z_1}	P_{Z_2}

with symmetry constraints:

$$P_{A_1|X_1Y_1} = P_{A_2|X_1Y_2}$$
$$P_{B_1|Y_1Z_1} = P_{B_2|Y_1Z_2}$$
$$P_{C_1|X_1Z_1} = P_{C_2|X_2Z_1}$$
$$P_{X_1} = P_{X_2}$$
$$P_{Y_1} = P_{Y_2}$$
$$P_{Z_1} = P_{Z_2}$$

Suppose $G' \in \text{inflations}(G)$

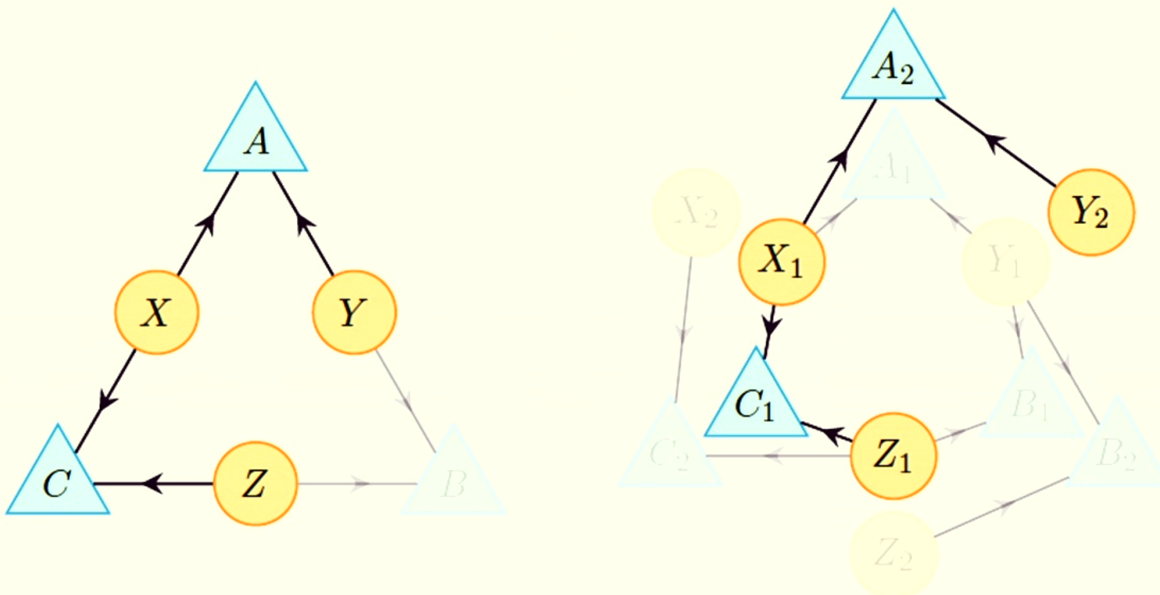
then

$$M' = \text{Inflation}_{G \to G'}(M)$$

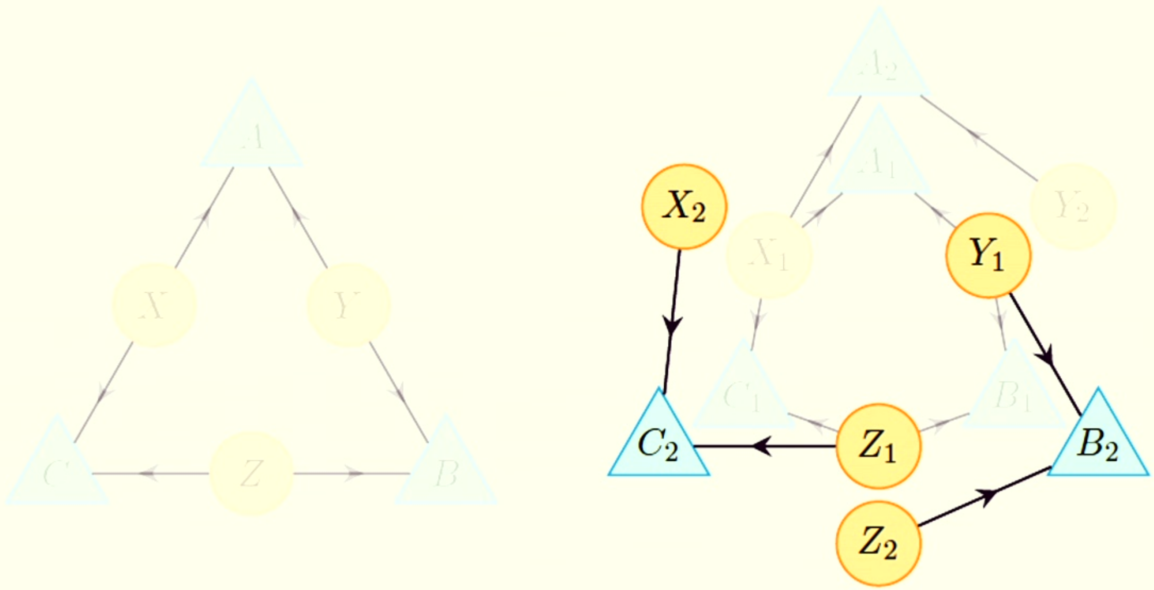
if and only if

$$\forall A_i, A_j \in \text{nodes}(G') : \ \mathrm{P}_{A_i | \mathrm{Pa}_{G'}(A_i)} = \mathrm{P}_{A_j | \mathrm{Pa}_{G'}(A_j)}.$$

Injectable sets of observed variables
in the inflation DAG

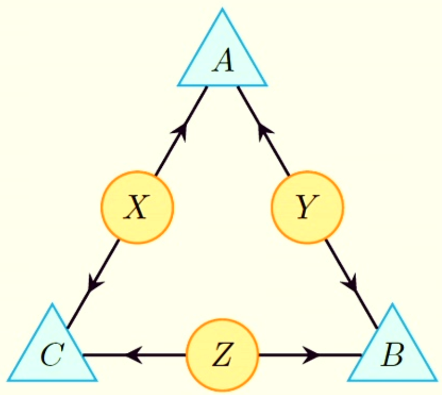


$\{A_2C_1\}$ is an injectable set



$\{B_2C_2\}$ is *not* an injectable set

model M on DAG G



$\mathrm{P}_{A|XY}$

$\mathrm{P}_{B|YZ}$

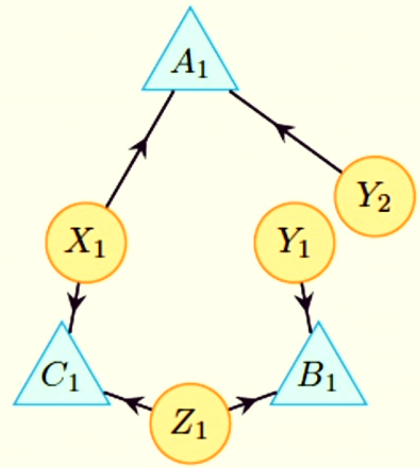
$\mathrm{P}_{C|XZ}$

P_{X}

P_{Y}

P_{Z}

M' = G $\rightarrow$ G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|Y_1Z_1}$

$\mathrm{P}_{C_1|X_1Z_1}$

P_{X_1}

P_{Y_1}

P_{Y_2}

P_{Z_1}

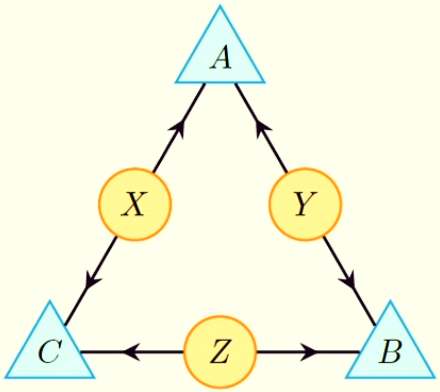
$\{A_1C_1\}$ is an injectable set

$$\mathrm{P}_{A_1C_1} = \sum_{X_1Y_2Z_1} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{C_1|X_1Z_1}\mathrm{P}_{X_1}\mathrm{P}_{Y_2}\mathrm{P}_{Z_1}$$

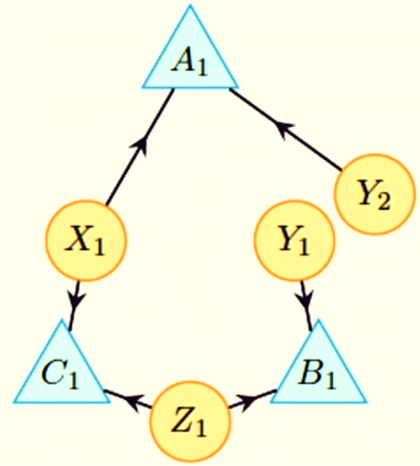
$$\mathrm{P}_{AC} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{C|XZ}\mathrm{P}_{X}\mathrm{P}_{Y}\mathrm{P}_{Z}$$

model M on DAG G

$M' = G \to G'$ Inflation of M



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|YZ}$
$\mathrm{P}_{C|XZ}$
P_{X}
P_{Y}
P_{Z}



$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

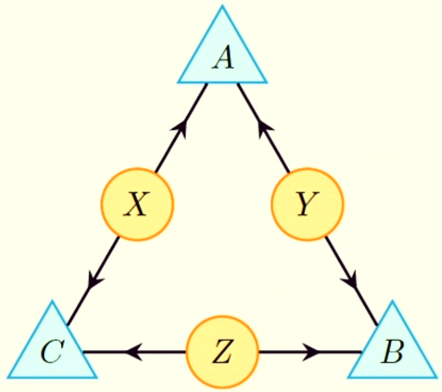
$\{A_1C_1\}$ is an injectable set

$$\mathrm{P}_{A_1C_1} = \sum_{X_1Y_2Z_1} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{C_1|X_1Z_1}\mathrm{P}_{X_1}\mathrm{P}_{Y_2}\mathrm{P}_{Z_1}$$

$$\mathrm{P}_{AC} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{C|XZ}\mathrm{P}_{X}\mathrm{P}_{Y}\mathrm{P}_{Z}$$

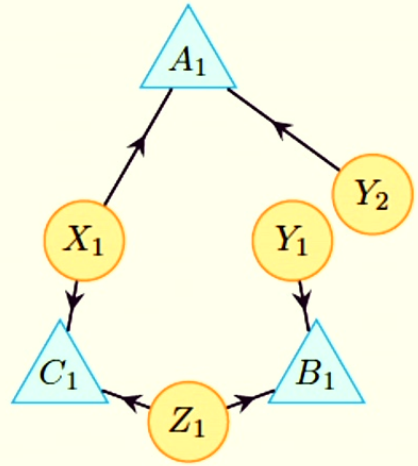
P_{AC} compatible with $M \implies P_{A_1C_1} = P_{AC}$ compatible with M'

model M on DAG G



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|YZ}$
$\mathrm{P}_{C|XZ}$
P_{X}
P_{Y}
P_{Z}

M' = G $\rightarrow$ G' Inflation of M

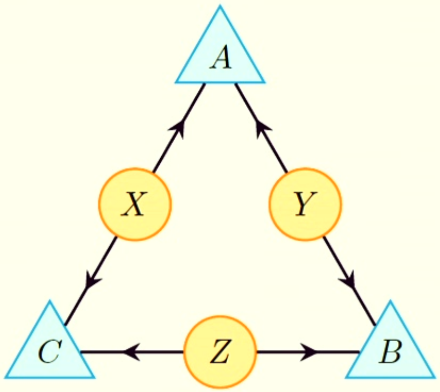


$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

$\{A_1B_1\}$ is *not* an injectable set

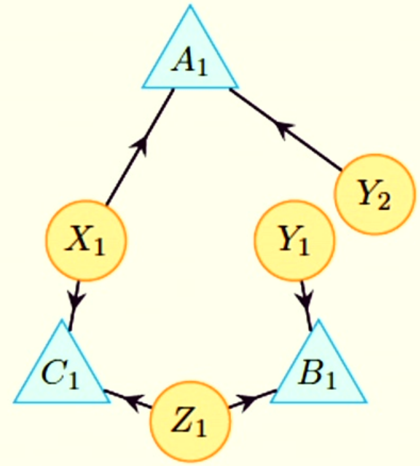
$$\mathrm{P}_{A_1B_1} = \sum_{X_1Y_1Y_2Z_1} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{B_1|Y_1Z_1}\mathrm{P}_{X_1}\mathrm{P}_{Y_1}\mathrm{P}_{Y_2}\mathrm{P}_{Z_1}$$

model M on DAG G



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|YZ}$
$\mathrm{P}_{C|XZ}$
P_{X}
P_{Y}
P_{Z}

M' = G → G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

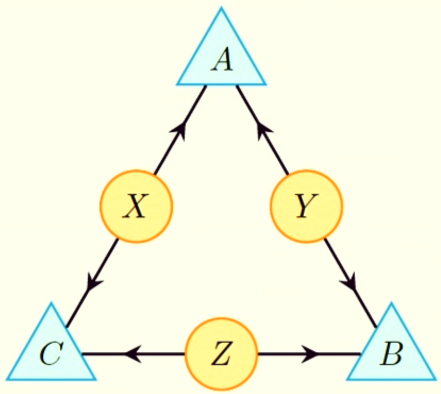
$\{A_1B_1\}$ is *not* an injectable set

$$\mathrm{P}_{A_1B_1} = \left(\sum_{X_1Y_2} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{Y_2}\mathrm{P}_{X_1}\right)\left(\sum_{Z_1Y_1} \mathrm{P}_{B_1|Y_1Z_1}\mathrm{P}_{Y_1}\mathrm{P}_{Z_1}\right)$$

$$\mathrm{P}_{AB} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{B|YZ}\mathrm{P}_X\mathrm{P}_Y\mathrm{P}_Z$$

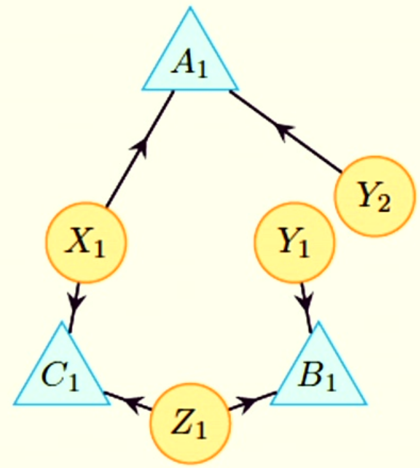
P_{AB} compatible with M $\not\Rightarrow$ $P_{A_1B_1} = P_{AB}$ compatible with M'

model M on DAG G



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|YZ}$
$\mathrm{P}_{C|XZ}$
P_{X}
P_{Y}
P_{Z}

M' = G $\rightarrow$ G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

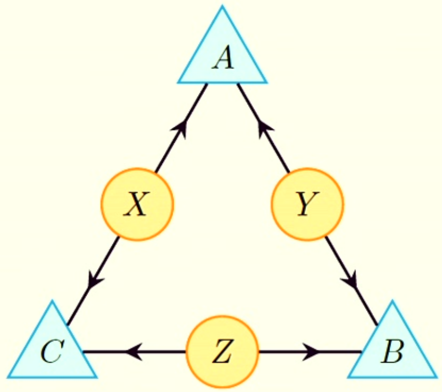
Injectable sets: $\{A_1\}, \{B_1\}, \{C_1\}, \{A_1C_1\}, \{B_1C_1\}$

$(P_A, P_B, P_C, P_{AC}, P_{BC})$ compatible with M $\Longrightarrow$ $(P_{A_1}, P_{B_1}, P_{C_1}, P_{A_1C_1}, P_{B_1C_1})$ compatible with M'

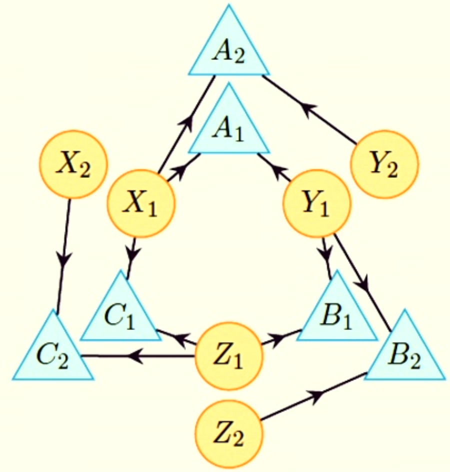
where

$\mathrm{P}_{A_1} = \mathrm{P}_A \qquad \mathrm{P}_{A_1C_1} = \mathrm{P}_{AC}$
$\mathrm{P}_{B_1} = \mathrm{P}_B \qquad \mathrm{P}_{B_1C_1} = \mathrm{P}_{BC}$
$\mathrm{P}_{C_1} = \mathrm{P}_C$

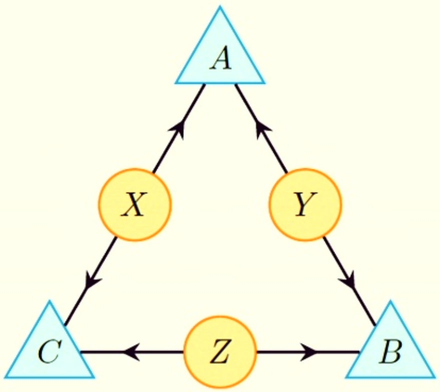
model M on DAG G



M' = G → G' Inflation of M



Injectable sets: $\{A_1\}, \{B_1\}, \{C_1\}, \{A_2\}, \{B_2\}, \{C_2\},$
$\{A_1 B_1\}, \{A_1, B_2\}, \{B_1 C_1\}, \{B_1, C_2\}, \{C_1, A_1\}, \{C_1, A_2\},$
$\{A_1 B_1 C_1\}$



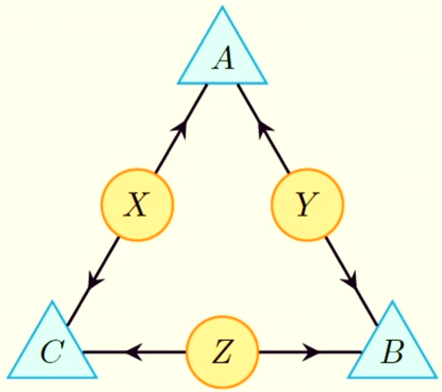
is incompatible with

$$P_{ABC} = \frac{1}{2}[000] + \frac{1}{2}[111]$$

Strategy: assume compatibility and derive a contradiction

Logic first given in:
Henson, Lal, Pusey, *New Journal of Physics* 16, 113043 (2014).

Causal model M



$\mathrm{P}_{A|XY}$

$\mathrm{P}_{B|YZ}$

$\mathrm{P}_{C|XZ}$

P_{X}

P_{Y}

P_{Z}

$P_{ABC} = \frac{1}{2}[000] + \frac{1}{2}[111]$

is compatible with M

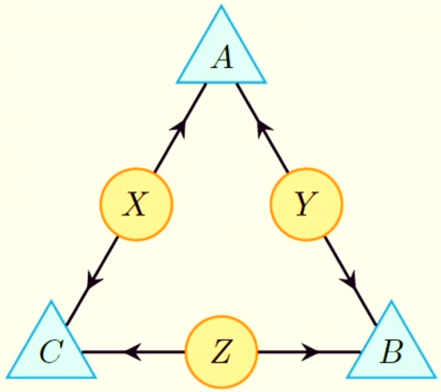
$\Longrightarrow$ (P_{AC}, P_{BC})

where $P_{AC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

$P_{BC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

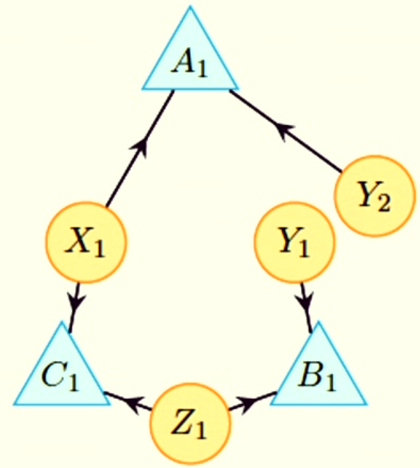
is compatible with M

Causal model M



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|YZ}$
$\mathrm{P}_{C|XZ}$
P_X
P_Y
P_Z

M' = G → G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

$$(P_{AC}, P_{BC})$$

where $P_{AC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

$P_{BC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is compatible with M

$$\Longrightarrow$$

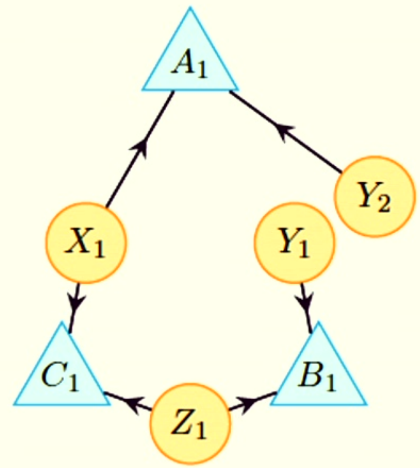
$$(P_{A_1C_1}, P_{B_1C_1})$$

where $\mathrm{P}_{A_1C_1} = \mathrm{P}_{AC}$

$\mathrm{P}_{B_1C_1} = \mathrm{P}_{BC}$

is compatible with M'

M' = G → G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

$$(P_{A_1C_1}, P_{B_1C_1})$$

where
$$P_{A_1C_1} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{B_1C_1} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$

is not compatible with M'

M′ = G → G′ Inflation of M

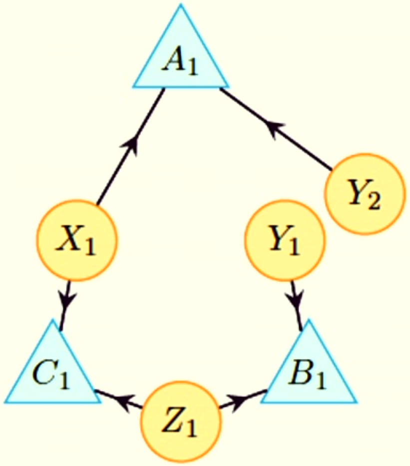
Proof:

If $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$
$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

then

$$P_{A_1B_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$$

(recall example 3 of marginal problem)



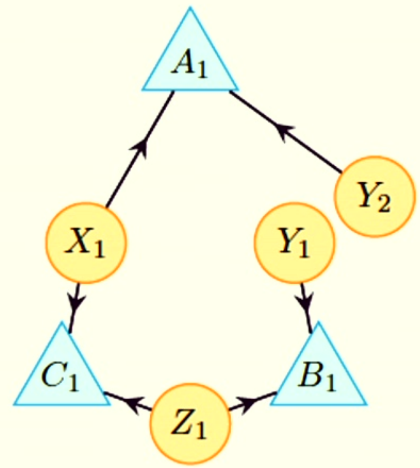
$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

$$(P_{A_1C_1}, P_{B_1C_1})$$

where $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$
$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is not compatible with M'

M′ = G → G′ Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

Proof:

If $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$
$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

then

$$P_{A_1B_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$$

(recall example 3 of marginal problem)

But this violates $A_1 \perp B_1$

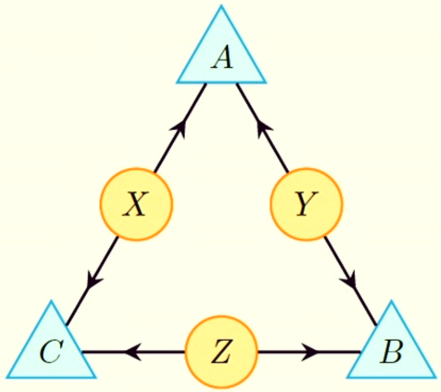
which is required by the d-separation relation $A_1 \perp_d B_1$

$$(P_{A_1C_1}, P_{B_1C_1})$$

where $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$
$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

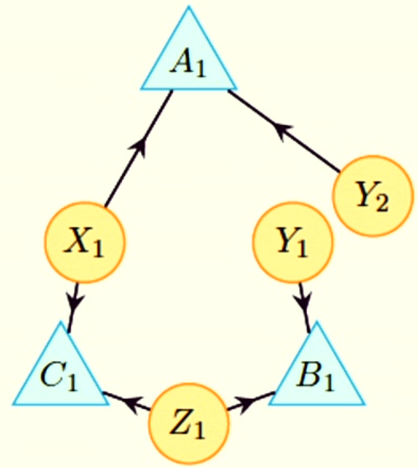
is not compatible with M'

Causal model M



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|YZ}$
$\mathrm{P}_{C|XZ}$
P_{X}
P_{Y}
P_{Z}

M′ = G $\rightarrow$ G′ Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

$$(P_{AC}, P_{BC})$$

where $P_{AC} = \frac{1}{2}[00] + \frac{1}{2}[11]$
$P_{BC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is not compatible with M

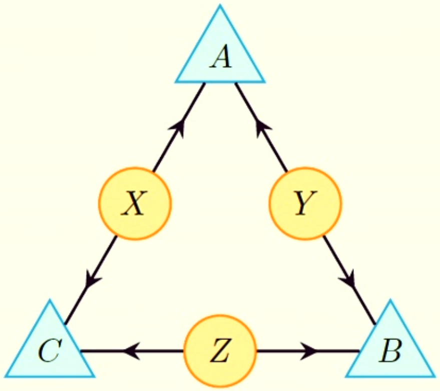
$\Longleftarrow$

$$(P_{A_1C_1}, P_{B_1C_1})$$

where $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$
$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is not compatible with M'

Causal model M



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|YZ}$
$\mathrm{P}_{C|XZ}$
P_{X}
P_{Y}
P_{Z}

$P_{ABC} = \frac{1}{2}[000] + \frac{1}{2}[111]$

is not compatible with M

$\Leftarrow$

(P_{AC}, P_{BC})

where $P_{AC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

$P_{BC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is not compatible with M

M' = G $\rightarrow$ G' Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$

$\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$$
is compatible with M

$$\Longrightarrow$$

$$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is compatible with M'

Deriving
causal compatibility inequalities
by the inflation technique

Let $I_{\mathcal{S}}$ be an inequality that acts on the family of distributions $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

such that whenever

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$ is compatible with M $\Longrightarrow$ $I_{\mathcal{S}}$ is **satisfied** for $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

Let $I_{\mathcal{S}}$ be an inequality that acts on the family of distributions $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

such that whenever

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$ is compatible with M $\Longrightarrow$ $I_{\mathcal{S}}$ is **satisfied** for $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

we say that

$I_{\mathcal{S}}$ is a **causal compatibility inequality** for model M

M’ = G $\rightarrow$ G’ Inflation of M

$$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$$

$$\mathcal{S}' \subseteq \text{InjectableSets}(G')$$

$$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$$

is compatible with M

$$\Longrightarrow$$

$$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$$

where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$

is compatible with M'

M′ = G → G′ Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$ $\qquad$ $\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$
is compatible with M

$\Longrightarrow$

$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is compatible with M'

$\Downarrow$

$I_{\mathcal{S}'}$ is **satisfied** for
$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$

M′ = G → G′ Inflation of M

$\mathcal{S} \subseteq \mathrm{ImagesInjectableSets}(G)$ $\qquad$ $\mathcal{S}' \subseteq \mathrm{InjectableSets}(G')$

$I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$
is compatible with M

$\Longrightarrow$

$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is compatible with M'

$\Downarrow$

$I_{\mathcal{S}'}$ is **satisfied** for
$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$

$\Longleftarrow$

$I_{\mathcal{S}}$ is **satisfied** for
$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

M′ = G $\rightarrow$ G′ Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$ $\quad\quad$ $\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

is compatible with M

$$\Downarrow$$

$I_{\mathcal{S}}$ is **satisfied** for
$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

$I_{\mathcal{S}}$ is a **causal compatibility inequality** for model M

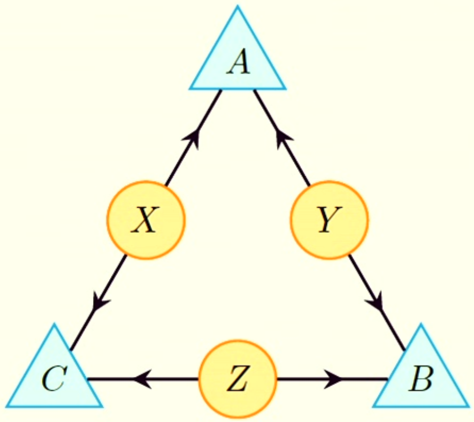
M' = G $\rightarrow$ G' Inflation of M

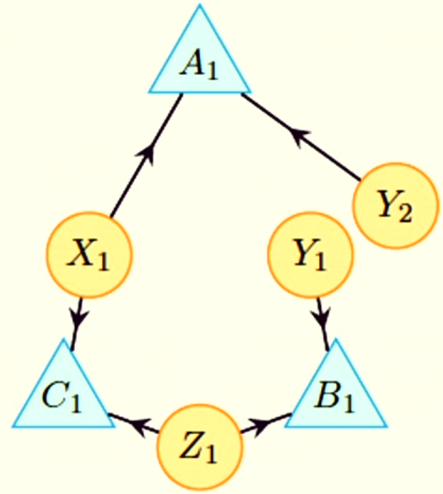
$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$

$\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$I_{\mathcal{S}}$ is a **causal compatibility inequality** for model M $\Longleftarrow$ $I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

binary A, B and C



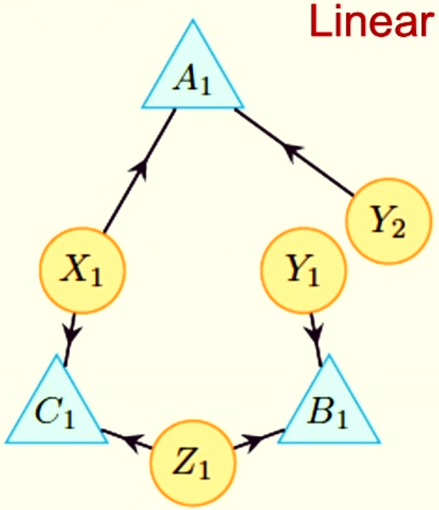


$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$
is a causal compatibility inequality for M

$P_{A_1} + P_{B_1} + P_{C_1} - P_{A_1} P_{B_1} - P_{B_1 C_1} - P_{A_1 C_1} \leq 1$
is a causal compatibility inequality for M'

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is a valid set of marginals $\Longrightarrow$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy
$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

Linear quantifier elimination

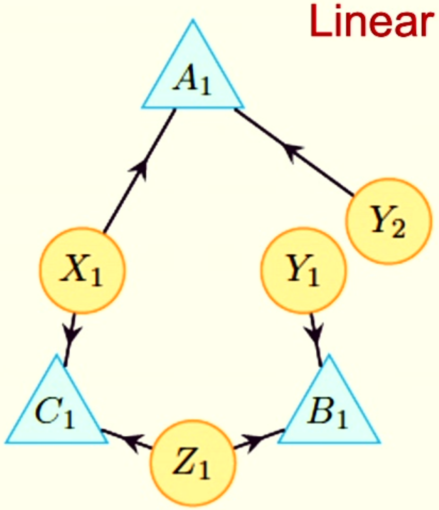


$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is compatible with M'

$\Longrightarrow A_1 \perp B_1 \Longrightarrow P_{A_1B_1} = P_{A_1}P_{B_1}$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is a valid set of marginals $\implies$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

Linear quantifier elimination

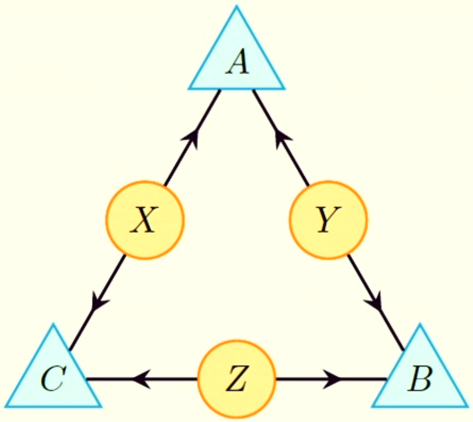


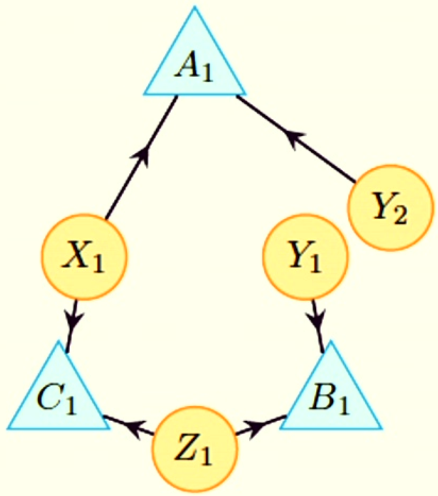
$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M'

$\implies A_1 \perp B_1 \implies P_{A_1B_1} = P_{A_1}P_{B_1}$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M' $\implies$ $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

binary A, B and C





$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

is a causal compatibility inequality for M

⇐

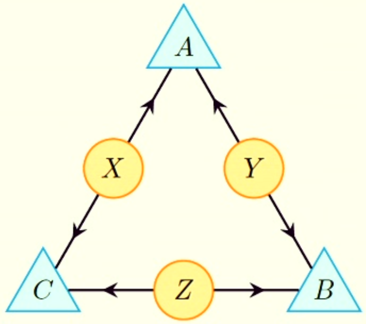
$$P_{A_1} + P_{B_1} + P_{C_1} - P_{A_1} P_{B_1} - P_{B_1 C_1} - P_{A_1 C_1} \leq 1$$

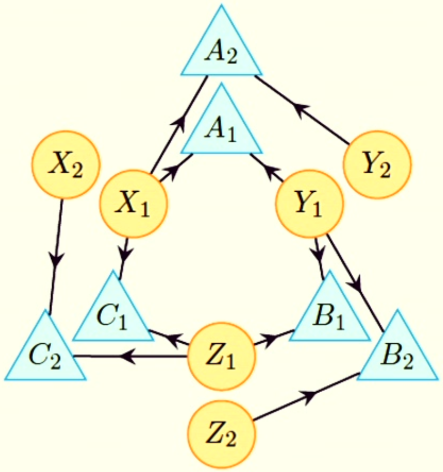
is a causal compatibility inequality for M'

rules out

$$P_{ABC} = \frac{1}{2}[000] + \frac{1}{2}[111]$$

Spiral inflation





$$P_A(1)P_B(1)P_C(1) \leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1) + P_{AC}(11)P_B(1) + P_{ABC}(000)$$

is a causal compatibility inequality for M

$\Leftarrow$

$$P_{A_2}(1)P_{B_2}(1)P_{C_2}(1) \leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1) + P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)$$

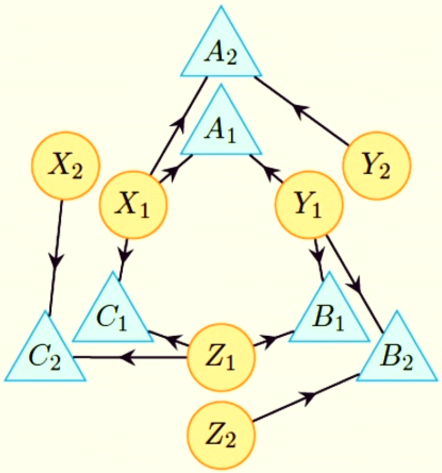
is a causal compatibility inequality for M'

$$(P_{A_2B_2C_2}, P_{A_1B_2C_2}, P_{A_2B_1C_2}, P_{A_2B_2C_1}, P_{A_1B_1C_1})$$

is a valid set of marginals $\Longrightarrow$

$$P_{A_2B_2C_2}(111) \leq P_{A_1B_2C_2}(111) + P_{B_1C_2A_2}(111) + P_{A_2C_1B_2}(111) + P_{A_1B_1C_1}(000)$$

Linear quantifier elimination



$$(P_{A_2B_2C_2}, P_{A_1B_2C_2}, P_{A_2B_1C_2}, P_{A_2B_2C_1}, P_{A_1B_1C_1})$$

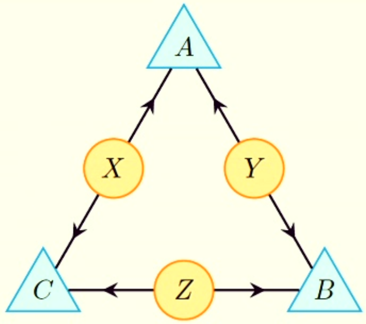
is compatible with M'

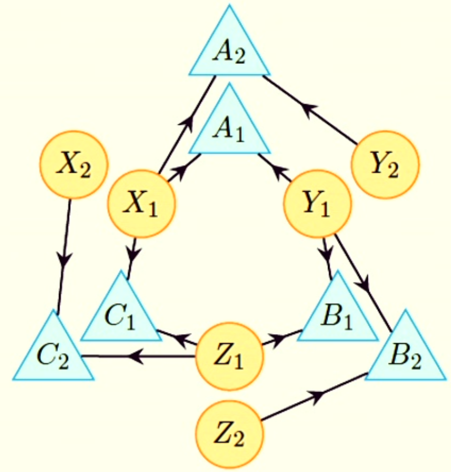
$$A_1B_2 \perp_d C_2 \implies P_{A_1B_2C_2} = P_{A_1B_2}P_{C_2},$$
$$B_1C_2 \perp_d A_2 \implies P_{B_1C_2A_2} = P_{B_1C_2}P_{A_2},$$
$$A_2C_1 \perp_d B_2 \implies P_{A_2C_1B_2} = P_{A_2C_1}P_{B_2},$$
$$A_2 \perp_d B_2 \perp_d C_2 \implies P_{A_2B_2C_2} = P_{A_2}P_{B_2}P_{C_2}$$

$$(P_{A_2B_2C_2}, P_{A_1B_2C_2}, P_{A_2B_1C_2}, P_{A_2B_2C_1}, P_{A_1B_1C_1})$$

is compatible with M' $\Longrightarrow$

$$P_{A_2}(1)P_{B_2}(1)P_{C_2}(1) \leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1) + P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)$$





$$\begin{aligned} &P_A(1)P_B(1)P_C(1) \\ &\leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1) \\ &+ P_{AC}(11)P_B(1) + P_{ABC}(000) \end{aligned}$$

⇐

$$\begin{aligned} &P_{A_2}(1)P_{B_2}(1)P_{C_2}(1) \\ &\leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1) \\ &+ P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000) \end{aligned}$$

rules out

$$P_{ABC} = \tfrac{1}{3}[001] + \tfrac{1}{3}[010] + \tfrac{1}{3}[100]$$

Linear inequality constraints from marginal compatibility (from linear quantifier elimination)

Polynomial inequality constraints for causal compatibility with the original DAG

⇐

+

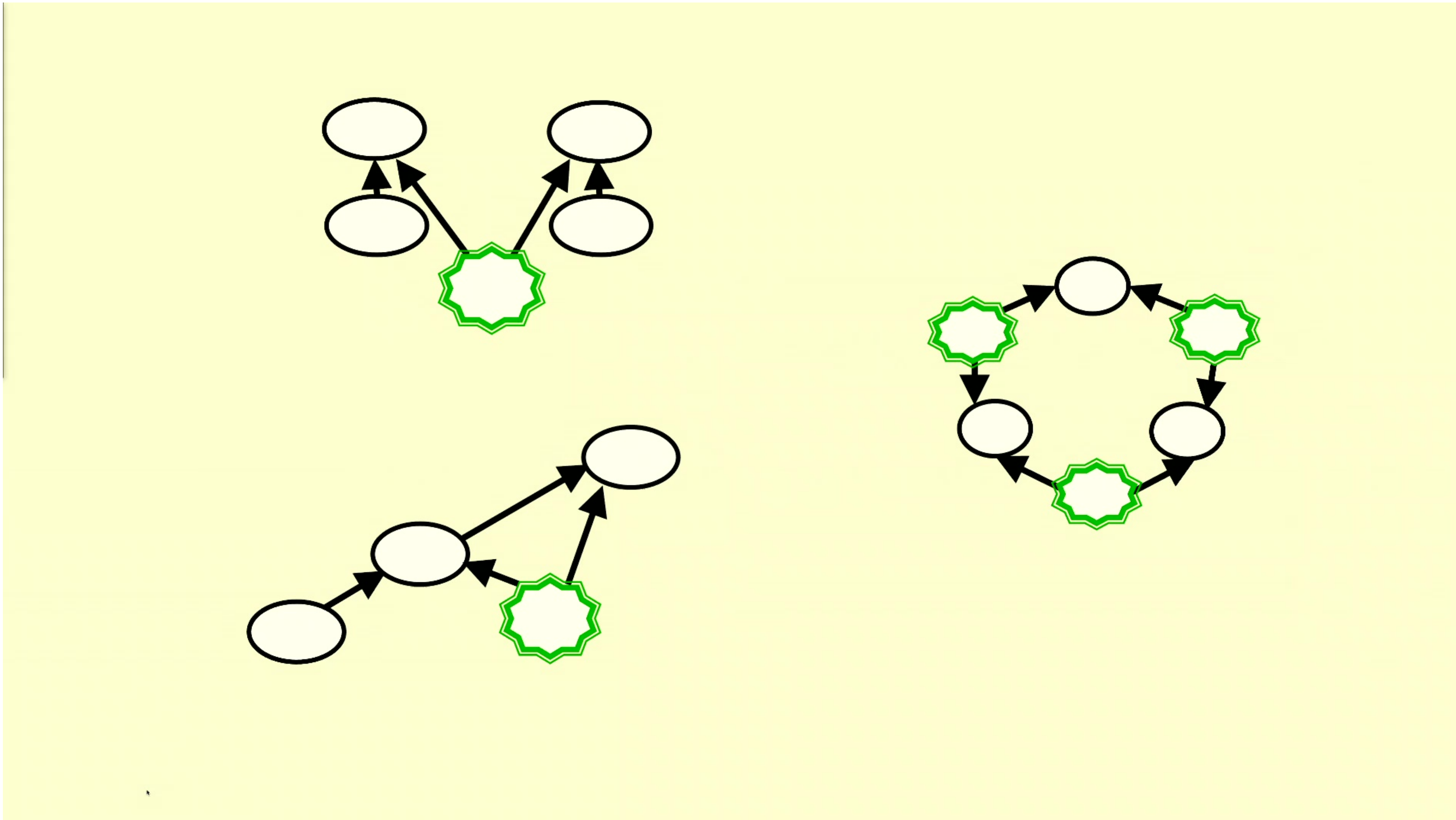
Polynomial equality constraints from causal compatibility with the inflated DAG (e.g., from d-separation relations)

The technique defines an algorithm for deriving causal compatibility inequalities and for testing compatibility

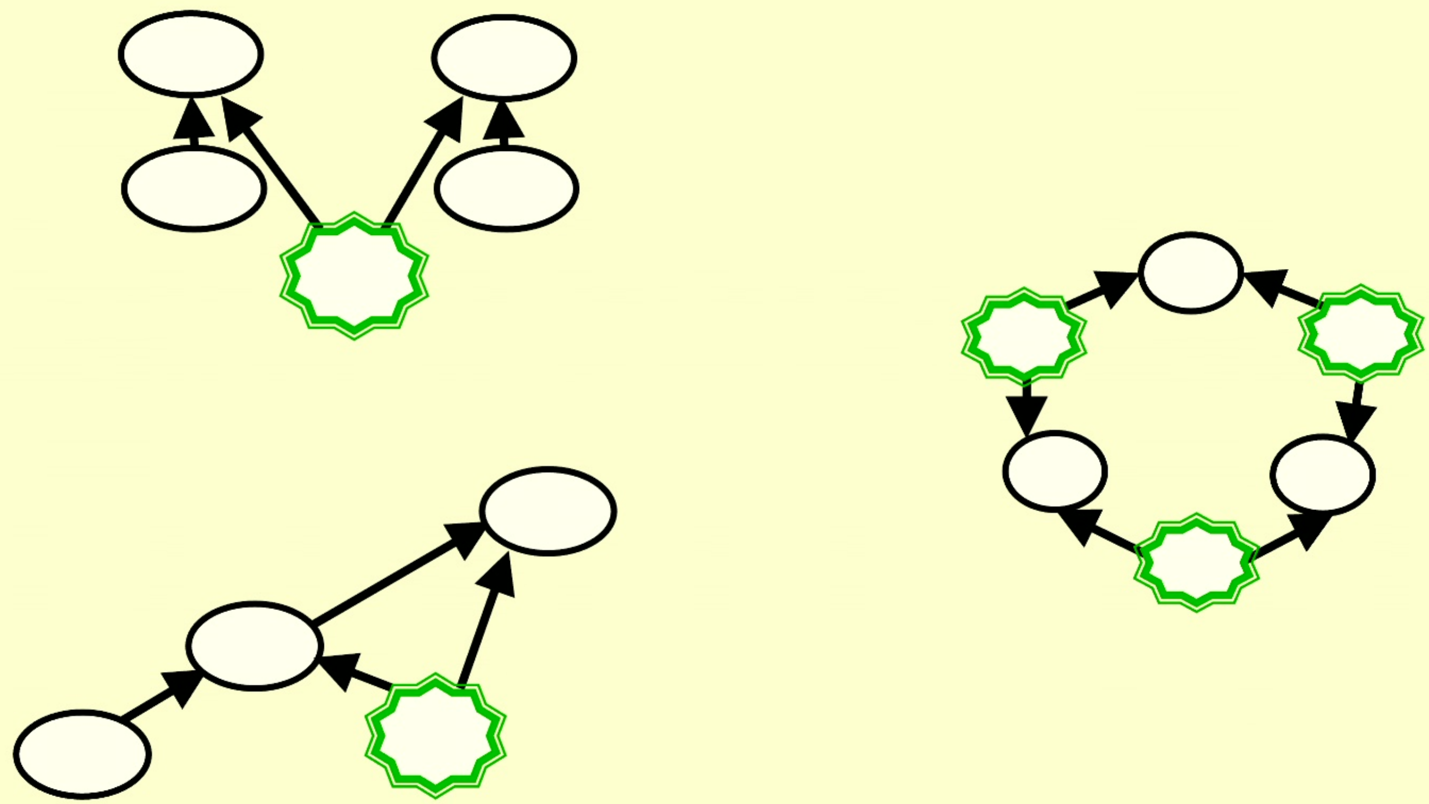
Proof that this provides a convergent heirarchy of tests:
Navascués & Wolfe, J. Causal Inf. 8(1) 70 (2020)

Approaches to Bell arguments that follow essentially
the logic of the inflation technique:

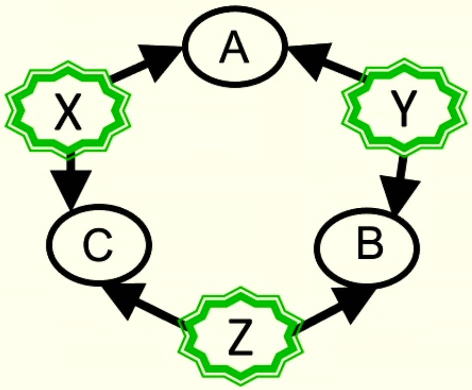
Fine's proof of CHSH inequalities
Hardy's proof of Bell's theorem
The Greenberger-Horne-Zeilinger proof of Bell's theorem
Bell's theorem by way of Kochen-Specker noncontextuality
Braunstein-Caves entropic inequalities
Symmetric extensions / shareability of local correlations
...



Causal compatibility in quantum causal models



Quantum triangle model



$\rho_{A|XY}$

$\rho_{B|YZ}$

$\rho_{C|XZ}$

ρ_X

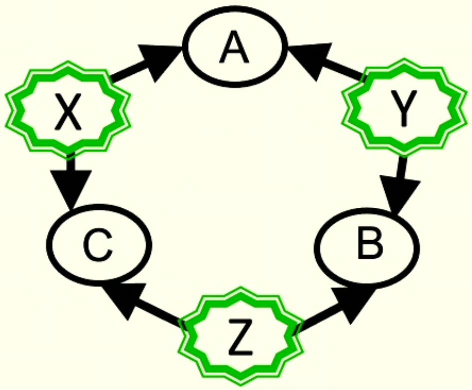
ρ_Y

ρ_Z

$$[\rho_{A|XY}, \rho_{B|YZ}] = 0$$
$$[\rho_{B|YZ}, \rho_{C|XZ}] = 0$$
$$[\rho_{A|XY}, \rho_{C|XZ}] = 0$$

$$P_{ABC} = \mathrm{Tr}_{XYZ}\left(\rho_{A|XY}\rho_{B|YZ}\rho_{C|XZ}\rho_X\rho_Y\rho_Z\right)$$

Quantum triangle model



$\rho_{A|X_A Y_A}$

$\rho_{B|Y_B Z_B}$

$\rho_{C|X_C Z_C}$

$\rho_{X_A X_C}$

$\rho_{Y_A Y_B}$

$\rho_{Z_B Z_C}$

$$P_{ABC} = \mathrm{Tr}_{X_A X_B Y_A Y_B Z_A Z_B}\left(\rho_{A|X_A Y_A}\rho_{B|Y_B Z_B}\rho_{C|X_C Z_C}\rho_{X_A X_C}\rho_{Y_A Y_B}\rho_{Z_B Z_C}\right)$$

Conventional assumption:
dimension of latent quantum system is arbitrary

Entropy cone technique

Von Neumann entropy

$$H(X) := -\mathrm{Tr}_X(\rho_X \log \rho_X)$$

satisfies

Submodularity

$$H(\mathbf{X}) + H(\mathbf{X}AB) \leq H(\mathbf{X}A) + H(\mathbf{X}B)$$

where A and B are systems not in the set **X**

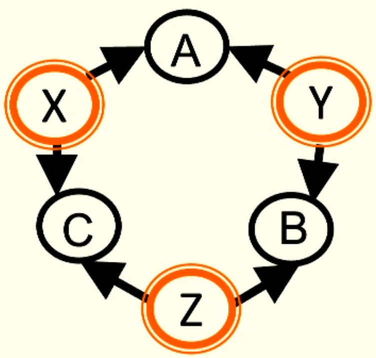
does not satisfy

Monotonicity

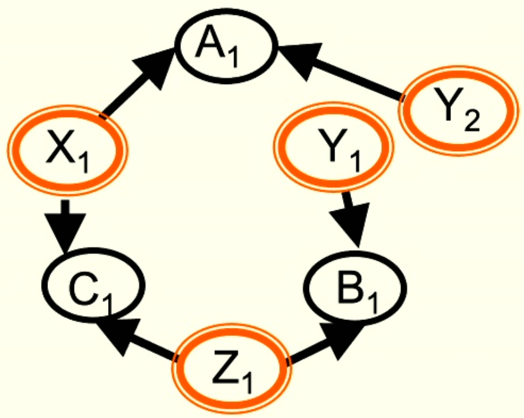
$$H(\mathbf{X}A) \geq H(\mathbf{X})$$

for every system A and sets of systems **X**

Instead, it satisfies a weaker condition (weak monotonicity)

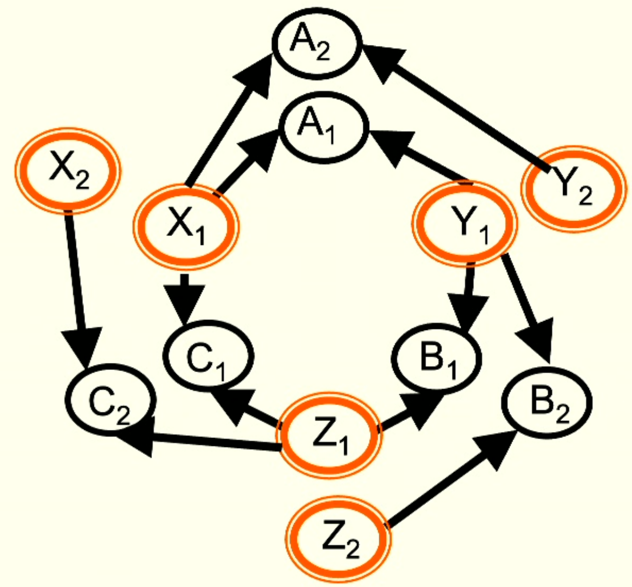


Triangle



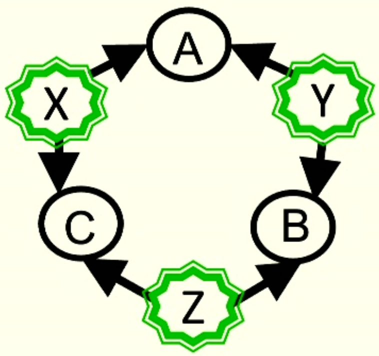
Cut inflation of Triangle

Non-fan-out

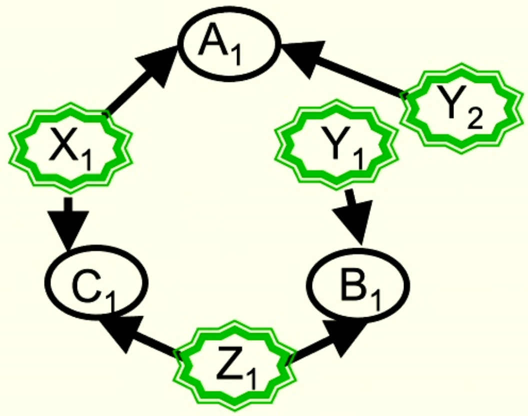


Spiral inflation of Triangle

Fan-out

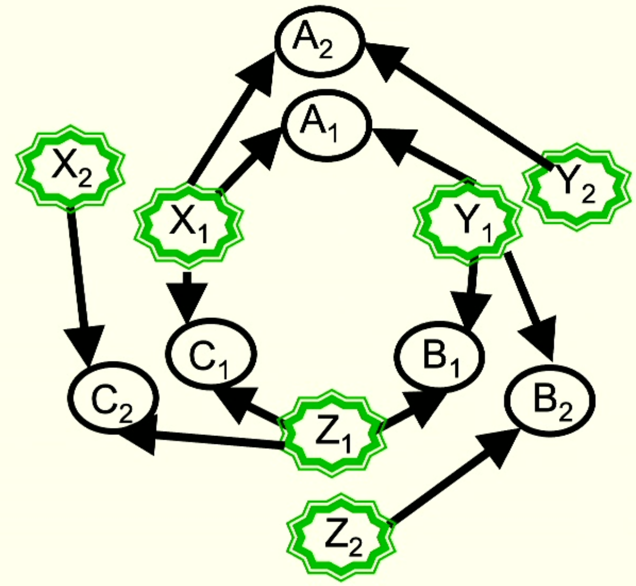


Quantum triangle



Cut inflation of Quantum Triangle

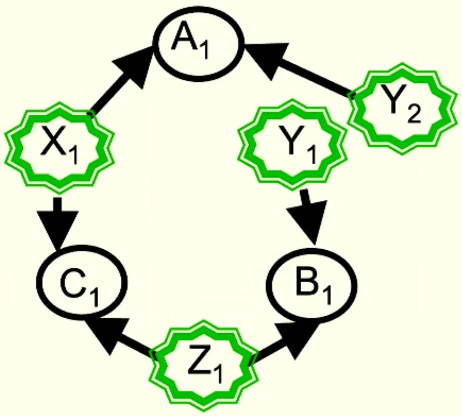
Non-fan-out



Spiral inflation of Quantum Triangle?

Fan-out

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is a valid set of marginals $\Longrightarrow$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$



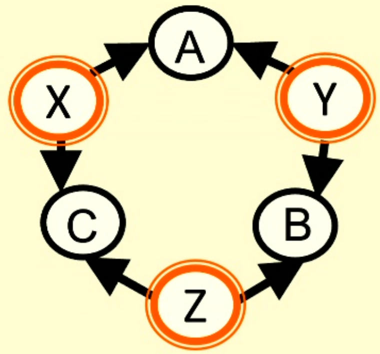
$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M'

$$\Longrightarrow A_1 \perp B_1 \Longrightarrow P_{A_1B_1} = P_{A_1}P_{B_1}$$

which is quantum valid

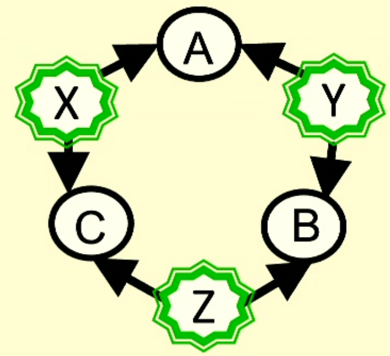
$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M' $\Longrightarrow$ $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

Classical Triangle model



$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

Quantum Triangle model

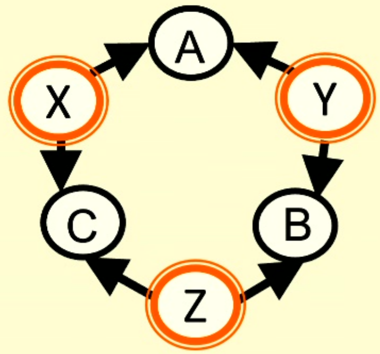


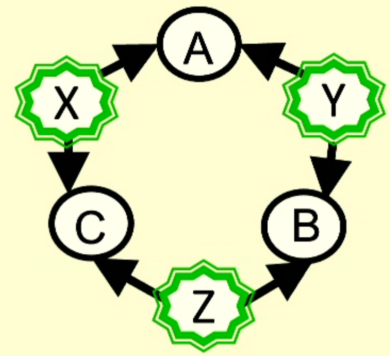
$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

This inequality was obtained from a non-fanout inflation and therefore holds for both quantum and classical latents

Classical Triangle model

Quantum Triangle model





$$P_A(1)P_B(1)P_C(1) \leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1) + P_{AC}(11)P_B(1) + P_{ABC}(000)$$

?

This inequality was obtained from a fanout inflation and therefore holds for classical latents, but might not hold for quantum latents

Inflation-adjacent technique: interruption a.k.a. single-world intervention graph

Quantum inflation technique

Wolfe, Pozas-Kerstjens, Grinberg, Rosset, Acín, Navascués, Phys. Rev. X **11**, 021043 (2021)

Generalization of NPA semidefinite programming hierarchy for Bell scenario

Navascués, Pironio, and Acín, New J. Phys. 10, 073013 (2008)

Quantum inflation technique
Wolfe, Pozas-Kerstjens, Grinberg, Rosset, Acín, Navascués, Phys. Rev. X **11**, 021043 (2021)

Generalization of NPA semidefinite programming hierarchy for Bell scenario
Navascués, Pironio, and Acín, New J. Phys. 10, 073013 (2008)

For convergence result, see:
Ligthart, Gachechiladze, and Gross, arXiv:2110.14659 (2021)

Thanks