Title: The Off-shell String Effective Action and Black Hole Entropy

Speakers: Amr Ahmadain

Series: Quantum Fields and Strings

Date: April 25, 2023 - 2:00 PM

URL: https://pirsa.org/23040092

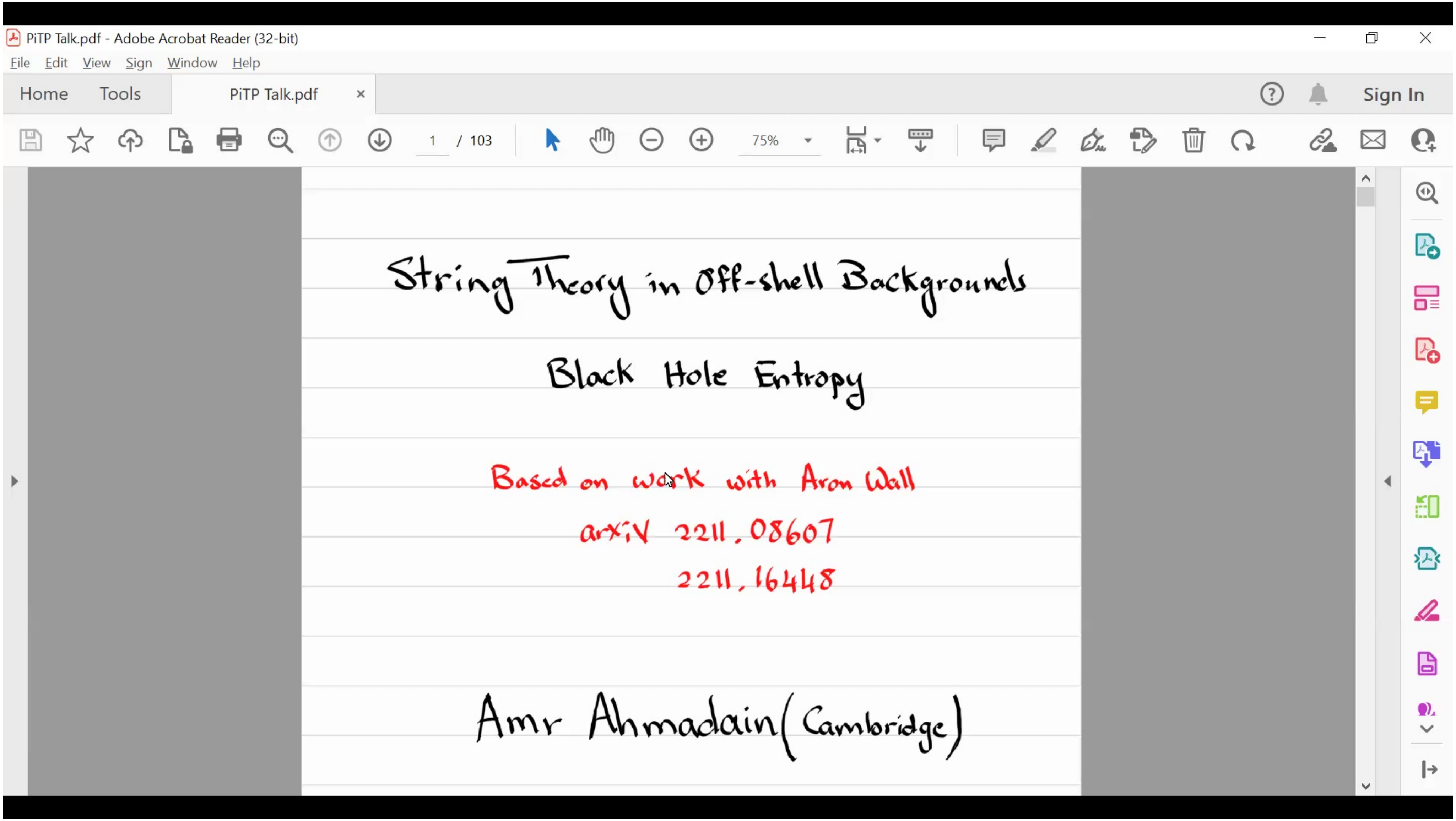
Abstract: Taking string theory off shell requires breaking Weyl invariance on the worldsheet, i.e. the worldsheet theory is now a QFT instead of a CFT. I will explain Tseytlin's first-quantized approach to taking the worldsheet theory off-shell in a consistent manner, with a particular emphasis on the subtleties involved in calculating the sphere amplitude. This approach allows for the derivation of a classical string action which gives rise to the correct equations of motion and S-matrix, to all orders in perturbation theory.

I'll also explain the underlying conceptual structure of the Susskind and Uglum black hole entropy argument. There I will show explicitly how the classical (tree-level) effective action and entropy S = A/4G_N may be calculated from the sphere diagrams.

Time permitting, I will discuss ongoing efforts to derive the Ryu-Takayanagi formula in string theory.

Based on arXiv:2211.08607 and arXiv:2211.16448.

Zoom link: https://pitp.zoom.us/j/93668017324?pwd=K2pxZEhjalRTWkVVbVRESCtRVDFmUT09


PiTP Talk.pdf - Adobe Acrobat Reader (32-bit)
String Theory in Off-shell Backgrounds
Black Hole Entropy
Based on work with Aron Wall
arXiV 2211.08607
2211.16448
Amr Ahmadain (Cambridge)

Plan of Talk

I. BH and String Information Theory

II. Entanglement Entropy in String Theory

III. The Off-shell Sphere Partition Function

IV. Key Results of arxiv 2211.08607

V. Gauge Orbits of $SL(2,C)$

VI. Ongoing Work & Future Directions

Quantum Information/Gravity

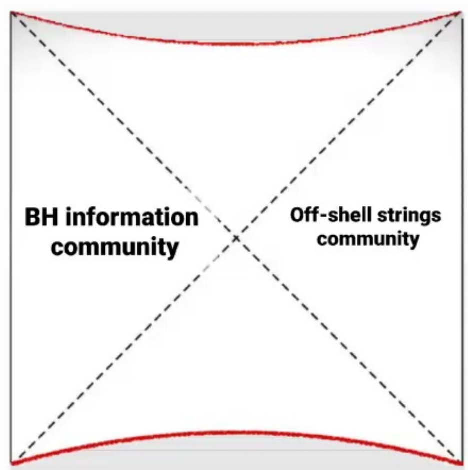
- Ryu-Takayangi holographic EE
- Entanglement Renorm. & TN PRD 86(2012)065007
- Firewall Paradox JHEP 2013(3):59
- Code Subspace & QEC JHEP 06(2013)085 JHEP 06(2015) 149
- ER = EPR arXiv:1306:0533
- Entanglement Wedge Reconstruction
- Quantum Extremal Surface JHEP. 2015(1):73
- Islands, Page Curve and the BH Info. Problem arXiv: 1905.08762 :2019 arXiv:1905.08255 :2019

WHAT IS NEXT ?

Do we need a theory of QG ??

- ALL of the Q.I. quantities are derived or conjectured in Semi-classical gravity.

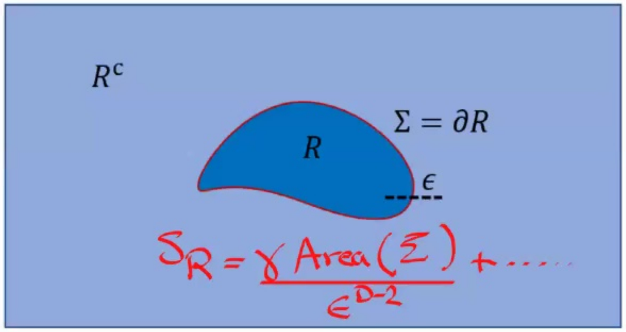
Can we learn more with
STRING THEORY ??



II. Entanglement Entropy in String Theory

Entanglement Entropy in QFTs

- A state in a local QFT has short-range correlations
- EE is sensitive to short-distance physics (correlations) of modes across entangling surface Σ
- EE is thus a UV-divergent quantity
- $S_R = -\mathrm{Tr}\, \rho_R \log \rho_R$ counts the # modes straddling Σ
- S_R is proportional area of boundary



The Replica Trick in QFTs

On an n-sheeted cover of spacetime with cuts corresponding in polar coordinates (ρ, τ) to $\tau = 2\pi k$, $k = 1, 2, \ldots n$, we get a flat cone C_n with angle deficit $2\pi(1-n)$

$$\boxed{\mathrm{Tr}\,\rho_A^n = Z[C_n]}$$

If we analytically continue to non-integer n

$$\boxed{S = -\mathrm{Tr}\,\rho_A \ln \rho_A = -(n\partial_n - 1)\ln \mathrm{Tr}\,\rho_A^n\Big|_{n=1}}$$

- Defining $W(n) = -\ln Z(n)$, with the polar coordinate τ periodic with period $2\pi n$ & $n \approx 1$, then

$$\boxed{S = (n\partial_n - 1)W(n)\Big|_{n=1}}$$

Susskind & Uglum Black Hole Entropy

(Phys Rev. D 50:2700, 1994)

- Consindered the Rindler entropy of the near-horizon region of a D-dim Schawarchild black hole

$$ds^2 = -\rho^2 dt^2 + d\rho^2 + \sum_{i=1}^{D-2} dx_i^2$$

- After analytic continuation to Euclidean time $\tau = -i t$

$$ds^2 = \rho^2 d\tau^2 + d\rho^2 + \sum_{i=1}^{D-2} dx_i^2$$

$$\boxed{\tau \sim \tau + \beta, \quad \beta = 2\pi}$$

$\Longrightarrow$ No conical singularity

The Semiclassical Approx. in Euclidean Canonical Quantum Gravity

- The partition function for canonical Q.G

$$Z(\beta) = N \int D[G] \int D[\phi] \, e^{-I[G,\phi]}$$

$$I[G,\phi] = I_{EH}[G] + I_{\phi}[\phi, G]$$

- Expand the metric around a saddle

$$G = \hat{G} + f$$

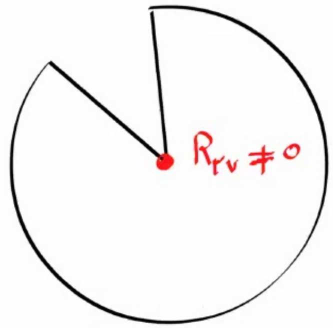
and expand $I[G,\phi] = [\hat{G} + f, \phi]$

$$= I_{EH}[\hat{G}] + I_{\phi}[\phi, \hat{G}] + I[f]$$

Then $Z(\beta)$ can be written as

$$Z(\beta) = e^{-\beta F} = e^{-I_{EH}[\hat{G}]} Z'$$

$$\boxed{Z' = N \int D[f] \int D[\phi] \, e^{-\left(I_{\phi}[\phi, \hat{G}] + I[f]\right)}}$$



- A conical manifold $\mathcal{M}_\beta$ is one where

$$\boxed{\tau \sim \tau + \beta} \quad \text{with}$$

Deficit angle: $\delta = 2\pi - \beta$, $\delta \lesssim 2\pi$

- At the conical tip, the Ricci scalar R is

$$\sqrt{G^{(2)}}\, R^{(2)} = 2(2\pi - \beta)\, \delta^2(x)$$

- Therefore the EH term is

$$I_{EH}[\hat{G}] = \frac{-(2\pi - \beta) A_\perp}{8\pi G_N} = \beta F$$

Entropy: $\boxed{S_{BH} = \beta^2 \frac{\partial F}{\partial \beta} = \frac{A_\perp}{4 G_N}}$

Entanglement Entropy as a 1-loop correction

arXiv. 1104.3712

- Consider a minimally coupled scalar

Computing $Z' = e^{-W'(G)}$ gives

$$W'(G) = -\frac{1}{192\pi^2\epsilon^2}\int_M R + \underbrace{Q(R)}_{\text{higher curvature}}$$

Renormalize G_B

$$\boxed{\frac{1}{G_R} = \frac{1}{G_B} + \frac{1}{12\pi\epsilon^2}}$$

- The leading UV divergence of entanglement entropy

$$\boxed{S = \frac{A(\Sigma)}{48\pi\epsilon^2}}$$

$$\Longrightarrow \boxed{S(G_R) = \frac{A(\Sigma)}{4G_R}}$$

Entanglement Entropy as a 1-loop correction

arXiv. 1104.3712

- Consider a minimally coupled scalar

Computing $Z' = e^{-W'(G)}$ gives

$$W'(G) = -\frac{1}{192\pi^2\epsilon^2}\int_M R + \underbrace{Q(R)}_{\text{higher curvature}}$$

Renormalize G_B

$$\boxed{\frac{1}{G_R} = \frac{1}{G_B} + \frac{1}{12\pi\epsilon^2}}$$

- The leading UV divergence of entanglement entropy

$$\boxed{S = \frac{A(\Sigma)}{48\pi\epsilon^2}}$$

$$\Longrightarrow \boxed{S(G_R) = \frac{A(\Sigma)}{4G_R}}$$

How does this scenario hold in

string theory ?

1. String partition function in conincal backgrounds.

2. Explicit counting of states

3. Renormalization of G_N

- String theory does NOT like this scenario

"Please do NOT take me off-shell"

Deficit angle: $\delta = 2\pi - \beta$, $\beta \lessgtr 2\pi$ ↳ arbitrary

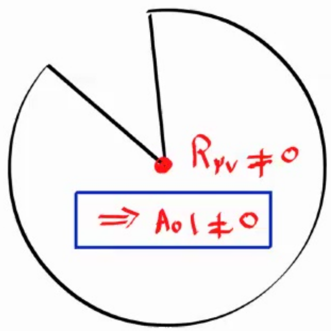
- At the cone tip, the Ricci scalar is

$$\boxed{\sqrt{G^{(2)}}\, R^{(2)} = 2(2\pi-\beta)\, \delta^2(x)}$$

And the EH action

$$\boxed{\int_{M_\beta} R = 2 A_\perp (2\pi-\beta)}$$

Can we compute $\int R$ from the worldsheet QFT?



nonzero graviton tadpole

Can we compute the string partition function in off-shell backgrounds?

The tree-level BH entropy in String Theory using the off-shell $\frac{\partial}{\partial \log \varepsilon}$ Tseytlin prescription.

$$
\begin{aligned}
S_{BH} &= \left(1-\beta\frac{\partial}{\partial\beta}\right)\left(\frac{\partial}{\partial\log\varepsilon}K_0\right)\Big|_{\beta=2\pi} \\
&= \left(1-\beta\frac{\partial}{\partial\beta}\right)(Z_0)\Big|_{\beta=2\pi} \\
&= \left(1-\beta\frac{\partial}{\partial\beta}\right)(-I_{EH})\Big|_{\beta=2\pi} \\
&= \left(1-\beta\frac{\partial}{\partial\beta}\right)\frac{2A_\perp}{8\pi G_N}(2\pi-\beta)\Big|_{\beta=2\pi} \\
&= \frac{A_\perp}{4G_N}
\end{aligned}
$$

$$G_N \sim (l_p)^{D-2} \sim g_s^2\,(l_s)^{D-2}$$

Can we compute the string partition function in off-shell backgrounds?

The tree-level BH entropy in String Theory using the off-shell $\frac{\partial}{\partial \log \varepsilon}$ Tseytlin prescription.

$$
\begin{aligned}
S_{BH} &= \left(1-\beta\frac{\partial}{\partial\beta}\right)\left(\frac{\partial}{\partial\log\varepsilon}K_0\right)\Big|_{\beta=2\pi} \\
&= \left(1-\beta\frac{\partial}{\partial\beta}\right)(Z_0)\Big|_{\beta=2\pi} \\
&= \left(1-\beta\frac{\partial}{\partial\beta}\right)(-I_{EH})\Big|_{\beta=2\pi} \\
&= \left(1-\beta\frac{\partial}{\partial\beta}\right)\frac{2A_\perp}{8\pi G_N}(2\pi-\beta)\Big|_{\beta=2\pi} \\
&= \frac{A_\perp}{4G_N}
\end{aligned}
$$

$$G_N \sim (l_p)^{D-2} \sim g_s^2 (l_s)^{D-2}$$

[arXiv 9404008098, 0111004]

The orbifold method is an alternative on-shell approach to calculating entropy in string theory.

- Target space is $\mathbb{R}^2/\mathbb{Z}_N \times \mathbb{R}^{D-2}$, N odd
- Deficit angle: $\delta = 2\pi - \beta = 2\pi\left(1-\frac{1}{N}\right)$

  The cone has opening angle $2\pi/N$
- Entropy $S_{g\geq 1} = \frac{d}{dN}\left(N Z_g(N)\right)\Big|_{N=1}$
- In Type II string theory, orbifold has twisted tachyons with $M_{T} = -\frac{4}{\alpha'}\left(1-\frac{k}{N}\right)$

  $k \in \{1, \ldots, N-1\}$

Edge Modes in String Theory ($\beta = 2\pi$)

- For an explicit state counting of $A/4G_N$ S&U proposed slicing the sphere along constant Euc. time s.t. the Hilbert space

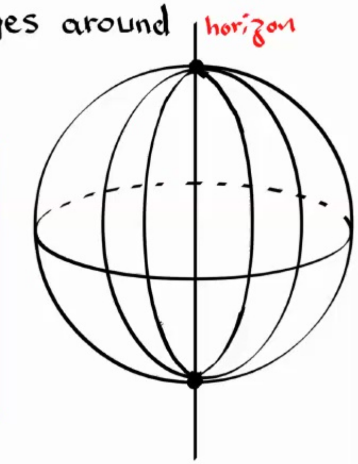
$$H \subseteq H_{in} \otimes H_{out}$$

- In Lorentzian signature, these would be the left & right wedges around bifurcation surface

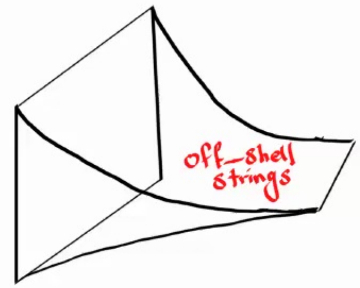
$$Z(\beta) \overset{?}{=} \mathrm{Tr}_{out}\, e^{-\beta K}$$

such that

$$S_{BH} \overset{?}{=} \mathrm{Tr}_{out}(\rho \log \rho^{-1})$$



(Off-shell) Strings in the Bulk ?



(1) Stringy LM proof of RT formula ?

$(AdS_3 \times S^3 \times T^4)_k$ for $k=1$?

(2) Replica trick in string theory ?

non-geometric backgrounds w/o U(1) symmetry ? ER = EPR ?

(3) α'-corrected and α'-exact backgrounds ?

(4) Factorization Puzzle ?

(5) Who knows ?

On-shell String Theory

The partition function in bosonic string

$$Z = \int \frac{[dX][dg]}{\text{Diff} \times \text{Weyl}} Z^{CFT}_{matter}[g]$$

- $\mathcal{L}_{CFT}$ is a Lagrangian that results in a conformally-inv. theory i.e. $c=0$ after including measure factors $\Rightarrow c=-26$
- Gauge fix Diff x Weyl to get b, c ghosts, fixing g up to moduli.
- Not really necessary for CFT to come from Lagrangian but simplifies discussion.

Suppose we would to derive for e.g.

the classical (tree-level) closed bosonic

string action I_0

$$I_0 = -\frac{1}{g_s^2}\int d^{26}X\sqrt{G}\,e^{-2\Phi}\left[4\nabla^2\Phi + R + \frac{1}{12}H^2 + O(\alpha')\right]$$

string coupling

dilaton

- Usual approach is to derive indirectly from $\beta_i = 0$ EOM conditions of the NLSM.

- Naively, I_0 should be obtained from a sphere diagram with zero vertex insertions!

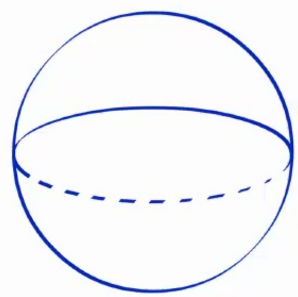
But this is a problem!

For genus-0 (sphere), CKG is $SL(2,\mathbb{C})$

$$\text{vol } SL(2,\mathbb{C}) = \infty$$

$$\downarrow$$

$$\boxed{I_0 = \frac{K_0}{\infty} = 0 \ (\text{mod boundary terms})}$$



$\sim O\left(\frac{1}{g_s^2}\right)$

- I_0 vanishes on-shell due to Φ EOM

$$-2e^{-2\Phi} = \partial_r \left(e^{-2\Phi} \frac{\delta \mathcal{L}}{\delta \partial_r \Phi} \right)$$

- We need to take the worldsheet off-shell.

Suppose we would to derive for e.g.

the classical (tree-level) closed bosonic

string action I_0

$$I_0 = -\frac{1}{g_s^2}\int d^{26}X \sqrt{G}\, e^{-2\Phi}\left[4\nabla^2\Phi + R + \frac{1}{12}H^2 + O(\alpha')\right]$$

string coupling

dilaton

- Usual approach is to derive indirectly from $\beta_i = 0$ EOM conditions of the NLSM.
- Naively, I_0 should be obtained from a sphere diagram with zero vertex insertions!

But this is a problem!

$$\boxed{-2\, e^{-2\Phi} \mathcal{L} = \partial_\mu \left(e^{-2\Phi} \frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} \right)}$$

- For an asymptotically flat solution, the boundary term at infinity using the lowest order terms in the action is

$$\boxed{\begin{aligned} I_0 &\sim -2 \int_{\partial M} d^{D-1}x \sqrt{h}\, e^{-2\Phi} K \\ &= -\frac{1}{8\pi G_N} \int_{\partial M} d^{D-1}y\, \partial_n \left(e^{-2\Phi} \sqrt{h} \right) \end{aligned}}$$

using the dilaton EOM!

- Is it possible to derive the sphere boundary term from the worldsheet?

- One way to define strings off-shell is string field theory.
- This talk follows a different approach, generalizing the work of Tseytlin which is a first-quantized NLSM approach.
- It takes the worldsheet theory to be a

QFT rather than CFT

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$(\beta_i \neq 0)$ $\qquad\qquad$ $(\beta_i = 0)$

$\boxed{\text{Is this allowed ?}}$

More Details On Gauge Fixing

On-shell

$$\langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle = \sum_T \int_{\mathcal{M}_T} \frac{\mathrm{d}^m t}{\Omega_{\mathrm{CKG}}} \int \frac{[\mathrm{D}\omega]}{\Omega_{\mathrm{Weyl}}} \Delta_{\mathrm{FP}} \left[e^{2\omega} \hat{g}(t)\right] \langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle^{\mathrm{M}}_{e^{2\omega}\hat{g}(t)}.$$

$$\langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle = \sum_T \int_{\mathcal{M}_T} \frac{\mathrm{d}^m t}{\Omega_{\mathrm{CKG}}} \cancel{\int \frac{[\mathrm{D}\omega]}{\Omega_{\mathrm{Weyl}}}} \Delta_{\mathrm{FP}} [\hat{g}(t)] \boxed{\langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle^{\mathrm{M}}_{\hat{g}(t)} e^{-(c_{\mathrm{M}}-26) S_L[g,\omega]}.}$$

$$\boxed{\begin{aligned} \langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle &= \sum_T \int_{\mathcal{M}_T} \frac{\mathrm{d}^m t}{\Omega_{\mathrm{CKG}}} \Delta_{\mathrm{FP}} [\hat{g}(t)] \langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle^{\mathrm{M}}_{\hat{g}(t)} \\ &= \sum_T \int_{\mathcal{M}_T} \frac{\mathrm{d}^m t}{\Omega_{\mathrm{CKG}}} \langle \mathscr{B}_1 \cdots \mathscr{B}_m \rangle^{\mathrm{gh}'}_{\hat{g}(t)} \langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle^{\mathrm{M}}_{\hat{g}(t)} \\ &= \sum_T \int_{\mathcal{M}_T} \frac{\mathrm{d}^m t}{\Omega_{\mathrm{CKG}}} \int [\mathrm{D}\Psi][\mathrm{D}b][\mathrm{D}'c] e^{-S_{\mathrm{CFT}}[\Psi,\hat{g}(t)] - S_{\mathrm{gh}}[b,c,\hat{g}(t)]} \\ &\qquad \times \mathscr{V}_1[\Psi, \hat{g}(t)] \cdots \mathscr{V}_n[\Psi, \hat{g}(t)] \mathscr{B}_1[b, \hat{g}(t)] \cdots \mathscr{B}_m[b, \hat{g}(t)]. \end{aligned} \qquad (1.32)}$$

Notice: $\int [\mathrm{D}'c]$ is ONLY over non-zero modes of "c" ghosts.

Off-shell

$$\langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle_\omega \equiv \sum_T \int_{\mathcal{M}_T} \frac{\mathrm{d}^m t}{\Omega_{\mathrm{CKG}}} \Delta_{\mathrm{FP}} \left[e^{2\omega} \hat{g}(t)\right] \boxed{\langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle^{\mathrm{M,QFT}}_{e^{2\omega}\hat{g}(t)},}$$

Gauge Fixing with zero modes

The n-point function with Δ^0_{FP} is

$$\langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle = \sum_T \int_{\mathcal{M}_T} \mathrm{d}^m t \Delta^0_{\mathrm{FP}} [\hat{g}(t)] \langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle^{\mathrm{M}}_{\hat{g}(t)}$$
$$= \sum_T \int_{\mathcal{M}_T} \mathrm{d}^m t \langle \mathscr{B}_1 \cdots \mathscr{B}_m \rangle^{\mathrm{gh}}_{\hat{g}(t)} \langle \mathscr{C}_1 \cdots \mathscr{C}_c \rangle^{\mathrm{gh}}_{\hat{g}(t)} \langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle^{\mathrm{M}}_{\hat{g}(t)}.$$

where

$$\mathscr{C}_i = \varepsilon_{ab} c^a(\tilde{\sigma}_i) c^b(\tilde{\sigma}_i)$$

$\tilde{\sigma}_i$ is gauge-fixed value of σ_i

On S^2:

$$\Delta^{\mathrm{SL}(2,\mathbb{C})}_{\mathrm{FP}}(\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3)^{-1} = 8 e^{-2\varphi(\tilde{\sigma}_1) - 2\varphi(\tilde{\sigma}_2) - 2\varphi(\tilde{\sigma}_3)}$$
$$\times \int_{\mathrm{SL}(2,\mathbb{C})} [\mathrm{d}A] \delta^2(\tilde{z}_1 - f_A(\tilde{z}_1)) \delta^2(\tilde{z}_2 - f_A(\tilde{z}_2)) \delta^2(\tilde{z}_3 - f_A(\tilde{z}_3)).$$

$$\Delta^{\mathrm{SL}(2,\mathbb{C})}_{\mathrm{FP}}(\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3)^{-1} = 8 \frac{e^{-2\varphi(\tilde{\sigma}_1) - 2\varphi(\tilde{\sigma}_2) - 2\varphi(\tilde{\sigma}_3)}}{|\tilde{z}_{12}|^2 |\tilde{z}_{23}|^2 |\tilde{z}_{31}|^2}.$$

Fix 3 points on-shell

- Off-shell, we can NOT fix 3 points:

We have deformed CFT $\longrightarrow$ QFT

$$\langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle_\omega \equiv \sum_T \int_{\mathcal{M}_T} \frac{\mathrm{d}^m t}{\Omega_{\mathrm{CKG}}} \Delta_{\mathrm{FP}}\left[e^{2\omega}\hat{g}(t)\right] \boxed{\langle \mathscr{V}_1 \cdots \mathscr{V}_n \rangle^{\mathrm{M,QFT}}_{e^{2\omega}\hat{g}(t)},}$$

- For $g \geq 1$, Ω_{CKG} is finite, so the deformation makes sense!

- It is only problematic on genus $g=0$ since Ω_{CKG} is ∞!

Consistent Definition of Off-shell Strings

$$Z(\omega) = \int \frac{[d\tau]}{CKG} \Delta_{FP}[\hat{g}(t) e^{2\omega}] \, Z^{QFT}_{matter}[e^{2\omega} \hat{g}(t)]$$

- Problem 1:

  Dependence on $\omega(z)$. Theory should NOT depend on choice of $\omega(z)$.

- Problem 2:

  How to make sense of non-compact $SL(2,C)$ CKG factor in non-conformal case?

It turns out both of these problems are solvable!

Problem 1 : Dependence on $\omega(z)$

The solution is to just pick $\omega(z)$ arbitrarily and show that different choices of ω lead to the same physics, related by a FIELD REDEFINITION of target space fields!

Field Redefs Primer

- To see why is so, consider any spacetime action (not necessarily string action) $I[\phi(x)]$ with $\delta I = \int d^D x \; E_i \, \delta\phi^i$ (+bdy term)
- Let us now do a 1st order field redef

$$\delta\phi^i = f^i[\phi] \iff \delta I = \int d^D x \; E_i f^i$$

Since the arrow is reversible, any variation $\propto$ EOM can be absorbed into field redef.

- But in string theory, EOM are linear combinations of β functions, which govern the dependence on scale $\omega(z)$

$$\frac{\delta}{\delta(\omega)} Z[\omega] = \langle\langle T(z) \rangle\rangle_\omega = \sum_a \beta^a \langle\langle O_a \rangle\rangle$$

- If we integrate along path $\omega_1 \longrightarrow \omega_2$, target differs by field redef:

1) If you pick ω_1 and I pick ω_2, our results are physically equivalent.

2) On the worldsheet, this is just

Renormalization !

Problem 2 :

- How to make sense of non-compact $SL(2,\mathbb{C})$ CKG factor in non-conformal case?

Solution :

$\boxed{\text{Gauge Orbits of } SL(2,\mathbb{C})}$

- Regularized $SL(2,\mathbb{C})$ volume
- Offshell Sphere prescription

Renormalization of the Möbius Volume

Jun Liu

and

Joseph Polchinski*

(Phys Lett B 203, 1,2 1988)

$$\boxed{\text{On-shell: } A_{0,0} = A_{0,1} = A_{0,2} = 0}$$

due to conformal invariance.

They noted that:

(1) The infinite volume can be offset

by divergent integration over positions of

ALL vertex operator insertions!

(2) $\mathrm{vol}\, SL(2,\mathbb{C}) = \underbrace{c_1 \epsilon^{-2}}_{3} + \underbrace{c_2 \ln \epsilon}_{2 \,\&\, 3}$

3 operators colliding

2 & 3 operators colliding

(3) This divergence of vol $SL(2,\mathbb{C})$ can be expressed as an IR divergence

$$\boxed{\frac{1}{L_0+\bar{L}_0} = \int_0^\infty ds\, e^{-s(L_0+\bar{L}_0)}}$$

is log divergent when $L_0+\bar{L}_0 = 0$

The Divergent Volume of SL(2,C)

- PSL$(2,\mathbb{C})$ group consists of translations, scalings and special conformal transform.

$$n_\alpha = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \quad a_\lambda = \begin{bmatrix} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{bmatrix}$$

$$n_\beta = \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}, \quad \begin{array}{l} \alpha, \beta \in \mathbb{C} \\ \lambda \in \mathbb{C}/[0, 2\pi i] \end{array}$$

- The Haar measure is given by

$$\int [dh] = \int_{\mathbb{C}} d^2\alpha \int_{\mathbb{C}} d^2\beta \int_{\mathbb{C}/[0,2\pi i]} d^2\lambda \, e^{-\lambda - \bar{\lambda}}$$

- Writing $\lambda = \frac{1}{2} \ln s + i\theta$,

$$\boxed{\int [dh] = \int_{\mathbb{C}} d^2\alpha \int_{\mathbb{C}} d^2\beta \int_0^\infty ds \int_0^{2\pi} d\theta}$$

which is clearly divergent !

(1) The infinite volume can be offset

by divergent integration over positions of

ALL vertex operator insertions !

$$\boxed{\begin{aligned}\int_{SL(2,\mathbb{C})}[dh] &= \int_{\mathbb{C}^3} d^2z_1 d^2z_2 d^2z_3 \frac{1}{|z_{12}|^2|z_{23}|^2|z_{31}|^2} \\ &= \int d^2\sigma_1 d^2\sigma_2 d^2\sigma_3 \sqrt{g(\sigma_1)}\sqrt{g(\sigma_2)}\sqrt{g(\sigma_3)} \frac{8}{C(\sigma_1,\sigma_2)C(\sigma_2,\sigma_3)C(\sigma_3,\sigma_1)}\end{aligned}}$$

$$C(\sigma_1, \sigma_2) = e^{\omega(\sigma_1)+\omega(\sigma_2)} |z_{12}|^2$$

is the conformal distance.

The Off-shell Prescription in a Nutshell

(1) 2d QFT NLSM + UV cutoff ϵ

(2) Compute partition function K_0

to some order in α' and $\log \epsilon$

by integrating over All insertions

We get Two types of log divergences:

- $SL(2,\mathbb{C})$ volume
- physical exchanges along internal prop.

– Log divergences in sigma model QFT

correspond to massless poles of a

low-energy EA in target space

(3) Use the local RG equation to renormalize

$$\boxed{\frac{d K_0}{d \log \epsilon} = \frac{\partial K_0}{\partial \log \epsilon} - \beta^i \frac{\partial K_0}{\partial \phi^i} = 0}$$

The Off-shell Sphere Action

$$\boxed{T1: \quad Z_0 = \frac{\partial}{\partial \log G} K_0}$$

T1 is a prescription for constructing off-shell low energy tree-level (classical) effection action for bosonic massless String modes perturbatively in α'.

around flat space vacuum

$$G_{\mu\nu} = \delta_{\mu\nu}\ , \quad \Phi = \text{const}\ , \quad D=26$$

- The prescription says the correct method to substract (renormalize) the log divergences is

$$\boxed{\text{NOT } \frac{1}{\log \epsilon} \quad \text{But } \frac{\partial}{\partial \log \epsilon}}$$

- n-point correlators has the following expansion

$$\boxed{K_{0,n} = \langle\langle V_{0,1} \ldots V_{0,n} \rangle\rangle = \sum_{m=1}^{n-1} \underbrace{B_n^{(m)}}_{\text{n-point Beta functions}} \ln^m \epsilon}$$

while n-point string amplitudes

$$\boxed{A_{0,n} = \frac{\partial}{\partial \log \epsilon} K_{0,n} = \sum_{m=1}^{n-2} B_n^{(m)} \ln^m \epsilon}$$

The sphere PF K_0 is expressed as

$$\boxed{K_0 \propto e^{1/\frac{D-26}{6} \cdot 2 \ln \epsilon} \int d^D y \sqrt{G}\, e^{-2\Phi} \left[1 + \frac{1}{2} \alpha' \ln \epsilon \left(R + 2 \nabla^2 \Phi \right) + \ldots \right]}$$

$$I_0^{T1} = \frac{2}{g_s^2} \int d^D x \sqrt{G}\, e^{-2\Phi}\, \tilde{\beta}^{\Phi}$$

where $\boxed{\tilde{\beta}^{\Phi} = \beta^{\Phi} - \frac{1}{4} G^{\mu\nu} \beta^{G}_{\mu\nu}}$

$$= c_0 - \frac{1}{4}\alpha' \left(R + 4\nabla^2\Phi - 4\partial_\mu\Phi\partial^\mu\Phi\right) + O(\alpha'^2)$$

& $c_0 = \frac{1}{6}(D-26)$ ($c_0 = 0$ for a string background)

is called the central charge action since

(1) it is leading order (in α')

$$\boxed{2\pi \langle\langle T \rangle\rangle = \tilde{\beta}^{\Phi} R^{(2)} + \cdots}$$

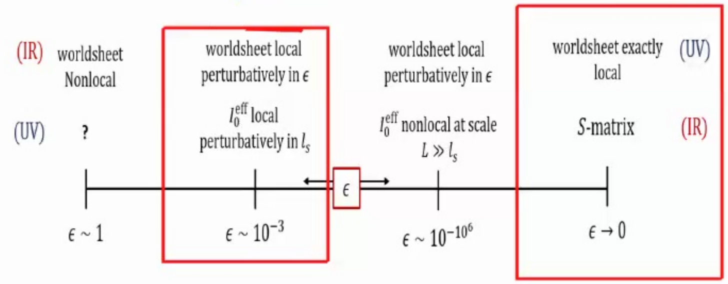
and is equal to the central charge of the CFT at the conformal point where $\beta^G = 0$ and $\tilde{\beta}^{\Phi} = \beta^{\Phi}$

(2) The EOM of I_0^{T1} are equivalent to

$$\boxed{\beta^G = 0 \quad \& \quad \beta^{\Phi} = 0 .}$$

Locality Scale: Role of UV cutoff

Chapter "6" Chapter "5"



Local action:	S-matrix:
• finite, real ϵ	• Complex $\epsilon \to 0$
• $n = 1, 2$	• $n \geq 3$
• away from poles	• near poles
from degeneration limit	near deg. limit
• Generic momenta	• Special momenta

II. Results of Our Work

Result # 1 [Chapter 6]

(1) Show that Tseytlin's prescriptions

$$\boxed{\begin{aligned} T1: \quad & Z_0 = \frac{\partial}{\partial \log \varepsilon} K_0 \\ T2: \quad & Z_0 = \left(\frac{\partial}{\partial \log \varepsilon} + \frac{1}{2}\frac{\partial}{(\partial \log \varepsilon)^2}\right) K_0 \end{aligned}}$$

for constructing the tree-lere (classical) off-shell low energy effective action from the worldsheet QFT is valid, to All orders "n" in conformal perturbation theory and to ALL orders in α'

Conformal Perturbation Theory

$$\mathcal{L} = \mathcal{L}_{CFT} + \sum_a \epsilon^{2(h_a-1)} \phi^a O_a$$

O_a → operators

$\mathcal{L}_{CFT}$: $c=0$

ϕ^a: a vector of fields in coupling space

$$V_a = \int d^2z \sqrt{g}\, O_a$$

$(h,\bar{h}) = (1,1)$ OR $(h,\bar{h}) \neq (1,1)$

marginal irrelevant

An n-point correlator on a genus-g W.S.

$$K_g(\tau) = \sum_{n=0}^{\infty} K_{g,n}(\tau)$$

$$\boxed{K_{g,n} = \sum_{a_1,\dots,a_n} \left(\prod_n \epsilon^{2(h_a-1)} \phi^a \right) \frac{1}{n!} \langle\langle V_{a_1} \cdots V_{a_n} \rangle\rangle_{CFT}}$$

Result #2 [Chapter 5]

In the limit that $\epsilon \to 0$, the TI FT2 prescriptions recover the Euclidean String S-matrix.

We also obtain the Lorentzian S-matrix by integrating over complex values of the UV cutoff ϵ and recover Witten's $i\varepsilon$ prescription for the internal propagator poles.

Result #3 [Chapter 5]

In the S-matrix regime $\log \epsilon^{-1} \to \infty$, we show that the effect of applying T1 or T2 is equivalent to modding out by the gauge orbits of $SL(2,\mathbb{C})$ acting on the position of "n" punctures.

IV. Gauge Orbits [Chapter 5]

- Define the pre-moduli space $PM_{0,n}$ as the space of possible insertion positions $(z_1, \dots, z_n)$ of vertex operators on the sphere with $z_i \neq z_j$

Then

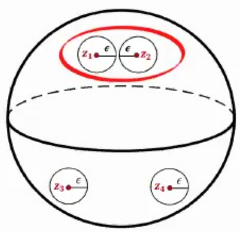
$$M_{0,n} := PM_{0,n} / SL(2,\mathbb{C})$$

with

$$\dim_{\mathbb{R}}(M_{0,n}) = 2n-6$$

- Now, we regulate $M_{0,n}$ with cutoff ϵ.

$PM^{(\epsilon)}_{0,n}$ is the regulated pre-moduli space satisfying the condition that NO TWO operator insertions are closer than 2ϵ on the sphere



$$|z_1 - z_2| \geq 2\epsilon$$

- Conformal invariance guarantees that the CFT amplitude density of conformal primaries

$$\left(\prod_n d^2 z_n\right) \langle\langle P_{i_1}(z_1) \dots P_{i_n}(z_n) \rangle\rangle$$

is invariant under $SL(2,\mathbb{C})$ action acting on ALL points simultaneously.

- $P\mathcal{M}^{(\epsilon)}_{0,n}$ is NOT invariant under $SL(2,\mathbb{C})$.

But, any TWO elements of $P\mathcal{M}^{(\epsilon)}_{0,n}$ can be related by an $SL(2,\mathbb{C})$ transformation if they lie on the same $SL(2,\mathbb{C})$ orbit

$$\boxed{\mathcal{M}^{(\epsilon)}_{0,n} := P\mathcal{M}^{(\epsilon)}_{0,n} / SL(2,\mathbb{C})}$$

Elements of $\mathcal{M}^{(\epsilon)}_{0,n}$ are gauge orbits Ω, equivalence classes!

Do we divide or quotient out by Vol(Ω)?

- Vol(Ω) is different for each $\Omega \in \mathcal{M}^{(\epsilon)}_{0,n}$ w.r.t. Haar measure on SL(2,C)
- Vol(Ω) depends on number of insertions "n" AND on the regulated cross ratio which, e.g. for $n=4$ is

$$\Psi = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_4-z_2)}$$

Normally $\Psi = \infty$ when $z_1 \to z_3$ or $z_4 \to z_2$

But $\Psi^{(reg)} < \frac{1}{4\epsilon^2}$ for $\mathcal{M}^{(\epsilon)}_{0,4}$

- Therefore, dividing is NOT the correct thing to do

$$Z[\omega] = \frac{\int [dx][d\tau][db][dc]}{Vol(SL(2,C))} Z_{QFT}$$

- One would have to compute Vol(Ω) for EACH gauge orbit Ω, which is not realistic!

- To show that, first calculate $\mathrm{Vol}(\Omega)$ of $\mathcal{M}^{(\epsilon)}_{0,n}$.

  The divergence in $SL(2,\mathbb{C})$ comes from the

  hyperbolic 3-manifold

$$\boxed{\frac{SL(2,\mathbb{C})}{SU(2)} = H_3}$$

- The cutoff ϵ is invariant under $SU(2)$
- Take H_3 to have unit curvature radius
- Pick a point "p", the origin of H_3
- Trace hyperbolic geodesic from a ray "r" in any one direction from "p".
- The ray "r" is parametrized by λ

Then, the regulated volume per solid angle is

$$\boxed{\mathrm{Vol}(\Omega) = \int_0^{\log(a/\epsilon) + O(\epsilon^2)} d\lambda \, \sinh^2(\lambda) = \frac{a^2}{8}\epsilon^{-2} \boxed{+\frac{1}{2}\log\epsilon} + b + O(\epsilon^2)}$$

Universal

$a \gg \epsilon$.

a & b depend on Ω, p & r $\Longrightarrow$ non-universal

Acting with T2 on $\mathrm{Vol}(\Omega)$ gives

$$\boxed{\lim_{\epsilon \to 0} \left(\frac{\partial}{\partial \log\epsilon} + \frac{1}{2} \frac{\partial^2}{(\log\epsilon)^2} \right) \mathrm{Vol}(\Omega) \propto 1}$$

- But this "1" is true for ALL gauge orbits, then the effect of T2 is mod out by the gauge symmetry!

VI. Ongoing Work & Future Directions

(1) Stringy LM proof of RT formula ?

$(AdS_3 \times S^3 \times T^4)_k$ for $k=1$?

(2) Replica trick in string theory ?

non-geometric backgrounds w/o U(1)

symmetry ? ER = EPR ?

(3) String WdW equation & T^2 Deformation

(4) Lorentzian backgrounds ?

(5) What is the boundary term of the

sphere partition function ?

THANK YOU!