Title: Integrable systems from Calabi-Yau categories

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Abstract: I will describe a general categorical approach to constructing Hamiltonian actions on moduli spaces. In particular cases, this specializes to give a ``universal" Hitchin integrable system as well as the Calogero-Moser system. Moreover, I will describe a generalization to higher dimensions of a classical result of Goldman which says that the Goldman Lie algebra of free loops on a surface acts by Hamiltonian vector fields on the character variety of the surface. A key input is a description of deformations of Calabi-Yau structures, which is of independent interest. This is joint work with Chris Brav.

Zoom link: https://pitp.zoom.us/j/92929253744?pwd=WGFNQmRJck5NdzFFdU8xcXRlN3RRQT09

Integrable systems from Calabi–Yau categories.

jt w/ C. Brav

2 Motivating examples:

1) Hitchin system: X Riemann surface (= algebraic curve)

$\mathrm{Higgs}_n :=$ rk n - Higgs bundles

$\{ V, \ \phi : V \to V \otimes \Omega^1)$

$$h: \mathrm{Higgs}_n \longrightarrow \bigoplus_{k=0}^{n} H^0(X, (\Omega^1)^{\otimes k})$$

symplectic stack.

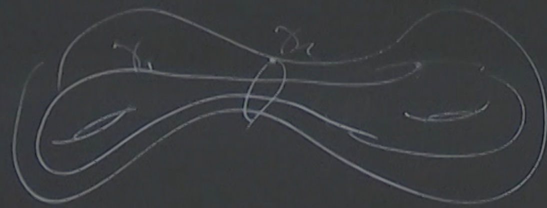
$$(V, \phi) \longmapsto \bigoplus_k \mathrm{tr}(\Phi^k)$$

2) Σ Riemann surface

character stack: $\mathrm{Char}_n(\Sigma)$

also a symplectic stack

Goldman: defined a Lie alg. generated by h'topy classes of free loops on Σ



$$[\gamma_1, \gamma_2] = \sum_{x \in \gamma_1 \cap \gamma_2} \pm 1 \cdot \gamma_1 \#_x \gamma_2$$

<u>Thm</u> (Goldman): The map

$$\mathcal{L} \longrightarrow \mathcal{O}_{\mathrm{Char}_n(\Sigma)}$$ "trace of holonomy"

Goldman Lie alg

is a map of Lie algebras.

Non-commutative geometry

$\mathcal{C}$ dg category

ex: 1) $\mathcal{C} = QCoh(T^*X)$

2) $\mathcal{C} = Rep(\pi_1 \Sigma)$ (if $g>0$)

(in general $C_*(\Omega\Sigma)$-mod)

Toën - Vaquié defined a moduli space of objects:

$\mathcal{M}_{\mathcal{C}}$ "derived stack"

parametrizes "finite dimensional objects in $\mathcal{C}$"

Examples:

1) $\mathcal{C} = \mathrm{QCoh}(T^*X)$

$$\mathcal{M}_{\mathcal{C}} = \{ \mathcal{E} \, , \; \Phi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1 \}$$

perfect complex

2) K connected top space

$\mathcal{C} = C_*(\Omega K)$ - mod

$\mathcal{M}_{\mathcal{C}}$ = "perfect local systems"

Pantev - Toën - Vezzosi - Vaquié defined the notion of n-shifted symplectic structure on a derived stack.

<u>Thm</u> (Brav - Dyckerhoff): Suppose $\mathcal{C}$ is smooth d-CY category then $\mathcal{M}_{\mathcal{C}}$ has a $(2-d)$-shifted symplectic structure.

<u>Ex.</u> 1) Y d-dim smooth CY variety
$\mathcal{C} = QCoh(Y)$ is a smooth d-CY category.

2) M cpct oriented d-dim'l mfld
$\mathcal{C} = C_*(\Omega M)$-mod is a smooth d-CY category.

If $\mathcal{M}$ is an n-shifted symplectic stack

$R\Gamma(\mathcal{M}, \mathcal{O}_{\mathcal{M}})[n]$ has a Poisson bracket.

<u>Goal</u>: understand this Lie algebra.

Given a DG category, have "Hochschild invariants" "forms"

$S^1 \circlearrowright HH_*(\mathcal{C})$ Hocshild homology

$\rightsquigarrow HC^-(\mathcal{C}) = HH(\mathcal{C})^{hS^1}$ neg. cyclic homology

$HC(\mathcal{C}) := HH(\mathcal{C})_{hS^1}$

$HP(\mathcal{C}) := HH(\mathcal{C})^{tS^1}$

$\exists$ natural map:

for any DG category $\mathcal{C}$ "trace of holonomy"

$$HC(\mathcal{C}) \longrightarrow \mathcal{O}_{\mathcal{M}_{\mathcal{C}}}$$

Examples: A alg

$$HC_0(A) = A/[A,A]$$

If $A = \mathbb{C}[\pi_1 \Sigma]$

$HC_0(A) =$ vector space gen. by free loops up to h'topy

Thm (Brav–Dyckerhoff) If $\mathcal{C}$ is a smooth d-CY dg category, then $HC(\mathcal{C})[2-d]$ has a natural structure of a dg Lie alg and the map

$$\underbrace{HC(\mathcal{C})[2-d]}\longrightarrow \mathcal{O}_{\mathcal{M}_{\mathcal{C}}}[2-d]$$

is naturally a map of Lie algebras.

How to prove this?

Recall: Lie algebras $\rightsquigarrow$ formal moduli problems.

Folklore thm (Gerstenhaber, ..., Lurie)

deformations of $\mathcal{C}$ are governed by the Lie alg $HH^*(\mathcal{C})[1]$

Thm (Brav – R.)

Let $\mathcal{C}$ be a smooth d-CY DG category

1) defs. of $\mathcal{C}$ as a CY category are governed by Lie alg.

$$HC^-(\mathcal{C})[1-d]$$

2) "exact" deformations of $\mathcal{C}$ as a CY category are governed by $HC(\mathcal{C})[2-d]$

exact CY deformation.

CY structure $\Theta \in HC^{-}_{n}(\mathcal{C}) \xrightarrow{(\ )} HP(\mathcal{C})$

exact CY deformation : CY deformation keeping the class $[\Theta]$ constant.

$HP(\mathcal{C}) := HH(\mathcal{C})$

Hitchin system:

$\mathcal{C} = QCoh(T^*X)$ 2-CY DG cat.

$HC(\mathcal{C}) \longrightarrow \mathcal{O}_{M_{\mathcal{C}}}$ map of Lie algebras

polyvector fields

some symmetries of $QCoh(T^*X)$ are given by tensoring by line bundles.

Lie algebra: $R\Gamma(X, \mathcal{O})[1] \longrightarrow HH^*(QCoh(T^*X))[1]$

Easy calculation.

this lifts to a map:

$$R\Gamma(T^*X, \mathcal{O})(1) \to \underline{HC(QCoh(T^*X))} \quad \text{(Serre duality)}$$

$$H^0(R\Gamma(T^*X, \mathcal{O})(1)) = H^1(T^*X, \mathcal{O}) = \text{functions on Hitchin base}$$

Q quiver

$\bar{Q}$ doubled quiver

S set of edges

$$\underset{S}{\bigoplus} k(t) \xrightarrow{m} k[\bar{Q}] \qquad t \to \sum_i e_i e_i^* - e_i^* e_i$$

1-CY

Obs: relative 2-CY str.

rel version of thm:

$HC(k(\bar{Q}))$ is a Lie alg. "necklace Lie alg"

$$
\begin{tikzcd}
\bigoplus k[t] \arrow[r] \arrow[d, "t \mapsto q"'] & k[\bar{Q}] \arrow[d] \\
\bigoplus k \arrow[r] & \tilde{\Pi}_q
\end{tikzcd}
$$

$\leftarrow$ deformed pre proj. alg.

$\uparrow$ 2CY

Upshot:

$HC(k[\bar{Q}])$ acts by Hamiltonian vector fields on $\mathcal{M}_{\hat{\Pi}_q}$.