Title: Asymptotic Joint Realizability of Properties of Quantum States

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Abstract: There are numerous properties of quantum states that one might be interested in characterizing, including statistical moments of observables such as expectations or variances, or more generally purities, entropies, probabilities, eigenvalues, symmetries, marginals, etc. Given a fixed collection of properties, the realizability problem aims to determine which value-assignments to those properties are jointly exhibited by at least one quantum state. In addition to the decision problem of realizability, one might also be interested in quantifying what proportion of quantum states possesses those property values.

Any property of a quantum state can always, at least in principle, be estimated empirically by suitably measuring an ensemble of many independently and identically prepared copies of that quantum state. The particular sequence of positive operator valued measures which estimates a given property is known as a property estimation scheme. The purpose of this talk is to discuss a strategy for tackling realizability problems by studying the large deviation behaviour of property estimation schemes.

The key idea of this approach is the following:
A given collection of properties is realized by a quantum state if and only if a random quantum state occasionally produces that collection of properties as estimates.
Under suitable conditions, this observation leads to a complete hierarchy of necessary conditions for realizability.

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Asymptotic realizability of properties of quantum states

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Quantum Foundations Talk
April 4th, 2023

A Concrete Example

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Pauli-X and Pauli-Z realizability

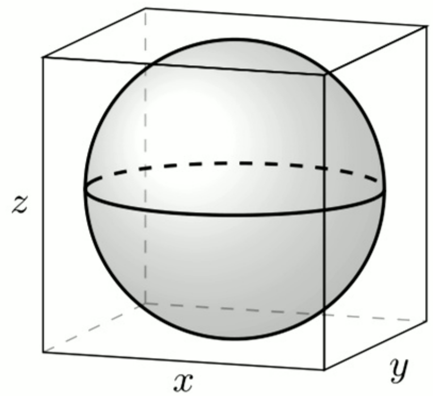
- pure qubit states $\mathcal{S} \simeq \mathcal{P}(\mathbb{C}^2)$
- interested in Pauli-X and Pauli-Z expectation values

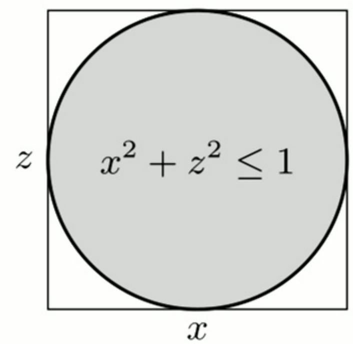
$$f_X(\psi) = \langle\psi|\sigma_X|\psi\rangle \quad \text{and} \quad f_Z(\psi) = \langle\psi|\sigma_Z|\psi\rangle$$

- Question: Which expectation values are jointly realizable?

$$(x, z) \overset{?}{\in} \{(f_X(\psi), f_Z(\psi)) \mid \psi \in \mathcal{P}(\mathbb{C}^2)\}$$

- Answer:

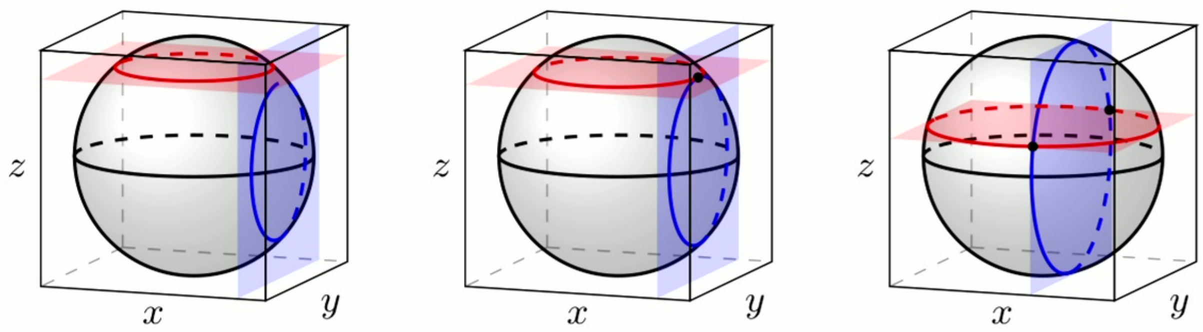




Joint realizability and intersections

- Consider fibers of f_X and f_Z

$$f_X^{-1}(x) = \{\psi \mid f_Z(\psi) = x\}$$
$$f_Z^{-1}(z) = \{\psi \mid f_Z(\psi) = z\}$$



(g) $(x,z) = (\frac{3}{4}, \frac{3}{4})$ $x^2 + z^2 = \frac{9}{8}$

(h) $(x,z) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ $x^2 + z^2 = 1$

(i) $(x,z) = (\frac{1}{4}, \frac{1}{4})$ $x^2 + z^2 = \frac{1}{16}$

- Another answer: (x,z) are jointly realizable if and only if

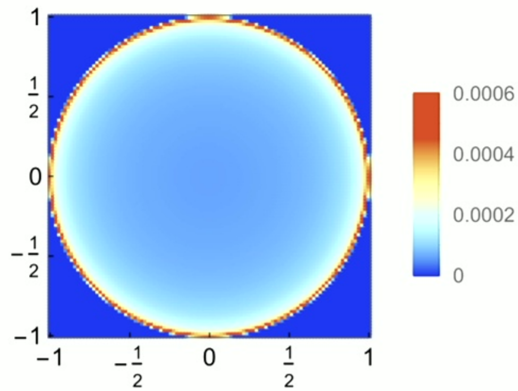
$$f_X^{-1}(x) \cap f_Z^{-1}(z) \neq \emptyset.$$

Quantitative realizability

- given uniform measure μ over $\psi \in \mathcal{P}(\mathbb{C}^2)$

$$\mathrm{d}\mu(\theta, \phi) = \frac{1}{4\pi} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

- Question: What proportion of states have $(f_X, f_Z)(\psi) \in \Delta \subset \mathbb{R}^2$?
- Answer: The pushforward measure $(f_X, f_Z)_* \mu$:



- probability *density* is given by (Zhang et al., 2022)

$$\frac{\mathrm{d}((f_X \times f_Z)_* \mu)}{\mathrm{d}x \, \mathrm{d}z} = \begin{cases} \frac{1}{2\pi\sqrt{1-x^2-z^2}} & x^2 + z^2 < 1 \\ 0 & x^2 + z^2 > 1 \end{cases}.$$

Realizable Properties of Quantum States

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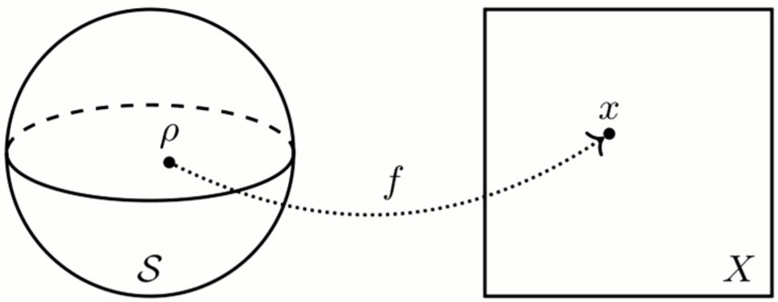
Abstract properties

Definition

A property of quantum states is a measurable function,

$$f : \mathcal{S} \to X$$



- $\mathcal{S}$ either pure or mixed states on $\mathcal{H} \cong \mathbb{C}^d$,
- X is a Polish space (a metrizable topological space that is separable and complete).

Examples of properties

property	f	$f(\rho)$	X
expectation value	$\langle \hat{O} \rangle$	$\langle \hat{O} \rangle_\rho = \mathrm{tr}[\rho \hat{O}]$	$\mathbb{R}$
event probability	$\mathrm{tr}[\Pi \cdot]$	$\mathrm{tr}[\Pi \rho]$	$[0,1]$
von Neumann entropy	S	$-\mathrm{tr}[\rho \ln \rho]$	$\mathbb{R}_{\geq 0}$
reduced state	$\mathrm{tr}_{\mathcal{B}}$	$\mathrm{tr}_{\mathcal{B}}[\rho_{\mathcal{AB}}] = \rho_{\mathcal{A}}$	$\mathcal{D}(\mathcal{A})$
eigenvalues	spec	$\mathrm{spec}(\rho) = \{\lambda_1, \ldots, \lambda_d\}$	$\Delta_d^{\downarrow}$
moment map	μ	$A \mapsto \mathrm{tr}(\rho \Phi(A))$	$\mathfrak{g}^*$
purity	pur	$\mathrm{pur}(\rho) = \mathrm{tr}[\rho^2]$	$[0,1]$
the whole state	id	ρ	$\mathcal{S}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The category of properties

- compose: $f : \mathcal{S} \to X$ and $g : X \to Y$

$$g \circ f : \mathcal{S} \to Y$$

- product: $f : \mathcal{S} \to X_1$ and $g : \mathcal{S} \to X_2$

$$(f, g) : \mathcal{S} \to X_1 \times X_2, \quad (f, g)(\rho) = (f(\rho), g(\rho))$$

- factorize: $h : \mathcal{S} \to X_1 \times X_2$,

$$(p_1 \circ h) : \mathcal{S} \to X_1, \quad (p_2 \circ h) : \mathcal{S} \to X_2$$

- a collection of properties is a property

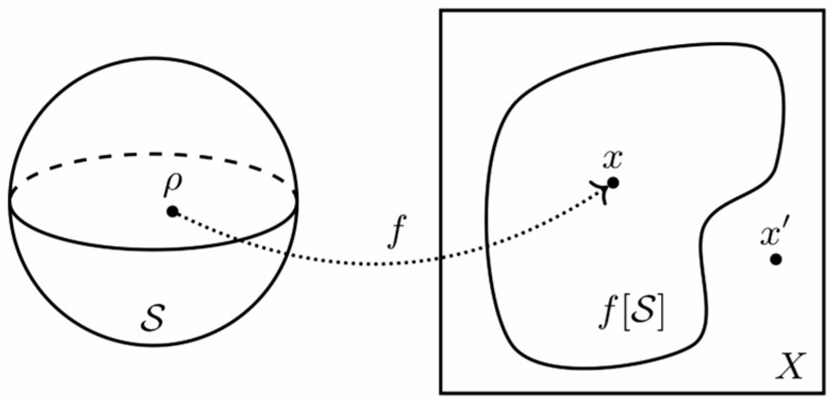
General realizability problems

Definition (Realizability Problem)

Realizable region of a property $f : \mathcal{S} \to X$

$$f[\mathcal{S}] = \{x \in X : \exists \rho \in \mathcal{S}, f(\rho) = x\} \subseteq X.$$

Realizability problem is to *characterize* $f[\mathcal{S}]$.



$x \in f[\mathcal{S}]$ is realizable and $x' \notin f[\mathcal{S}]$ is unrealizable.

Examples of realizability

- uncertainty relations

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$$

- necessary condition for realizability problem of

$$(\Delta\hat{A}, \Delta\hat{B}, |\langle[\hat{A},\hat{B}]\rangle|) : \mathcal{S} \to \mathbb{R}^3_{\geq 0}$$

- quantum marginals problem realizability problem of

$$(\mathrm{tr}_A, \mathrm{tr}_B, \mathrm{tr}_C) : \mathcal{S} \to \mathcal{D}_{BC} \times \mathcal{D}_{AC} \times \mathcal{D}_{AB}$$

- quantum entropy inequalities realizability problem for

$$S(\rho_{\mathcal{ABC}}) + S(\rho_{\mathcal{B}}) \leq S(\rho_{\mathcal{AB}}) + S(\rho_{\mathcal{BC}})$$

- realizable region is moduli space $\mathcal{S}/G$ of G-invariants $f_i : \mathcal{S} \to \mathbb{C}$

$$(f_1, \ldots, f_k)\,[\mathcal{S}] \cong \mathcal{S}/G$$

- interconversion in resource theories: reversible-irreversible decomposition of free operation $F = I \circ R$

$$F(\sigma) = \sigma' \iff (\sigma, \sigma') \in (R^{-1}, I)[\mathcal{S}]$$

Techniques for realizability

- loads of techniques
- if $\mathcal{S} = \mathcal{D}(\mathcal{H})$ and $f = \mathcal{S} \to \mathbb{R}^k$ is *linear* function $f(\rho) = \langle F, \rho \rangle$, then realizability is a SDP feasibility problem
- if $f : \mathcal{S} \to \mathbb{C}^k$ has *polynomial* components $f_i(\rho)$, in non-linear quantifier elimination from computational algebraic geoemtry applies
- this talk: an asymptotic approach based on estimation theory

Estimating Pauli-X expectation value

- $f_X(\psi) = \langle\psi|\sigma_X|\psi\rangle$ approximated by measuring eigenspaces of σ_X
- For $n \in \mathbb{N}$, define:

$$V_{n;\hat{x}} := \{|\phi\rangle \in (\mathbb{C}^2)^{\otimes n} \mid \exp(i\theta\sigma_X)^{\otimes n} |\phi\rangle = \exp(i\theta n\hat{x}) |\phi\rangle\} \subseteq (\mathbb{C}^2)^{\otimes n}$$

- Define f_X-estimation scheme $E_n^X : \Sigma([-1,+1]) \to \mathcal{E}((\mathbb{C}^2)^{\otimes n})$

$$E_n^X(\Delta) = \sum_{\hat{x}\in\Delta} P_{V_{n;\hat{x}}}$$

- Probability of obtaining estimate $\hat{x} \in \Delta$

$$\mathrm{tr}(E_n^X(\Delta)\psi^{\otimes n})$$

- Explicit form $k = n(\hat{x}+1)/2$ and $p = (x+1)/2$

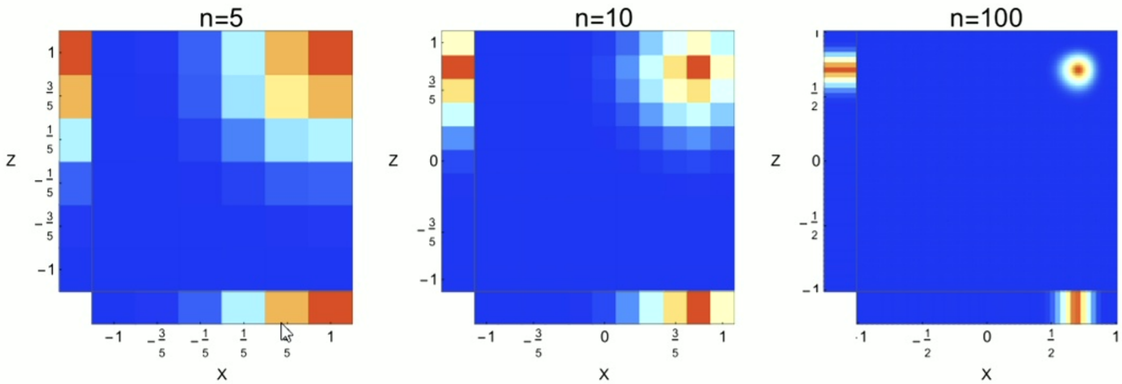
$$\mathrm{tr}(E_n^X(\{\hat{x}\})\psi^{\otimes n}) = \binom{n}{k} p^k (1-p)^{n-k}$$

Jointly estimating Pauli-X and Pauli-Z expectation values

- $f_X(\psi)$ and $f_Z(\psi)$ jointly estimated by $E_n^X(\Delta_X) \otimes E_n^Z(\Delta_Z)$
- Joint probability of obtaining estimate $(\hat{x}, \hat{z}) \in \Delta_X \times \Delta_Z$

$$\mathrm{tr}(E_n^X(\Delta_X) \otimes E_n^Z(\Delta_Z)\psi^{\otimes 2n})$$

- Example: if $(f_X(\psi), f_Z(\psi)) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ distribution of estimates



Property Estimation Schemes

Definition ((Keyl, 2006))
Let $f : \mathcal{S} \to X$ be a property. A sequence $E = (E_n)_{n\in\mathbb{N}}$ of POVMs

$$E_n : \Sigma(X) \to \mathcal{E}(\mathcal{H}^{\otimes n})$$

is an *f-estimation scheme* if for all $\rho \in \mathcal{S}$ and bounded continuous $g : X \to \mathbb{R}$

$$\lim_{n\to\infty} \int_X g(x)\mathrm{tr}(E_n(\mathrm{d}x)\rho^{\otimes n}) = \int_X g(x)\delta_{f(\rho)}(\mathrm{d}x) = g(f(\rho))$$

where $\delta_{f(\rho)}$ is the Dirac measure at $f(\rho)$

$$\delta_{f(\rho)}(\Delta) = \begin{cases} 1 & f(\rho) \in \Delta \\ 0 & f(\rho) \notin \Delta \end{cases}$$

Examples of property estimation schemes

Property	Estimation Scheme
Is $\|\phi\rangle$?	$E_n(\{\text{Yes}\}) = \|\phi\rangle\langle\phi\|^{\otimes n}$, $E_n(\{\text{No}\}) = \mathbb{1}^{\otimes n} - \|\phi\rangle\langle\phi\|^{\otimes n}$
Is Pure?	$E_n(\{\text{Yes}\}) = P_{(n)}$, $E_n(\{\text{No}\}) = \mathbb{1}^{\otimes n} - P_{(n)}$
Yes iff $\rho^2 = \rho$	$P_{(n)}$ is projector onto $\text{Sym}^n(\mathbb{C}^d) \subseteq (\mathbb{C}^d)^{\otimes n}$
Expectation	$E_n(\{\hat{v}\}) = \sum_{\{n_i\}\| \sum_i n_i v_i = n\hat{v}} \frac{n!}{\prod_i n_i!} T_n\{\bigotimes_{i=1}^d \|v_i\rangle\langle v_i\|^{\otimes n_i}\}$
$\text{tr}(V\rho)$	$V = \sum_{i=1}^d v_i \|v_i\rangle\langle v_i\|$, T_n sum over permutations
Spectra	$F_n(\{\frac{\lambda}{n}\}) = P_\lambda$
$\text{spec}(\rho)$	P_λ projector onto $V_\lambda \otimes M_\lambda^d$
State	$Q_n(\{\frac{\lambda}{n}\} \times \text{d}U) = \dim(\mathcal{V}_\lambda^d)\mathbb{1}_{\mathcal{W}_\lambda} \otimes \|\phi_\lambda^U\rangle\langle\phi_\lambda^U\| \,\text{d}U$
$\text{id}(\rho)$	$\|\phi_\lambda^U\rangle = U\|\phi_\lambda\rangle$ twirled highest weight vector in $\mathcal{V}_\lambda^d$
Marginal	$E_n(\text{d}\rho_A) = Q_n(\text{d}\rho_A) \otimes \mathbb{1}_B^{\otimes n}$
$\text{tr}_B(\rho_{AB})$	where Q_n is the state estimation scheme above
Continuous f	$F_n(\text{d}x) = Q_n(f^{-1}[\text{d}x])$
$f(\rho)$	where Q_n is the state estimation scheme above

Joint estimation schemes

property	estimation scheme
$f : \mathcal{S} \to X$	$F_n : \Sigma(X) \to \mathcal{E}(\mathcal{H}^{\otimes n})$
$g : \mathcal{S} \to Y$	$G_n : \Sigma(Y) \to \mathcal{E}(\mathcal{H}^{\otimes n})$

Lemma (Commuting)

If F_n and G_n commute, i.e. $[F_n(\Delta_X), G_n(\Delta_Y)] = 0$, then

$$E_n(\Delta_X \times \Delta_Y) \coloneqq \{F_n(\Delta_X), G_n(\Delta_Y)\}$$

is an (f, g)-estimation scheme.

Lemma (Non-Commuting)

Let $\lambda \in (0, 1)$. Then

$$(F \cup_\lambda G)(\Delta_X \times \Delta_Y) \coloneqq F_{\lfloor \lambda n \rfloor}(\Delta_X) \otimes G_{n - \lfloor \lambda n \rfloor}(\Delta_Y)$$

is an (f, g)-estimation scheme.

- Open question: Is there a better way?

Approximate quantitative realizability

Definition

Let $\mu : \Sigma(\mathcal{S}) \to [0,1]$ be a probability measure.
The μ-de Finetti state for $n \in \mathbb{N}$ is

$$D_n^\mu := \int_{\mathcal{S}} \mu(\mathrm{d}\rho)\rho^{\otimes n}$$

Theorem

Let $f : \mathcal{S} \to X$ be a property and $E = (E_n)_{n\in\mathbb{N}}$ an f-estimation scheme. Then for all bounded continuous $g : X \to \mathbb{R}$

$$\lim_{n\to\infty} \int_X g(x)\mathrm{tr}(E_n(\mathrm{d}x)D_n^\mu) = \int_X g(x)(f_*\mu)(\mathrm{d}x).$$

Applied to Pauli-X, Z realizability

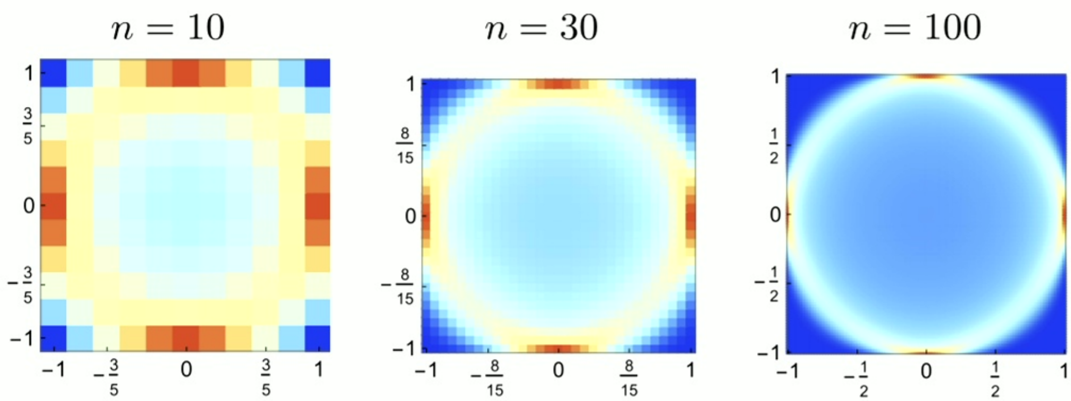
- Define E^X and E^Z as f_X- and f_Z-estimation schemes as before

$$E_n^{X,Z} := E_{\lfloor n/2 \rfloor}^{X} \otimes E_{n-\lfloor n/2 \rfloor}^{Z}.$$

- Let $\mu : \Sigma(\mathcal{S}) \to [0,1]$ be uniform $\mathrm{SU}(2)$-Haar prob. measure.

$$D_n^\mu = \frac{P_{\mathrm{Sym}^n(\mathbb{C}^d)}}{\mathrm{tr}(P_{\mathrm{Sym}^n(\mathbb{C}^d)})}$$

- Plots of $\mathrm{tr}(E_n^{X,Z}(\{\hat{x}, \hat{z}\})D_n^\mu)$ below



- Crucial question: How to distinguish between realizable and unrealizable property values?

Large Deviation Theory

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Large deviation theory

- Concerned with the exponential decay of probabilities of rare events

$$p_n(\Delta) \approx \exp(-nI(\Delta))$$

- for physicist-friendly treatment (Touchette, 2011)
- for rigorous treatment (Dembo and Zeitouni, 2010) and (Dupuis and Ellis, 2011)
- generalizes Laplace's method of steepest descent

$$\lim_{n \to \infty} \frac{1}{n} \ln \int_0^1 \exp(nh(x))\,\mathrm{d}x = \max_{x \in [0,1]} h(x)$$

Rate functions & the Laplace principle

Definition

A function $I : X \to [0, \infty]$ is called a rate function if for all $c \in [0, \infty)$, its lower-level-sets,

$$\{x : I(x) \leq c\} \subseteq X,$$

are *compact* subsets of X.

Definition

A sequence of measures $(\eta_n)_{n\in\mathbb{N}}$ on X satisfies the Laplace principle (LP) with rate function $I : X \to [0, \infty]$ if for all bounded continuous functions $g : X \to \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n} \ln \int_X \eta_n(\mathrm{d}x) e^{ng(x)} = \sup_{x\in X} (g(x) - I(x)).$$

The Laplace principle for estimation schemes

Lemma

Let $E = (E_n)_{n\in\mathbb{N}}$ *be a sequence of POVMs and* $\rho \in \mathcal{S}$.
If $\mathrm{tr}(E_n(\cdot)\rho^{\otimes n})$ *satisfies the LP with rate function* $I_\rho : X \to [0,\infty]$

$$\lim_{n\to\infty} \frac{1}{n} \ln \int_X \mathrm{tr}(E_n(\cdot)\rho^{\otimes n}) e^{ng(x)} = \sup_{x\in X} (g(x) - I_\rho(x)).$$

and

$$I_\rho(x) = 0 \quad \textit{if and only if} \quad f(\rho) = x,$$

then E *is an* f*-estimation scheme.*

- Remark: All previously mentioned estimation schemes are of this form.
- General theory [Botero, Christandl, and Vrana, 2021] and (Franks and Walter, 2020)

The Laplace principle for estimation schemes

Lemma

Let $E = (E_n)_{n\in\mathbb{N}}$ be a sequence of POVMs and $\rho \in \mathcal{S}$.
If $\operatorname{tr}(E_n(\cdot)\rho^{\otimes n})$ satisfies the LP with rate function $I_\rho : X \to [0,\infty]$

$$\lim_{n\to\infty} \frac{1}{n} \ln \int_X \operatorname{tr}(E_n(\cdot)\rho^{\otimes n}) e^{ng(x)} = \sup_{x\in X}(g(x) - I_\rho(x)).$$

and

$$I_\rho(x) = 0 \quad \textit{if and only if} \quad f(\rho) = x,$$

then E is an f-estimation scheme.

- Remark: All previously mentioned estimation schemes are of this form.
- General theory (Botero, Christandl, and Vrana, 2021) and (Franks and Walter, 2020)

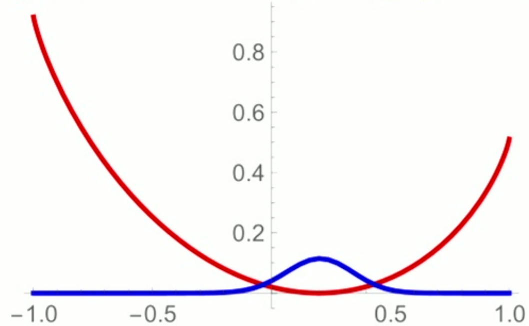
Rate function of Pauli-X estimation scheme

- Let E_n^X be the aforementioned f_X-estimation scheme
- For each $\rho \in \mathcal{S}$ the sequence

$$\mathrm{tr}(E_n^X(\cdot)\rho^{\otimes n})$$

- satisfies the Laplace principle with rate function $I_\rho = R(x|f_X(\rho))$
- example for $f_X(\rho) = \frac{1}{5}$ and $n = 50$:

$I_\rho(x)$ and $\mathrm{tr}(E_n^X(\{x\})\rho^{\otimes n})$



- where $R(x|x') : [-1,+1] \times [-1,+1] \to [0,\infty]$ is

$$R(x|x') = \frac{1}{2}\left[(1+x)\ln\frac{1+x}{1+x'} + (1-x)\ln\frac{1-x}{1-x'}\right]$$

The Fundamental Theorem

Inclusivity of de Finetti states

Definition

A prob. measure $\mu : \Sigma(\mathcal{S}) \to [0,1]$ is inclusive if there exists a $\gamma_\mu : \mathbb{N} \to \mathbb{R}_{>0}$ such that

$$\lim_{n\to\infty} n^{-1} \ln \gamma_\mu(n) = 0,$$

and for all n and all $\sigma \in \operatorname{supp}(\mu)$,

$$\gamma_\mu(n) D_n^\mu \geq \sigma^{\otimes n}.$$

- Examples:
  - $\mu = \lambda\delta_\rho + (1-\lambda)\delta_\sigma$ has $\operatorname{supp}(\mu) = \{\rho, \sigma\}$ and $\gamma_\mu(n) = \max\{\lambda, 1-\lambda\}$
  - μ Hilbert-Schmidt measure has $\operatorname{supp}(\mu) = \mathcal{D}(\mathbb{C}^d)$ and $\gamma_\mu(n) = \binom{n+d^2-1}{n}$
  - μ Haar measure $\operatorname{supp}(\mu) = \mathcal{P}(\mathbb{C}^d)$ and $\gamma_\mu(n) = \binom{n+d-1}{n}$
- Open problem: Find example of non-inclusive measure μ?

Cantor sequences

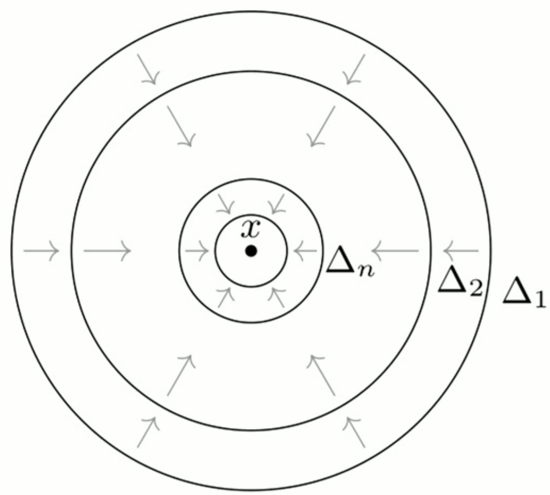
Definition

A Cantor sequence in complete metric space X is a sequence

$$\Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_n \supseteq \cdots$$

of non-empty, compact, and nested sets such that

$$\lim_{n\to\infty} \operatorname{diam}(\Delta_n) = 0, \quad \text{where} \quad \operatorname{diam}(\Delta_n) = \sup\{d(x,y) \mid x, y \in \Delta_n\}.$$



Cantor's intersection theorem implies $\bigcap_{n\in\mathbb{N}} \Delta_n = \{x\}$.

A fundamental theorem for realizability

Theorem

Let $f : \mathcal{S} \to X$ be a property and $(E_n)_{n\in\mathbb{N}}$ an estimation scheme satisfying LP. Let $\mu : \Sigma(\mathcal{S}) \to [0,1]$ be an inclusive probability measure.

If $x \in X$ is unrealizable, then for all Cantor sequences $(\Delta_n)_{n\in\mathbb{N}}$ converging to x

$$\lim_{n\to\infty} \gamma_\mu(n)\mathrm{tr}(E_n(\Delta_n)D_\mu^n) = 0.$$

If $x \in X$ is realizable, then there exists Cantor sequences $(\Delta_n)_{n\in\mathbb{N}}$ converging to x such that

$$\liminf_{n\to\infty} \gamma_\mu(n)\mathrm{tr}(E_n(\Delta_n)D_\mu^n) \geq 1.$$

- Intuition: x realizable iff x is occasionally an estimate of random state.
- Remark: It is sometimes possible to construct a sequence Δ_n.

Summary & Future Work

Summary:

- examples of property realizability problems in quantum theory
- explored one strategy using property estimation theory
- property estimation schemes provide unified framework
- "realizable iff occasionally estimated"

Future Work:

- place bounds on n needed to solve approximate realizability problems
- methods for calculating $\mathrm{tr}(E_n(\Delta)D_\mu^n)$
- better strategies for non-commutative estimation schemes
- understand relationship with analytic calculation of $f_*\mu$

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