Title: Mathematical Physics Lecture (230404)

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Collection: Mathematical Physics - Elective (2022/2023)

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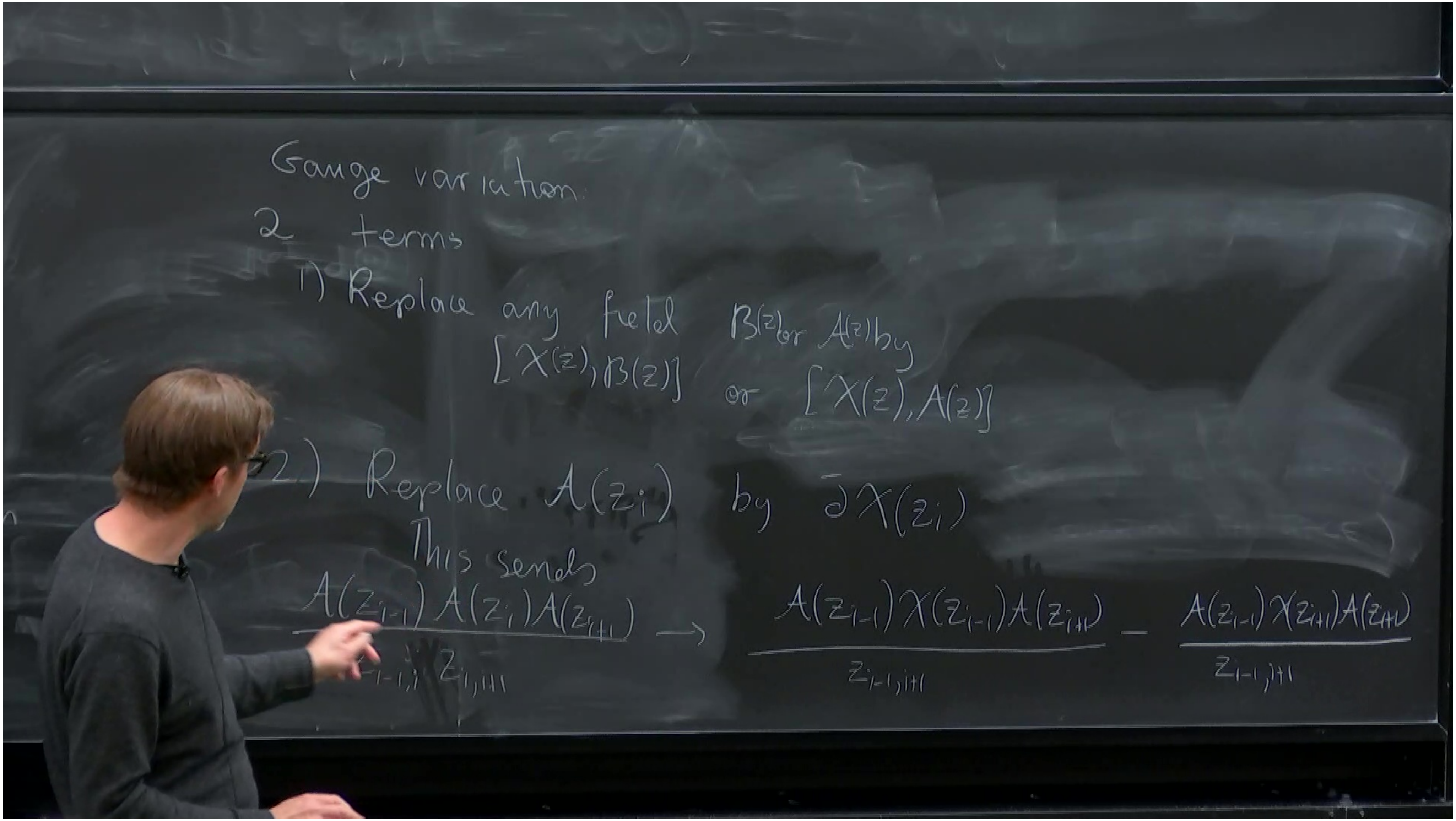
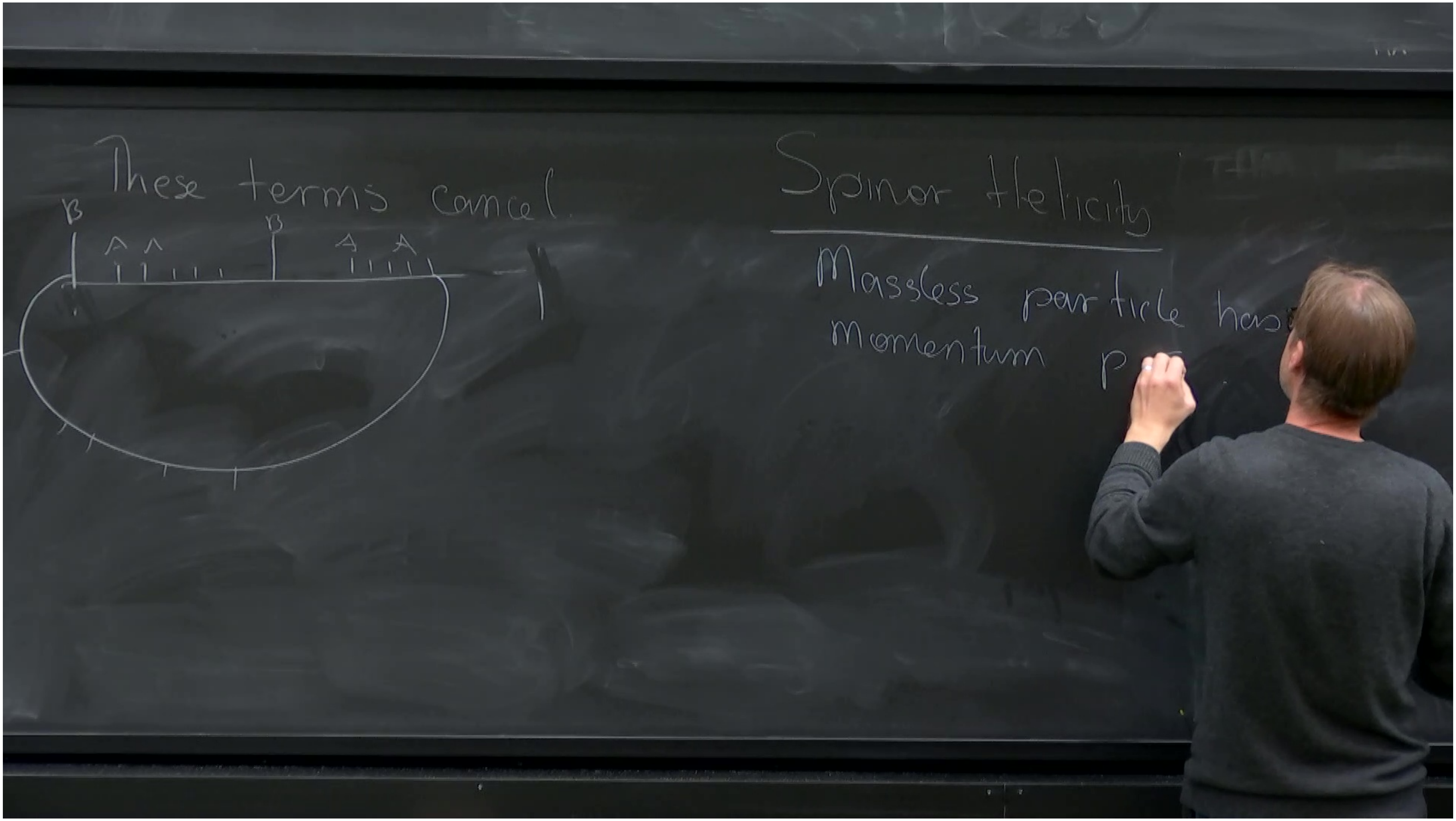
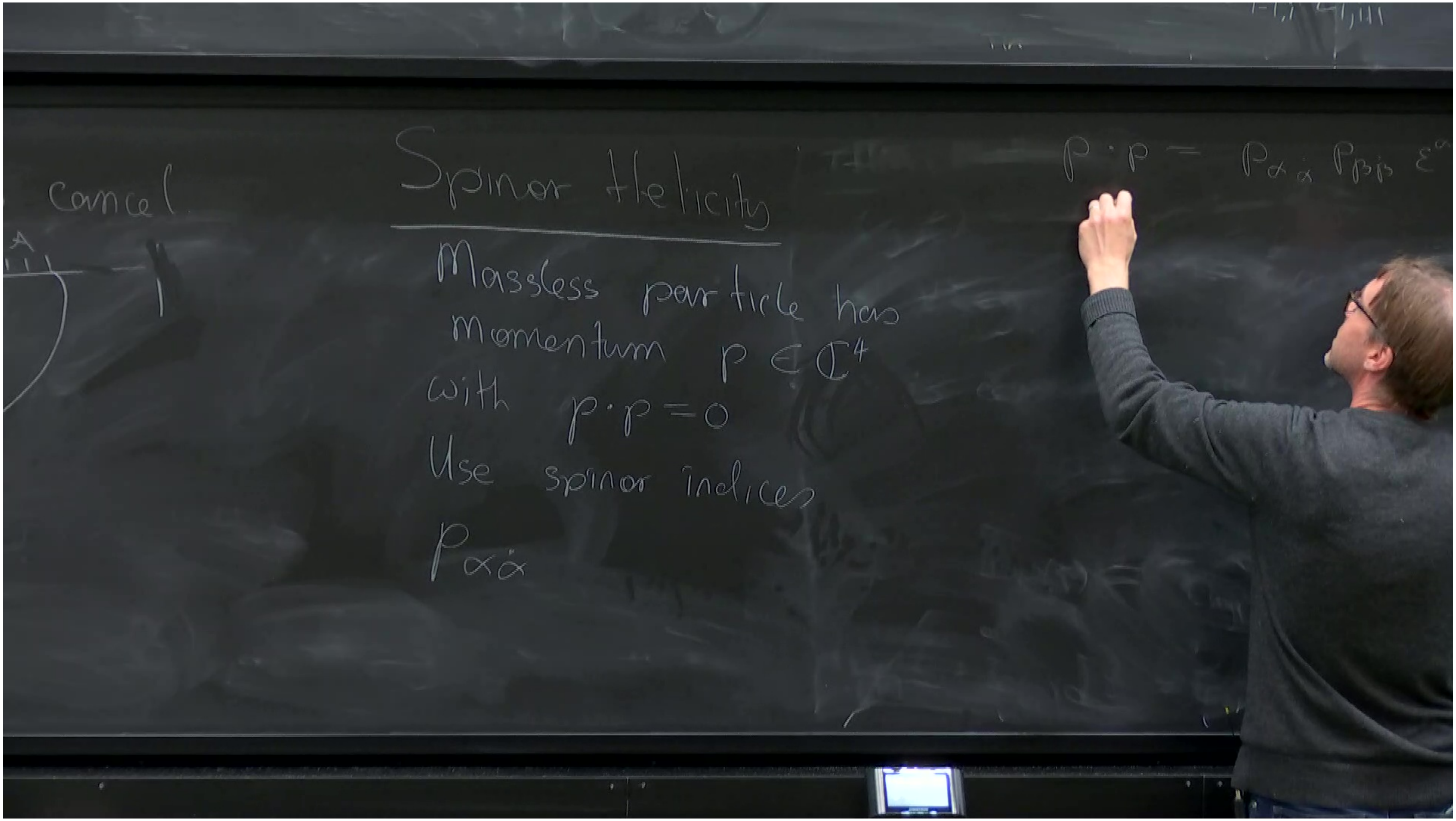
## Last time

We built a gauge invariant operator on $\mathbb{P}\mathbb{T}$ by the formula

$$\sum_{n=2}^{\infty}\sum_{m=2}^{n}\int_{z_1\ldots z_n}\frac{tr\left(B(z_1)A(z_2)\ldots B(z_m)\ldots A(z_n)\right)}{z_{12}\,z_{23}\ldots z_{n1}}\; z_{1m}^{4}$$


Gauge variation:
2 terms
1) Replace any field $B(z)$ or $A(z)$ by
$[\chi(z), B(z)]$ or $[\chi(z), A(z)]$
2) Replace $A(z_i)$ by $\partial\chi(z_i)$
This sends
$\frac{A(z_{i-1})A(z_i)A(z_{i+1})}{z_{i-1,i}\,z_{i,i+1}} \longrightarrow \frac{A(z_{i-1})\chi(z_{i-1})A(z_{i+1})}{z_{i-1,i+1}} - \frac{A(z_{i-1})\chi(z_{i+1})A(z_{i+1})}{z_{i-1,i+1}}$


These terms cancel
B
A A
B
A A
Spinor Helicity
Massless particle has
momentum p


cancel
Spinor Helicity
Massless particle has momentum $p \in \mathbb{C}^4$
with $p \cdot p = 0$
Use spinor indices
$p_{\alpha\dot{\alpha}}$
$p \cdot p = p_{\alpha\dot{\alpha}} p_{\beta\dot{\beta}} \varepsilon$

## Helicity

...ss particle has
...ntum $p \in \mathbb{C}^4$
$p \cdot p = 0$
...pinor indices

$$p \cdot p = p_{\alpha\dot\alpha}\, p_{\beta\dot\beta}\, \varepsilon^{\alpha\beta}\, \varepsilon^{\dot\alpha\dot\beta}$$

For any such $p$,
$\exists$ a pair of spinors
$\lambda_\alpha, \tilde\lambda_{\dot\alpha}$
so that

$$p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$$

This is unique up to $\lambda \to c\lambda$, $\tilde\lambda \to c^{-1}\tilde\lambda$

$p^{(1)}, \ldots, p^{(n)}$

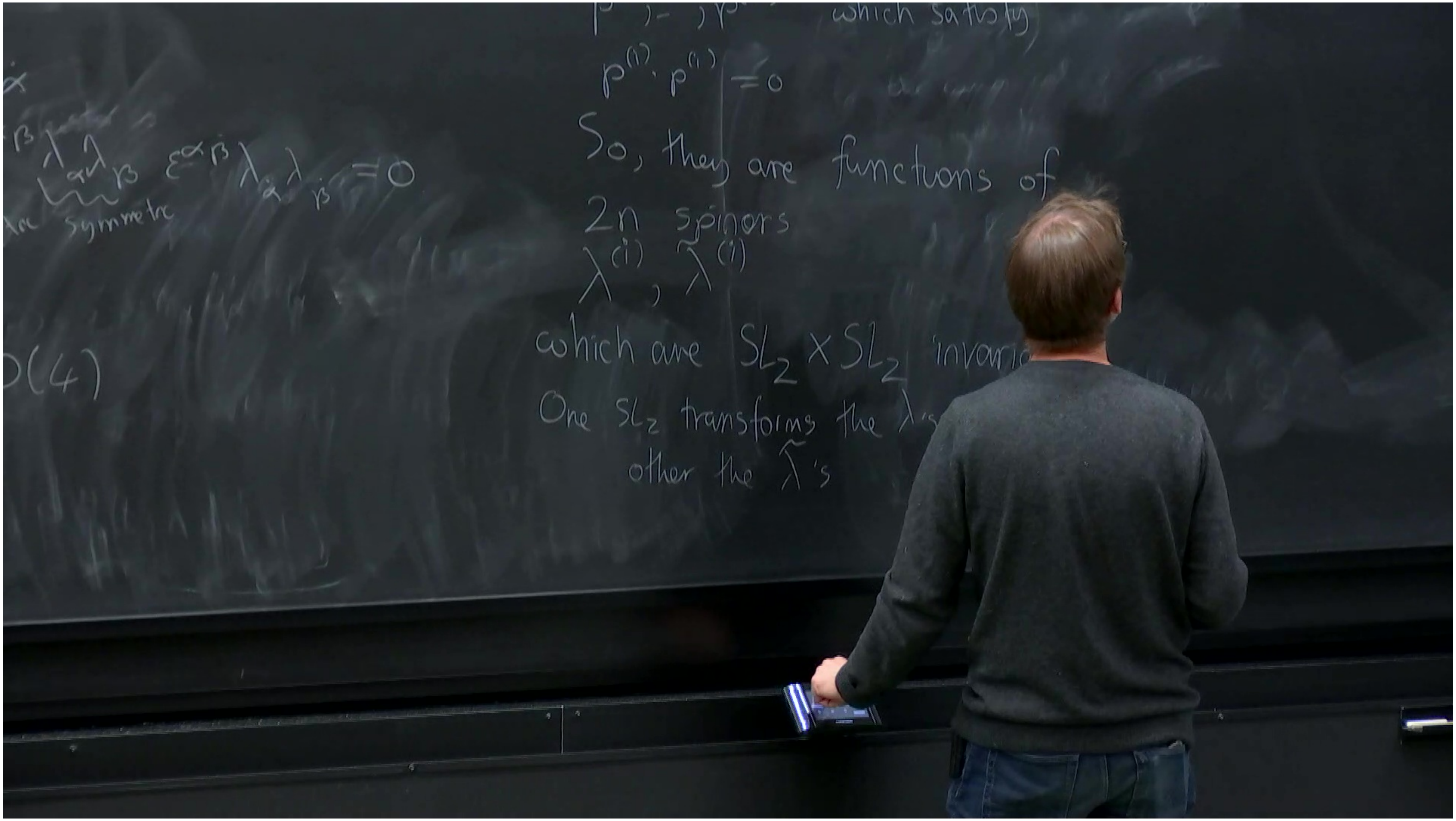
If $p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}}$

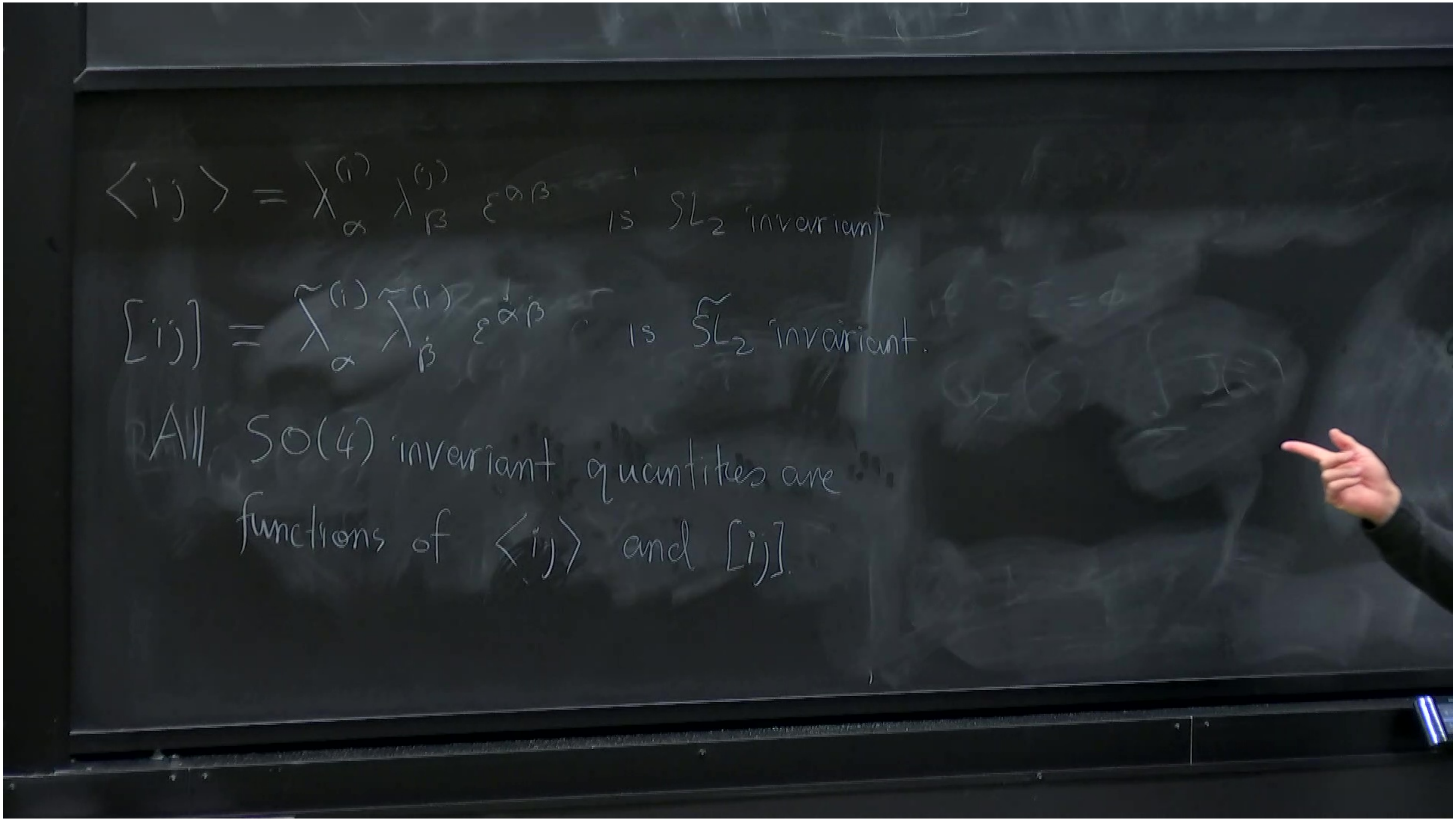
then $p \cdot p = \varepsilon^{\alpha\beta}\lambda_{\alpha}\lambda_{\beta}\,\varepsilon^{\dot{\alpha}\dot{\beta}}\lambda_{\dot{\alpha}}\lambda_{\dot{\beta}} = 0$

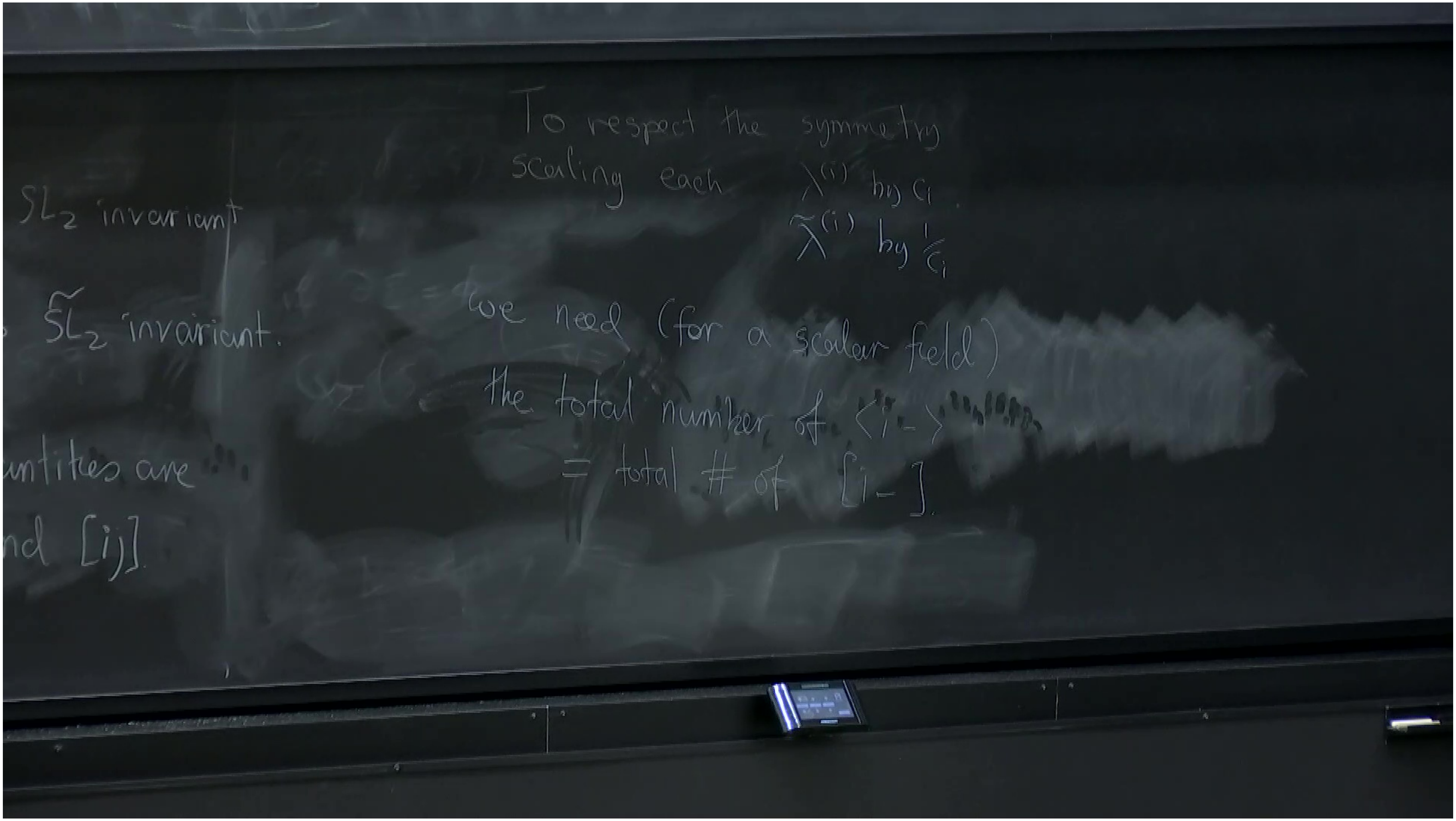
anti-symmetric    symmetric

Amplitudes are SO(4)

invariant functions of


which satisfy
So, they are functions of
2n spinors
which are SL2 x SL2 invarian
One SL2 transforms the λ's
other the λ̃'s
symmetric


$\langle ij \rangle = \lambda^{(i)}_{\alpha} \lambda^{(j)}_{\beta} \varepsilon^{\alpha\beta}$ is $SL_2$ invariant
$[ij] = \tilde{\lambda}^{(i)}_{\dot\alpha} \tilde{\lambda}^{(i)}_{\dot\beta} \varepsilon^{\dot\alpha\dot\beta}$ is $\widetilde{SL}_2$ invariant.
All $SO(4)$ invariant quantities are functions of $\langle ij \rangle$ and $[ij]$.


To respect the symmetry
scaling each $\lambda^{(i)}$ by $c_i$
$\tilde{\lambda}^{(i)}$ by $\frac{1}{c_i}$
We need (for a scalar field)
the total number of $\langle i - \rangle$
$=$ total # of $[i - ]$.
$SL_2$ invariant
$\tilde{SL}_2$ invariant.
untites are
nd $[ij]$.

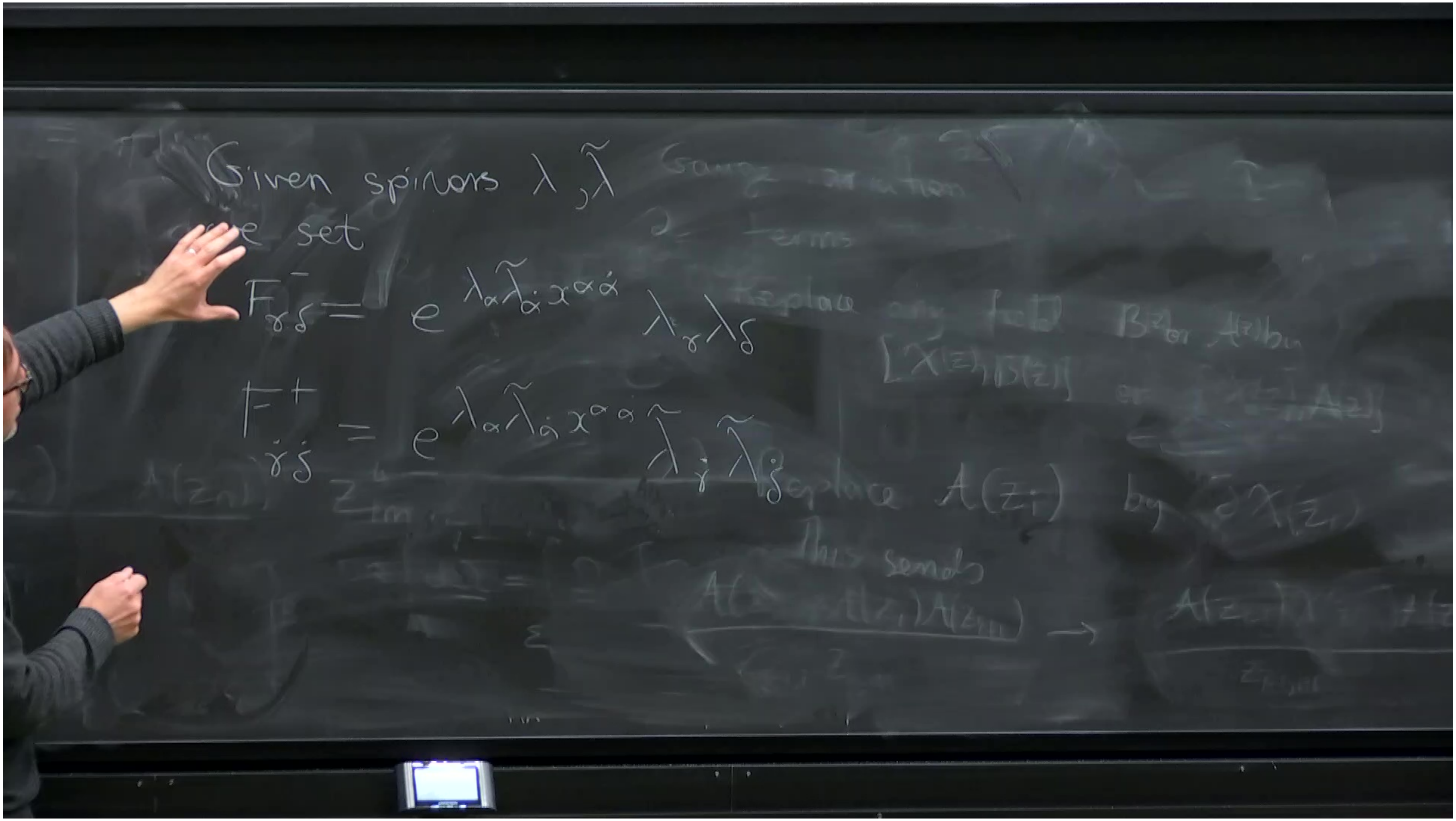
## Gauge field

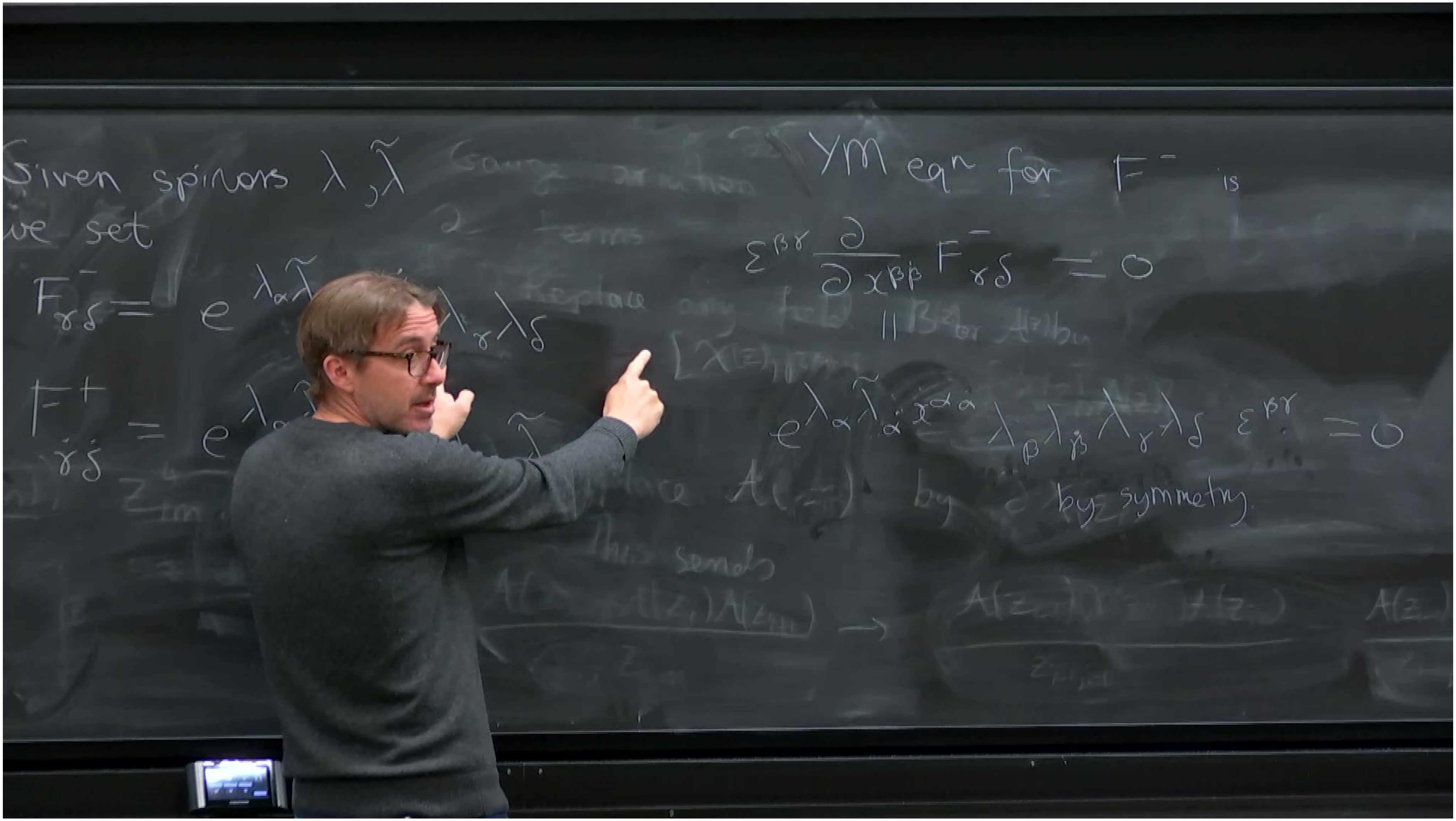
State of momentum $p$
of -ve helicity is
a gauge field $A$
with $F_+ = 0$

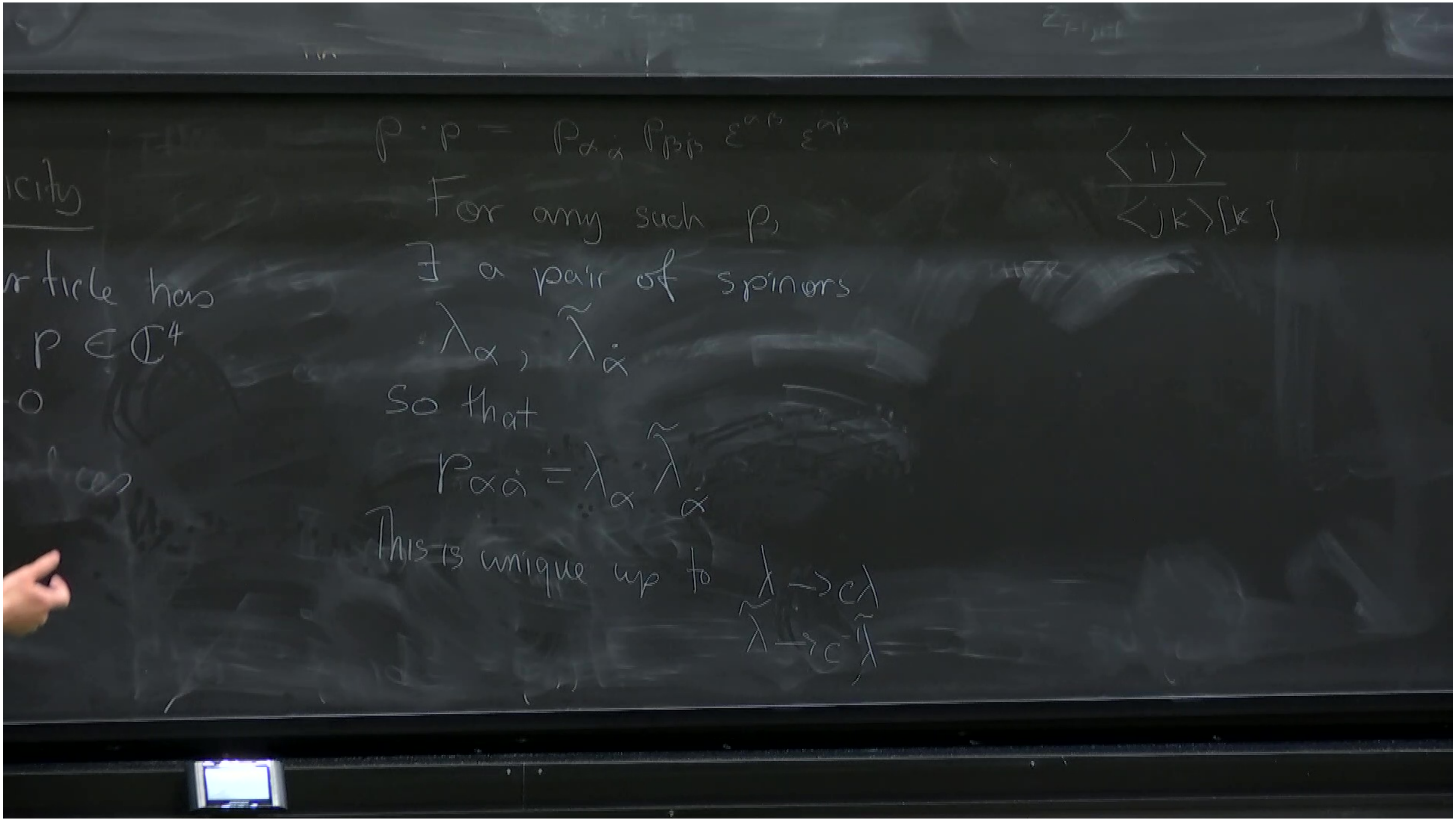
$$F_-^{\alpha\beta} = e^{ip\cdot x}\left(\tilde{\ }_{\alpha\beta}\right)$$

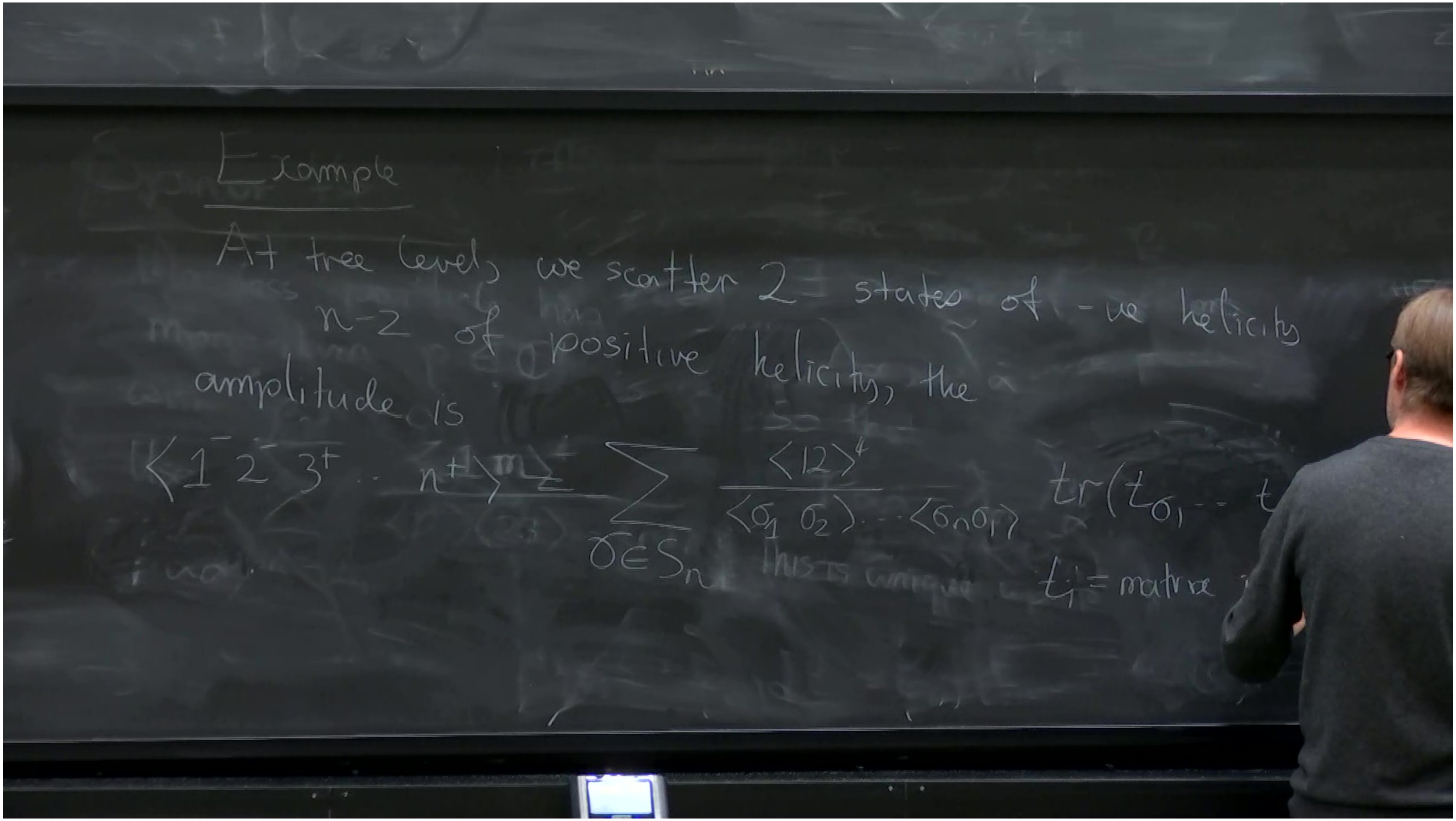
some expression

Given


Given spinors $\lambda, \tilde{\lambda}$
one set
$F^{-}_{\gamma\delta} = e^{\lambda_\alpha \tilde{\lambda}_{\dot\alpha} x^{\alpha\dot\alpha}} \lambda_\gamma \lambda_\delta$
$F^{+}_{\dot\gamma\dot\delta} = e^{\lambda_\alpha \tilde{\lambda}_{\dot\alpha} x^{\alpha\dot\alpha}} \tilde{\lambda}_{\dot\gamma} \tilde{\lambda}_{\dot\delta}$
Replace $A(z_i)$ by
This sends


Given spinors λ, λ̃
we set
YM eqn for F⁻ is
by symmetry
this sends


icity
rticle has
$p \in \mathbb{C}^4$
0
$p \cdot p = p_{\alpha\dot\alpha}\, p_{\beta\dot\beta}\, \varepsilon^{\alpha\beta}\, \varepsilon^{\dot\alpha\dot\beta}$
For any such p,
∃ a pair of spinors
$\lambda_\alpha, \tilde\lambda_{\dot\alpha}$
So that
$p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$
This is unique up to $\lambda \to c\lambda$
$\tilde\lambda \to c^{-1}\tilde\lambda$
$\frac{\langle ij \rangle}{\langle jk \rangle [k}$


Example
At tree level, we scatter 2 states of -ve helicity
n-2 of positive helicity, the
amplitude is
⟨1⁻ 2⁻ 3⁺ ... n⁺⟩ = Σ σ∈Sn ⟨12⟩⁴ / ⟨σ1 σ2⟩ ... ⟨σn σ1⟩ tr(tσ1 ... t
this is unique
ti = matrix

## Example

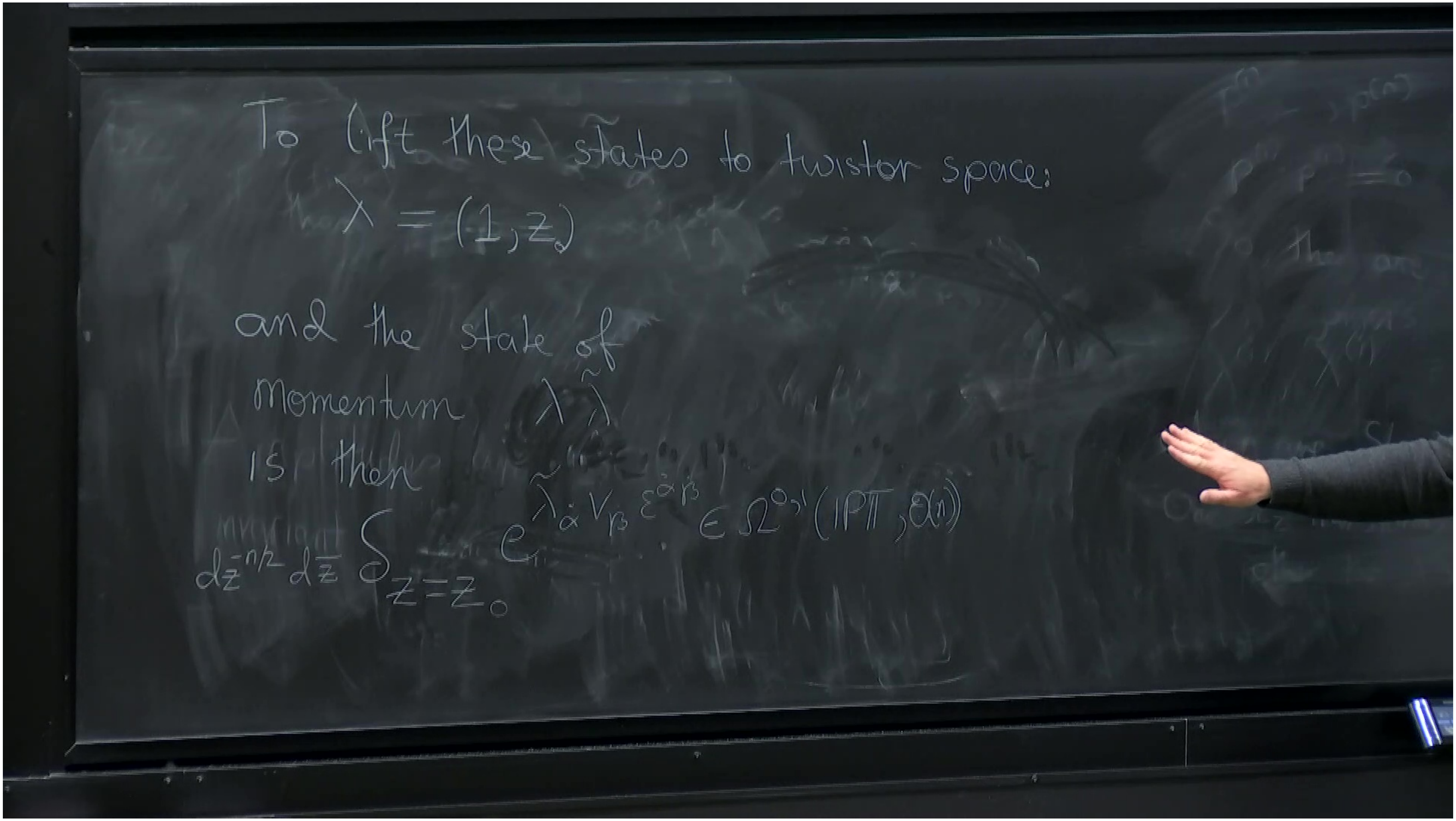
At tree level, we scatter 2 states of -ve helicity
n-2 of positive helicity, the amplitude is

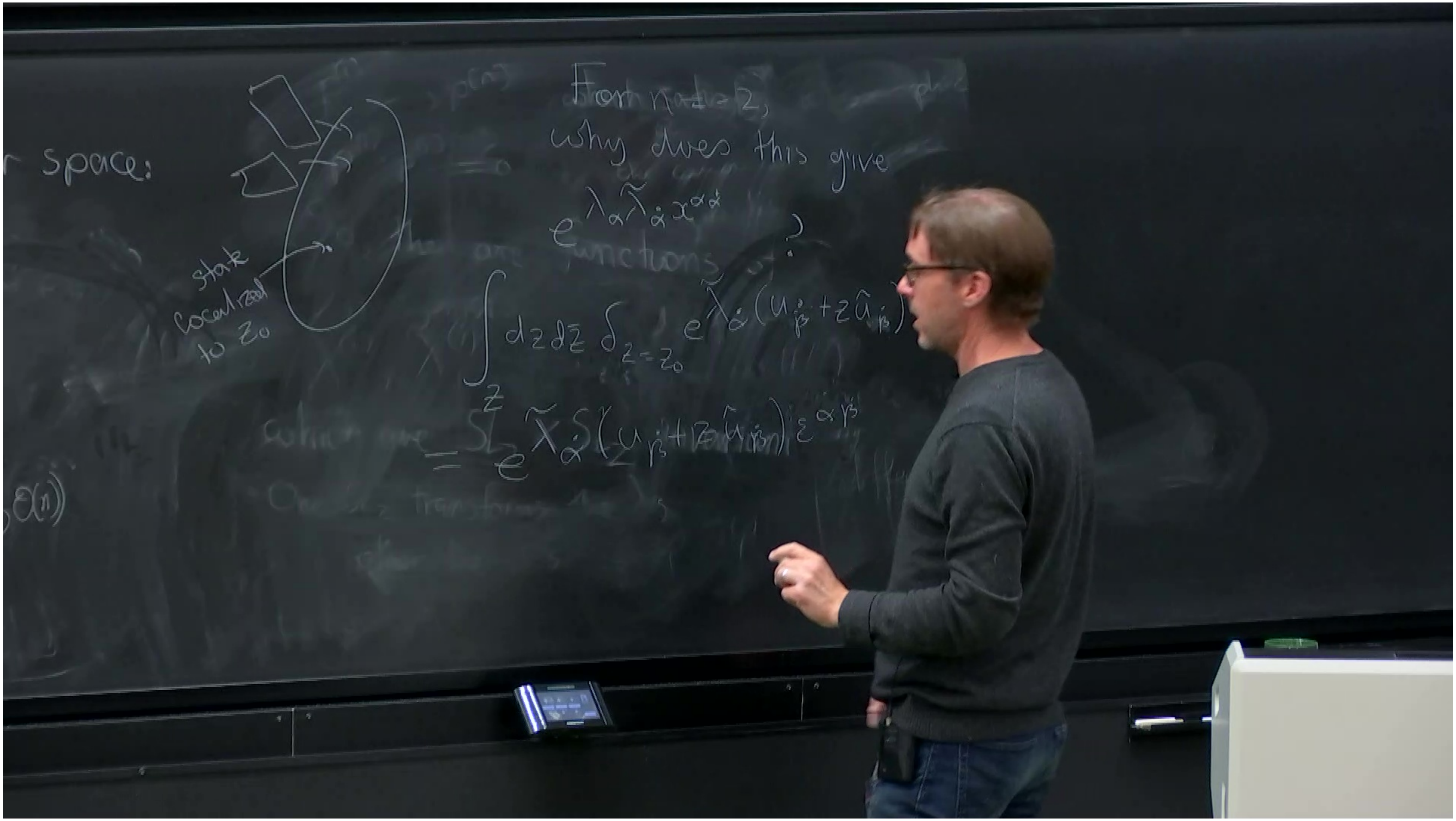
$$\langle 1^- 2^- 3^+ \cdots n^+ \rangle = \sum_{\sigma \in S_n} \frac{\langle 12 \rangle^4}{\langle \sigma_1 \sigma_2 \rangle \cdots \langle \sigma_n \sigma_1 \rangle} \, tr(t_{\sigma_1} \cdots t_{\sigma_n})$$

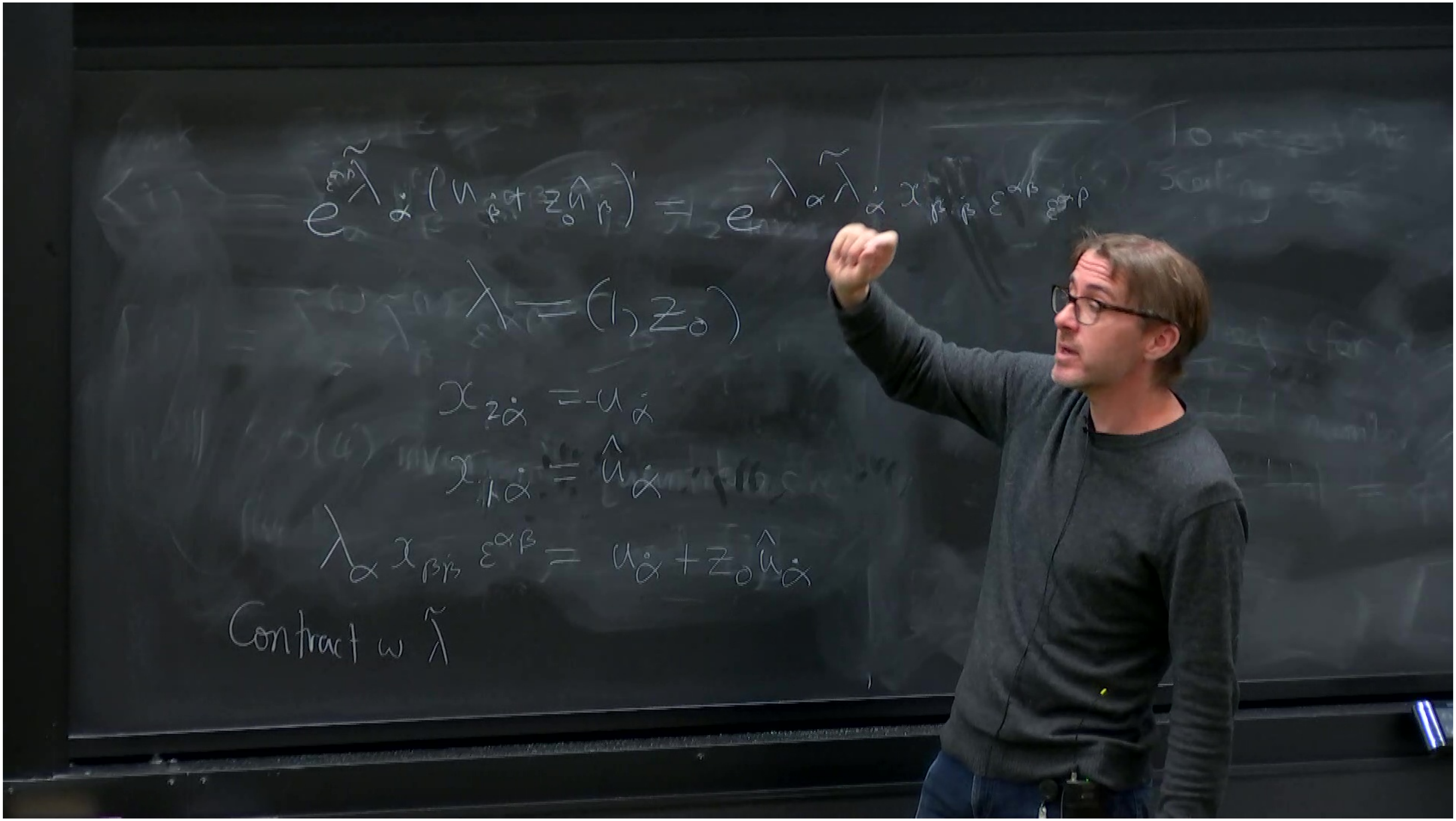
this is unique

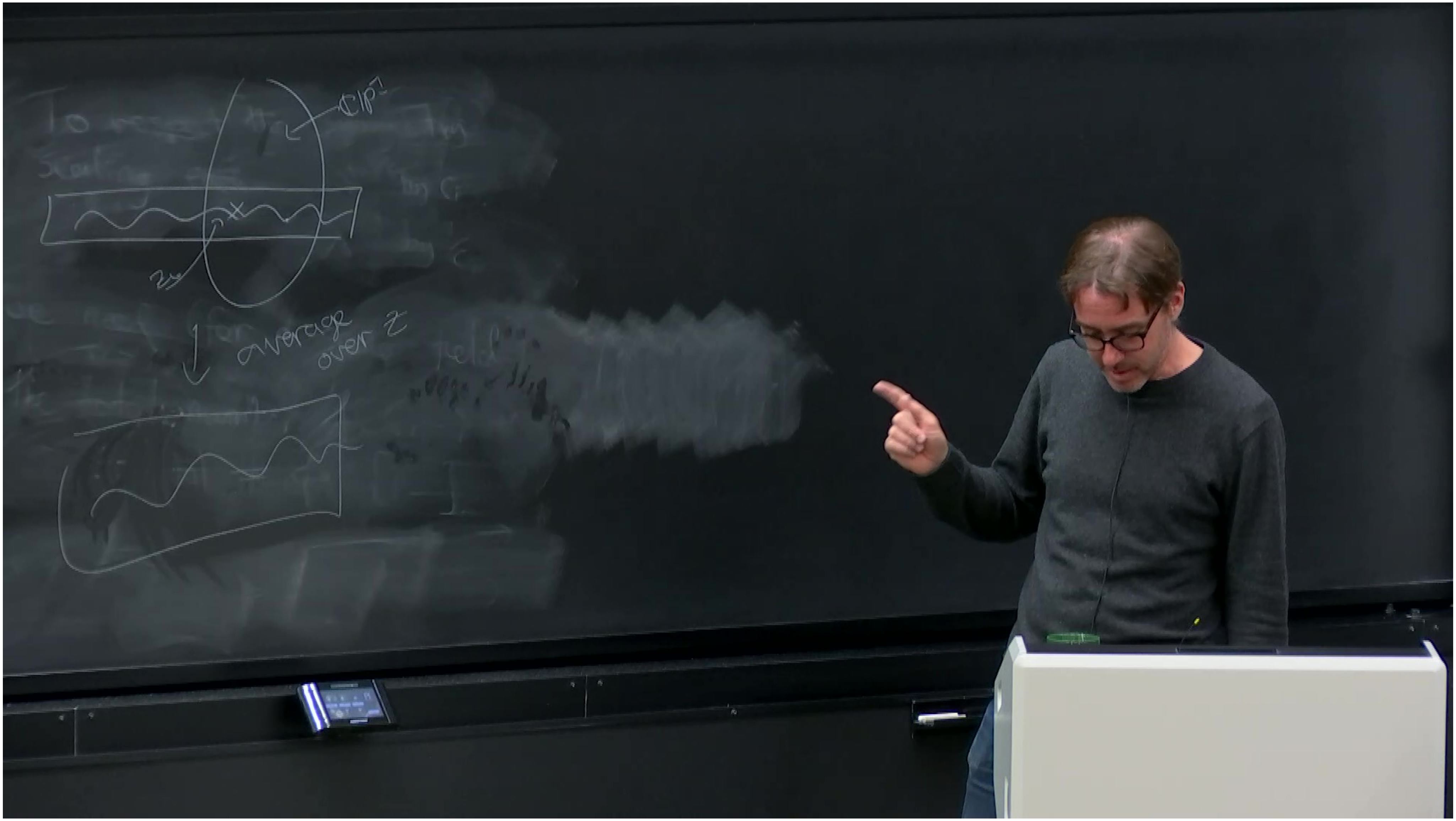
As before.

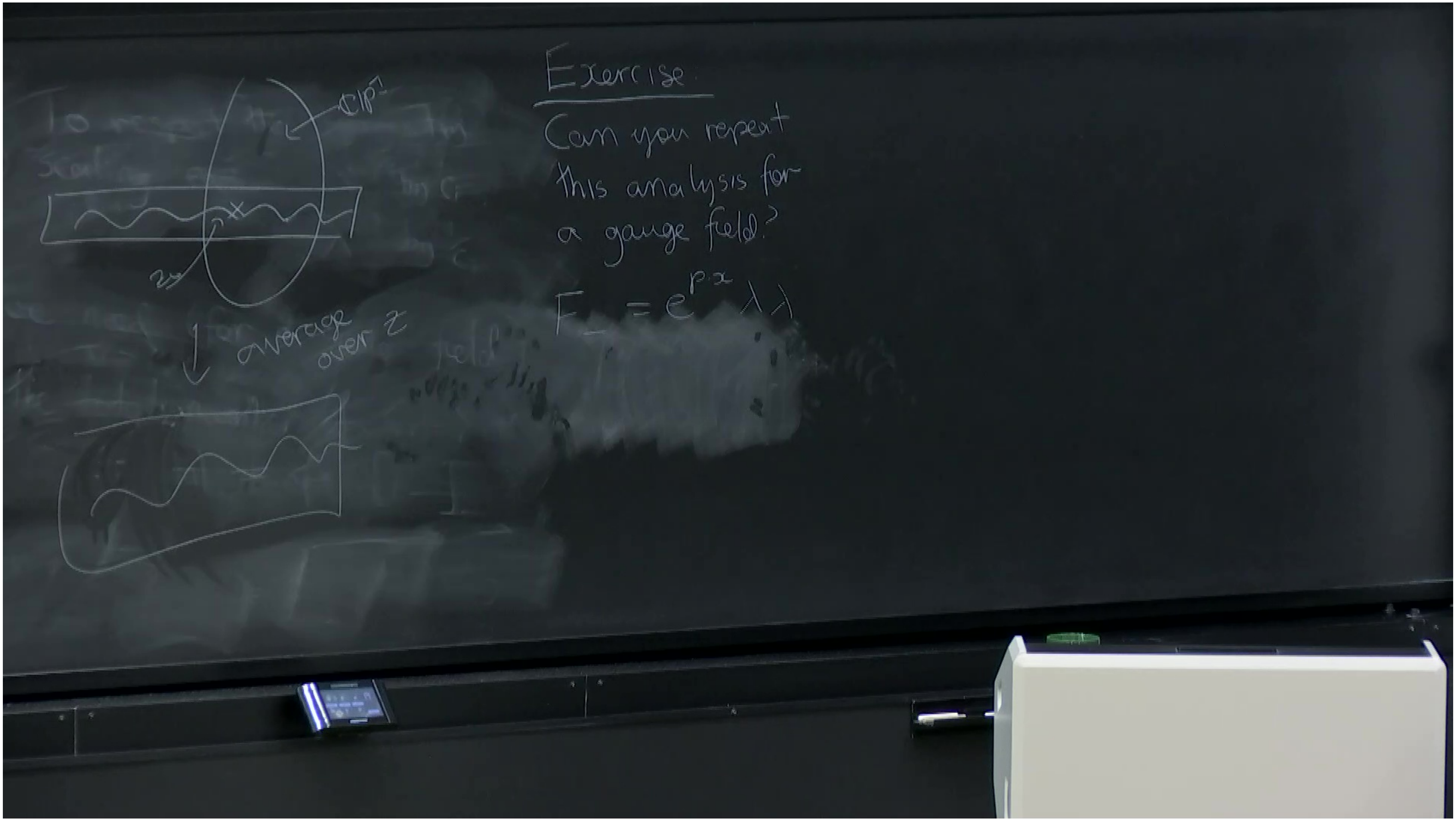
$t_i$ = matrix in $u(n)$ for the ith state

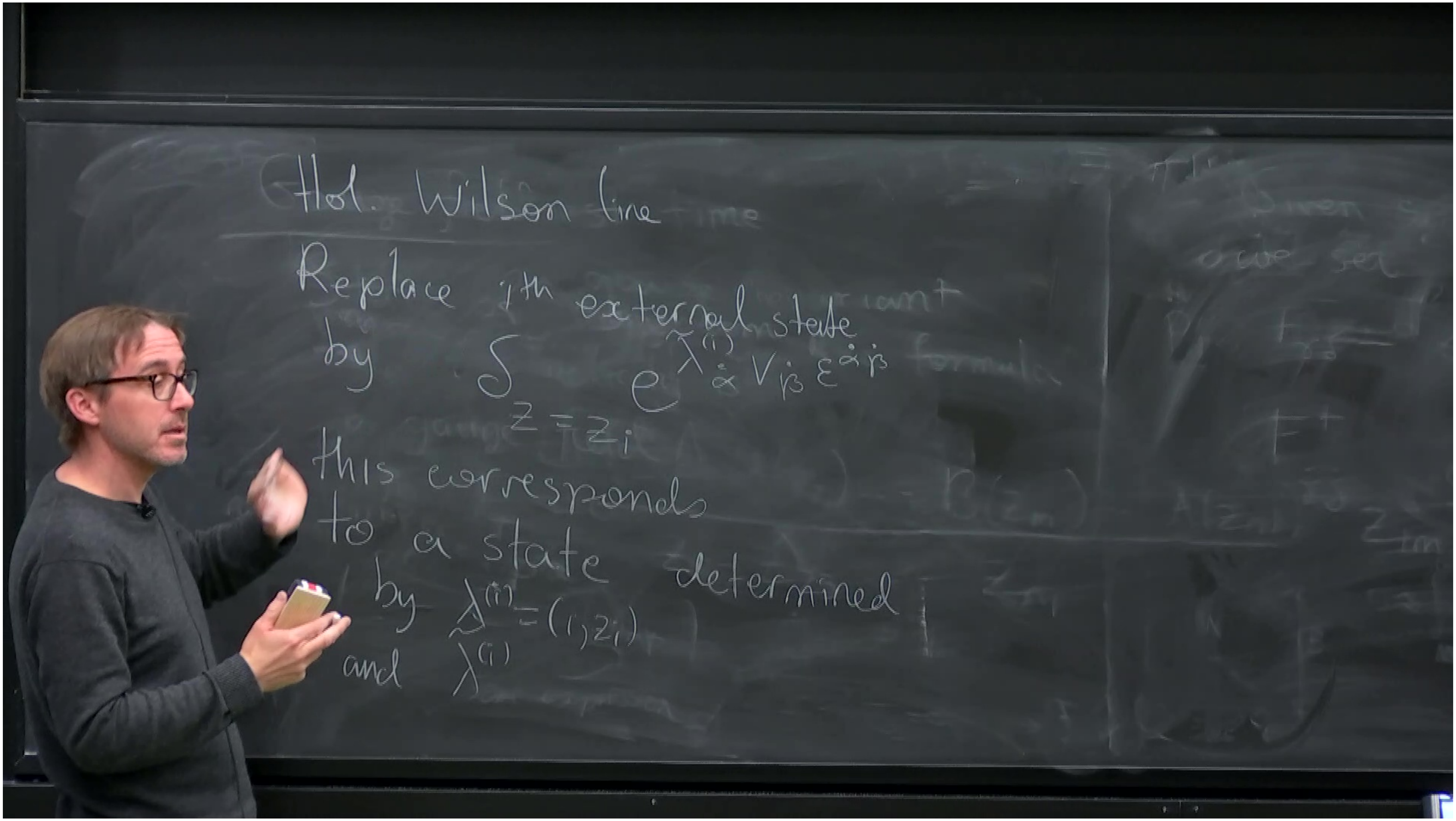

To lift these states to twistor space:
λ = (1, z)
and the state of
momentum λ λ̃
is then
dz^{-n/2} dz̄ ∮_{z=z₀} e^{λ̃_α̇ v_β ε^{α̇β}} ∈ Ω^{0,1}(PT, O(n))

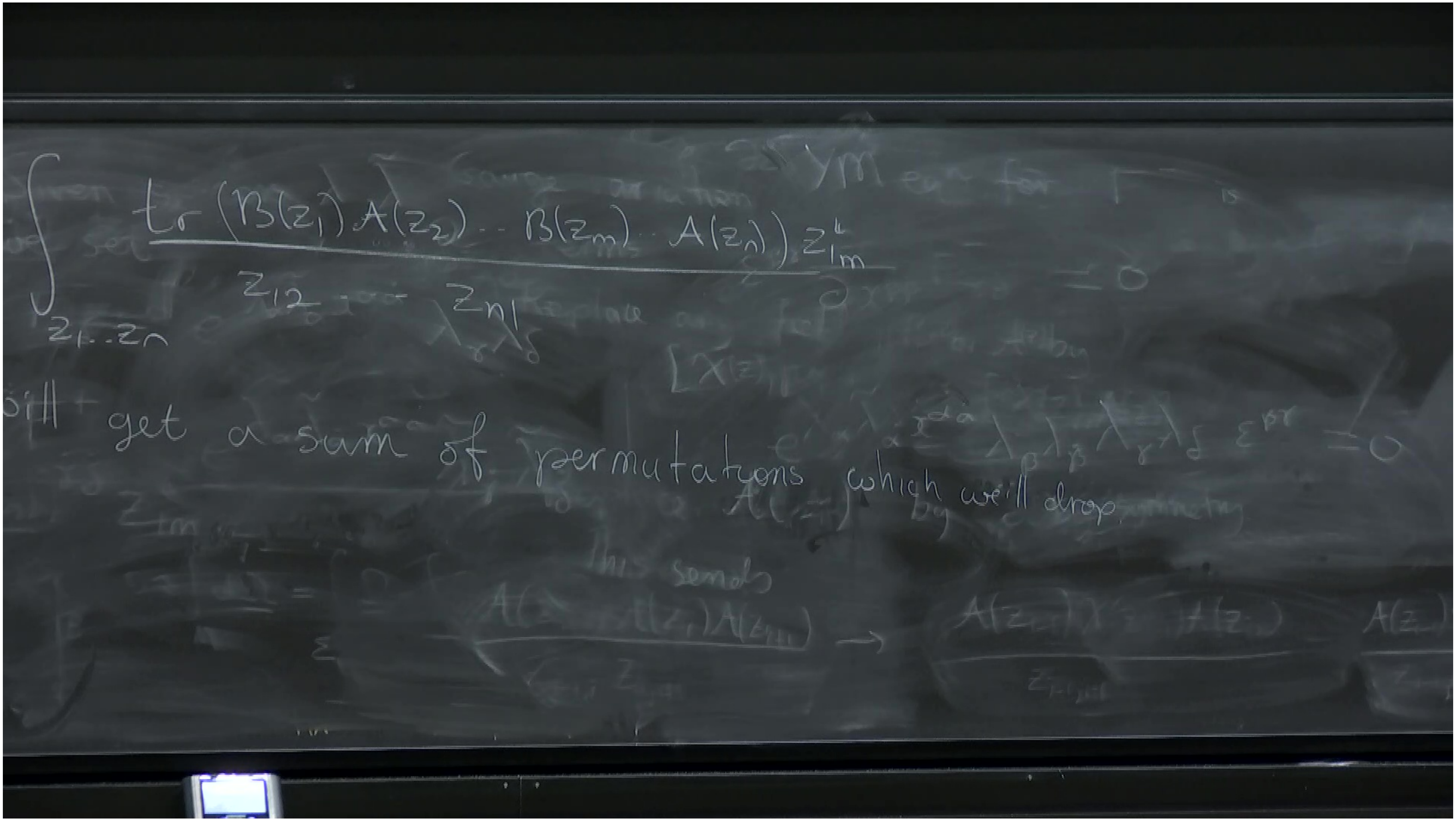

space:
state
localized
to z₀
why does this give
functions

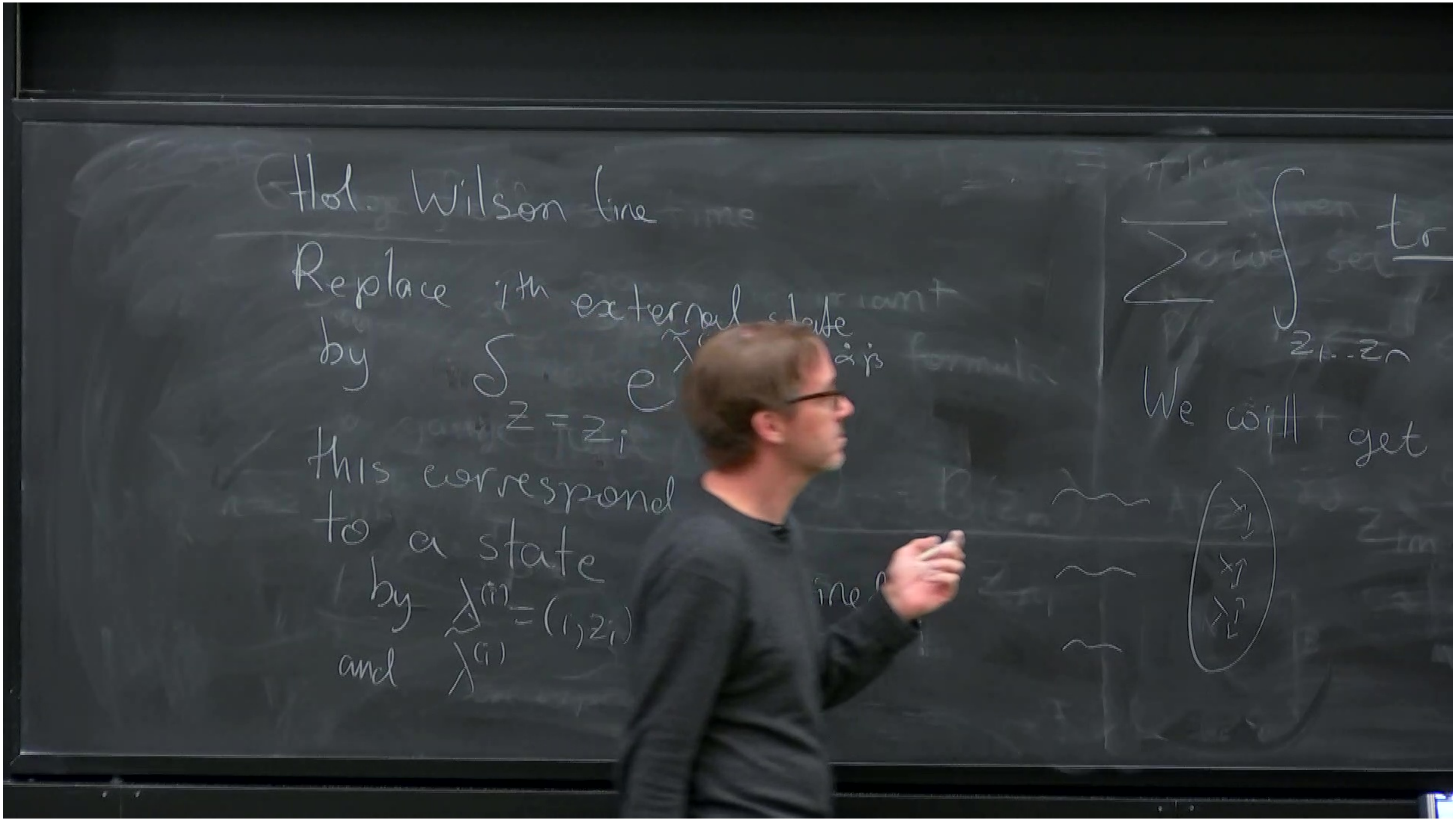

λ = (1, z0)
Contract w λ̃

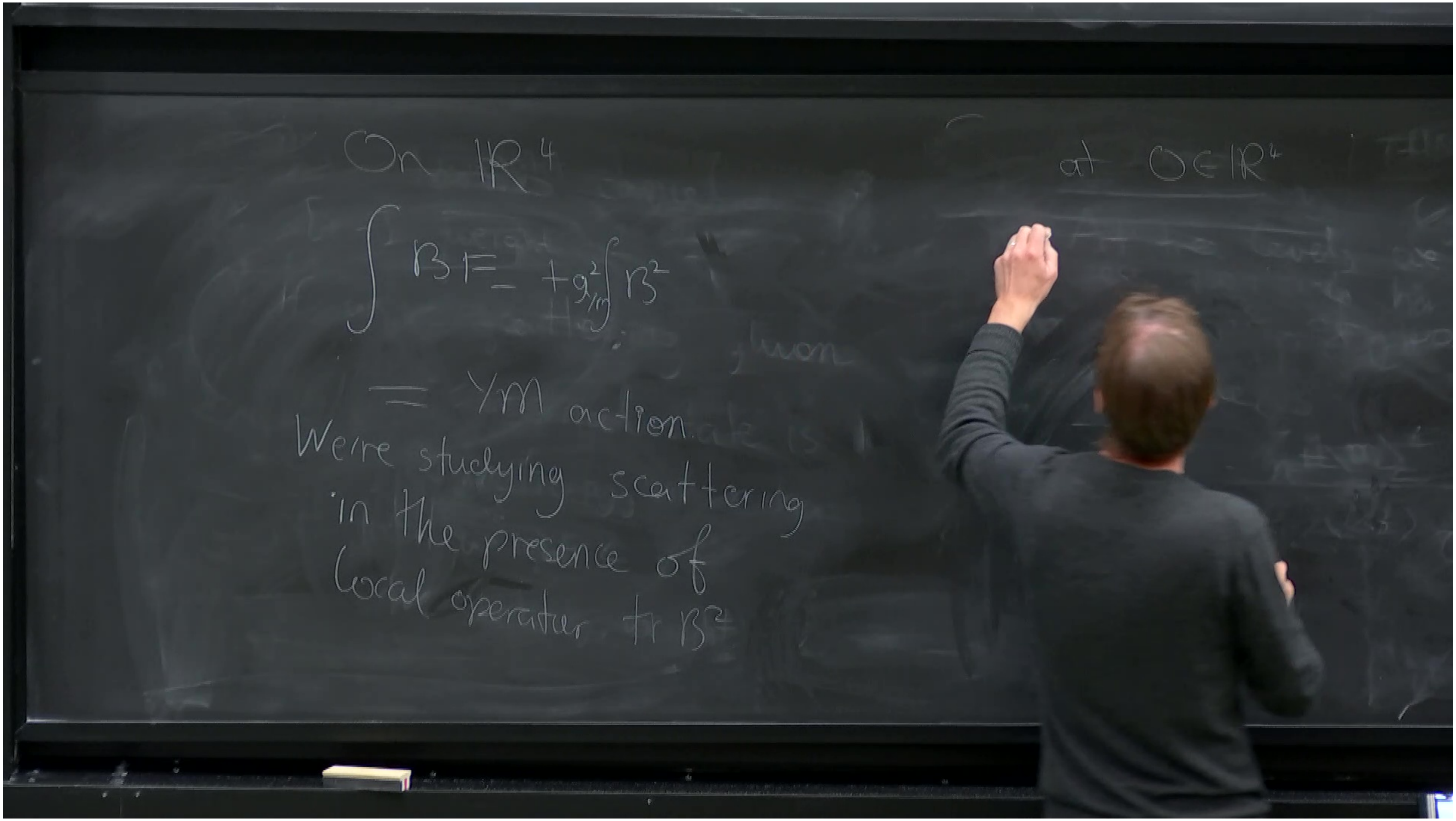

Clip!
average over z

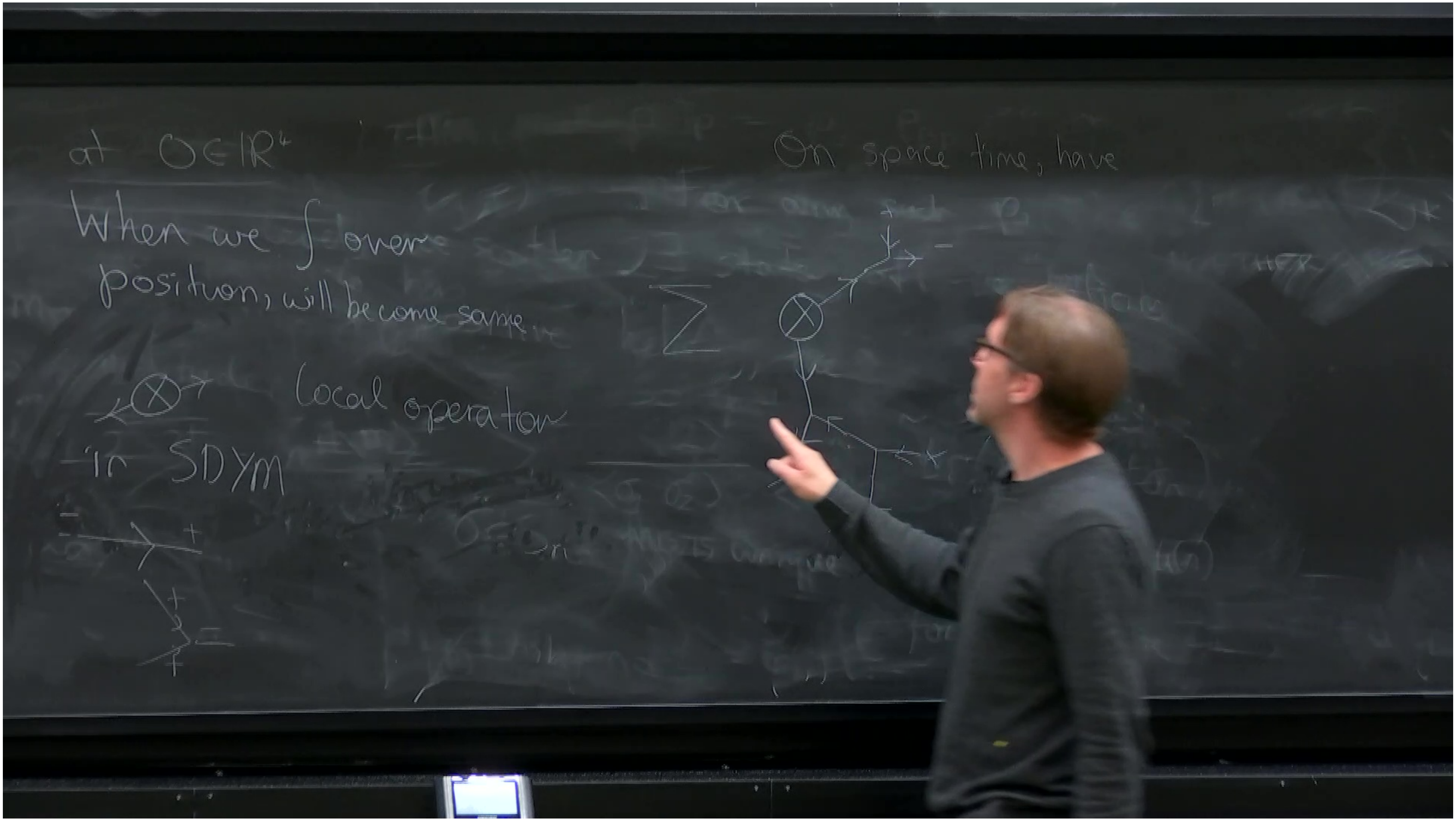

Exercise
Can you repeat this analysis for a gauge field?
F = e^{p·x} λλ
Clip!
average over z

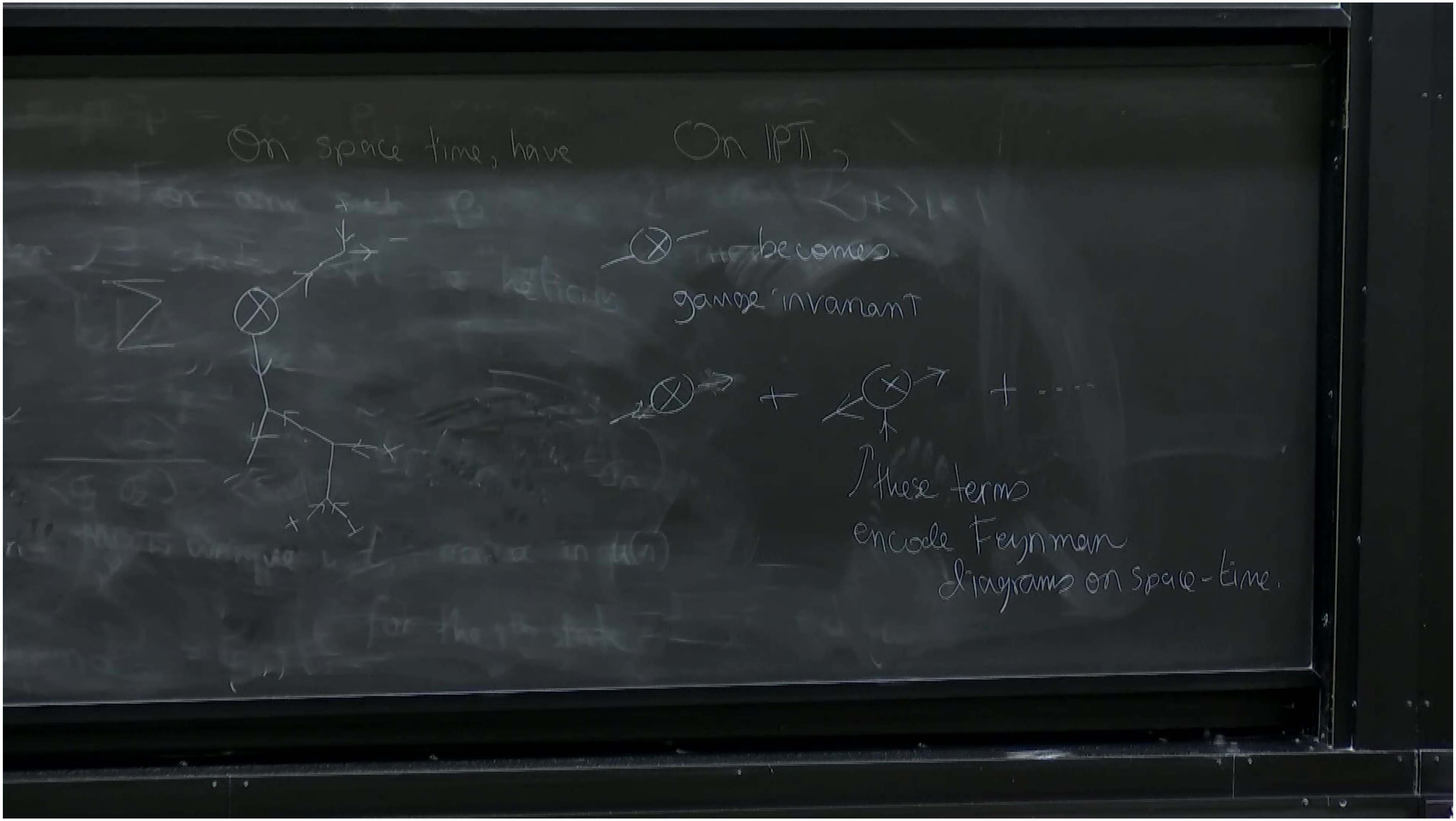

Hol. Wilson line
Replace $i^{th}$ external state
by $\int_{z=z_i} e^{\tilde{\lambda}^{(i)}_{\dot{\alpha}} V_{\dot{\beta}} \varepsilon^{\dot{\alpha}\dot{\beta}}}$
this corresponds
to a state determined
by $\tilde{\lambda}^{(i)} = (1, z_i)$
and $\lambda^{(i)}$

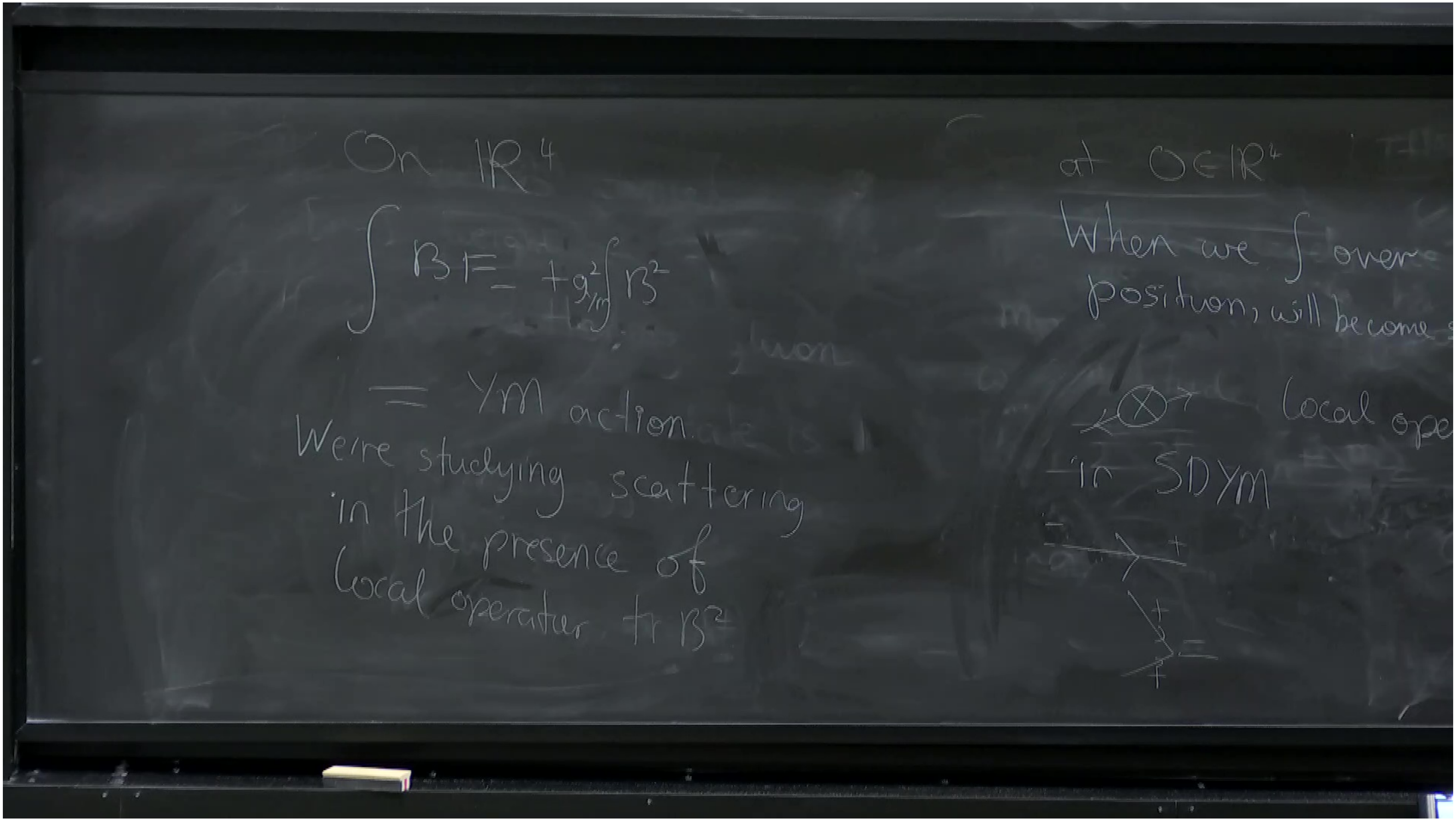

$\int_{z_1 \ldots z_n} \frac{tr(B(z_1) A(z_2) \ldots B(z_m) \ldots A(z_n)) \, z_{1m}^4}{z_{12} \ldots z_{n1}} = 0$
get a sum of permutations which we'll drop


Hol. Wilson line
Replace ith external state
by
$z = z_i$
this correspond
to a state
by $\tilde{\lambda}^{(i)} = (1, z_i)$
and $\lambda^{(i)}$
We will get
$z_1 \ldots z_n$


On ℝ⁴
∫ BF + g²YM ∫ B²
= YM action.
We're studying scattering
in the presence of
local operator tr B²
at 0 ∈ ℝ⁴


at 0 ∈ ℝ⁴
When we ∫ over
position, will become same
local operator
in SDYM
On space time, have


On space time, have
On IPT,
becomes
gauge invariant
+
+ ....
these terms
encode Feynman
diagrams on space-time.


On $\mathbb{R}^4$
$\int BF + g^2_{YM} \int B^2$
= YM action
We're studying scattering
in the presence of
local operator tr $B^2$
at $0 \in \mathbb{R}^4$
When we $\int$ over
position, will become
local ope
in SDYM