Title: Mathematical Physics Lecture (230404)

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Collection: Mathematical Physics - Elective (2022/2023)

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PORTAL

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  (Cartan's calculus)
- Chpt 1 & 2A
  (revised!)
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- Today → Tutorial +
  HW 1

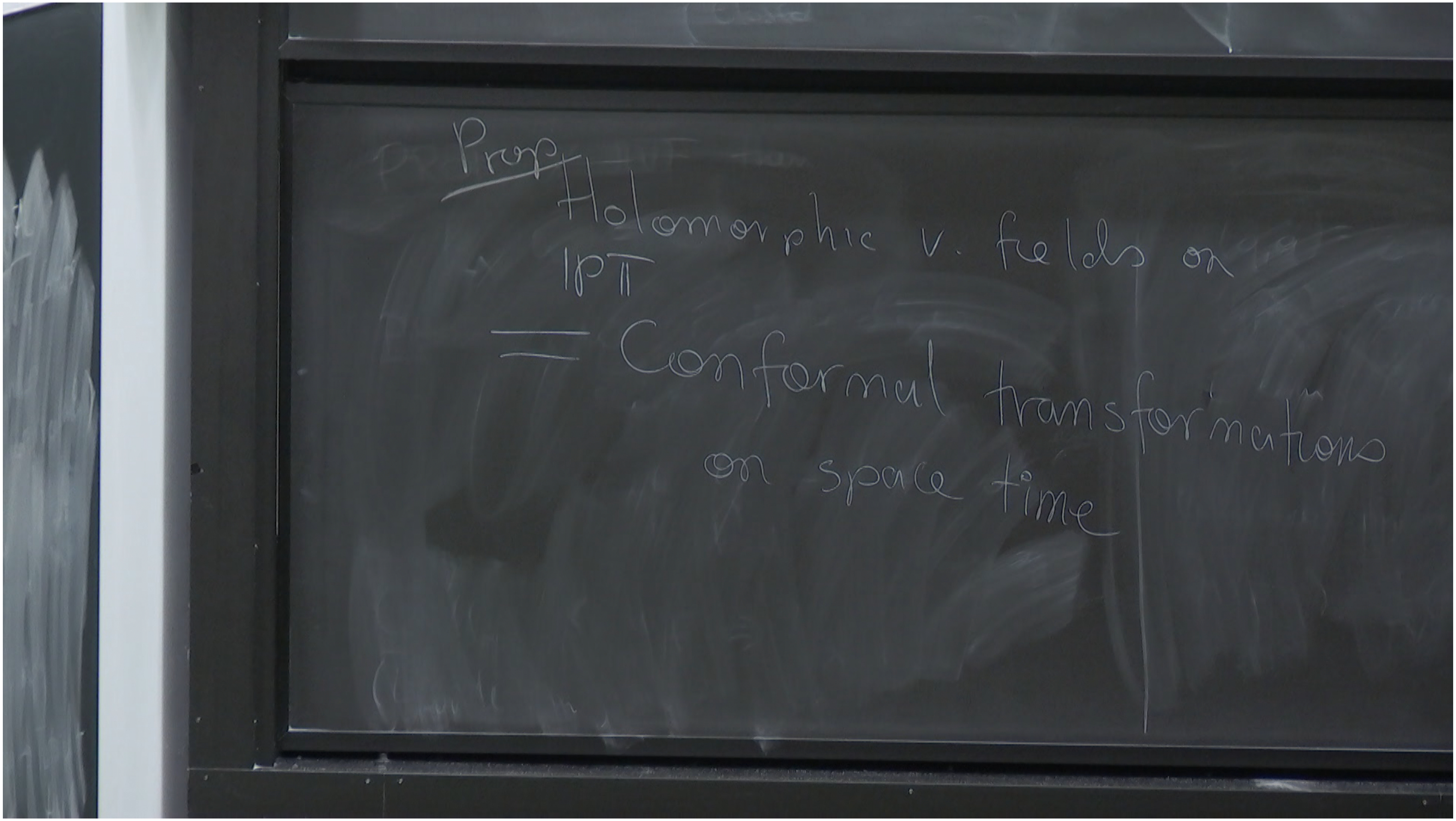
$$V_1 \longrightarrow \tilde{v}_1/\tilde{z}$$

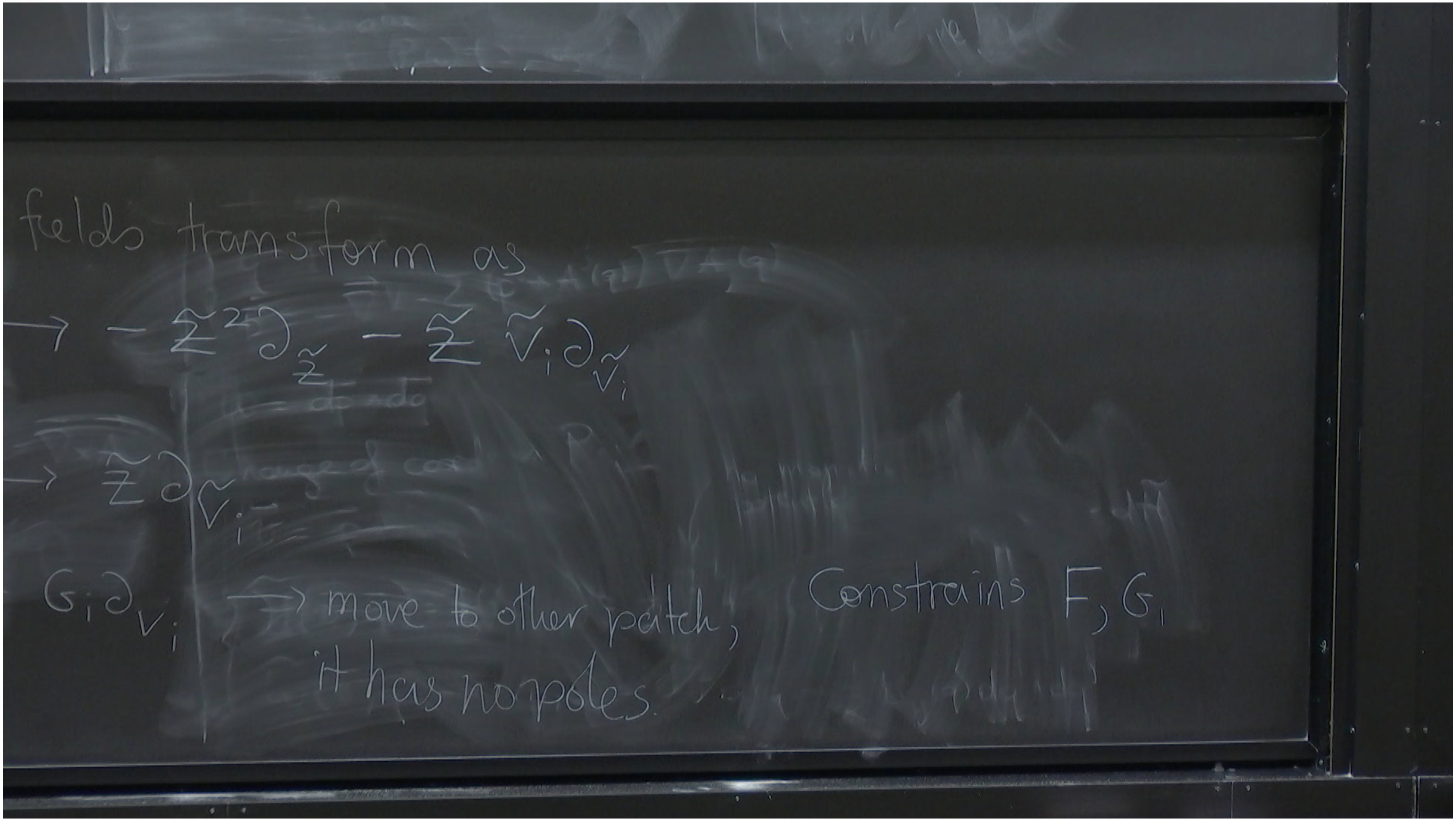
$$V_2 \longrightarrow \tilde{v}_2/\tilde{z}$$

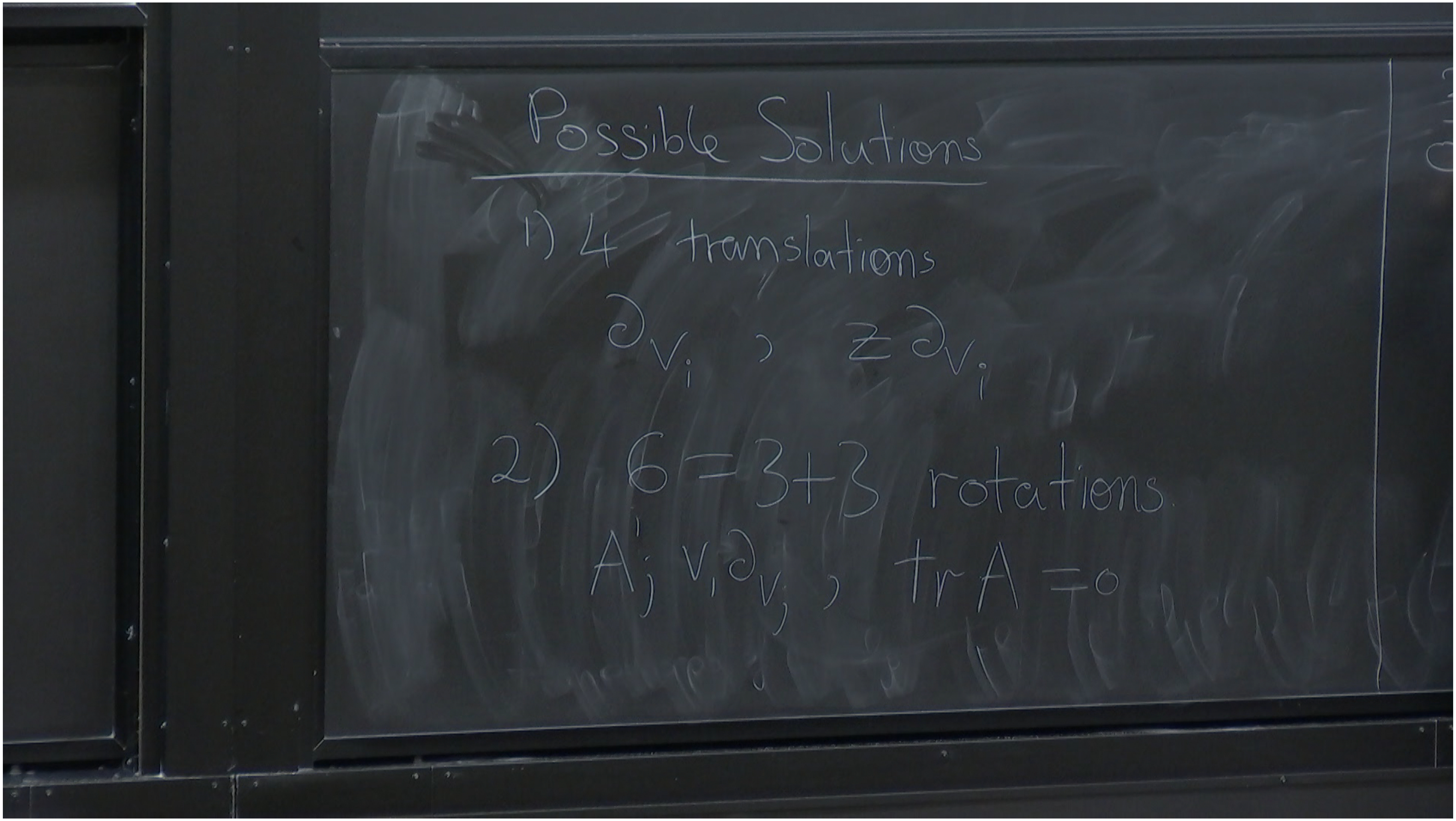
$$z \longrightarrow 1/\tilde{z}$$

V_1 has a pole at $z=\infty$

$\tilde{z}=0$


Prop
Holomorphic v. fields on
PT
= Conformal transformations
on space time


fields transform as
→ − Z̃² ∂Z̃ − Z̃ Ṽi ∂Ṽi
→ Z̃ ∂Ṽi
Gi ∂Vi
⟹ move to other patch, it has no poles.
Constrains F, Gi


Possible Solutions
1) 4 translations
∂_{v_i} , $z\partial_{v_i}$
2) 6 = 3+3 rotations
$A_j^i v_i \partial_{v_j}$, $\mathrm{tr} A = 0$

3 rotations.

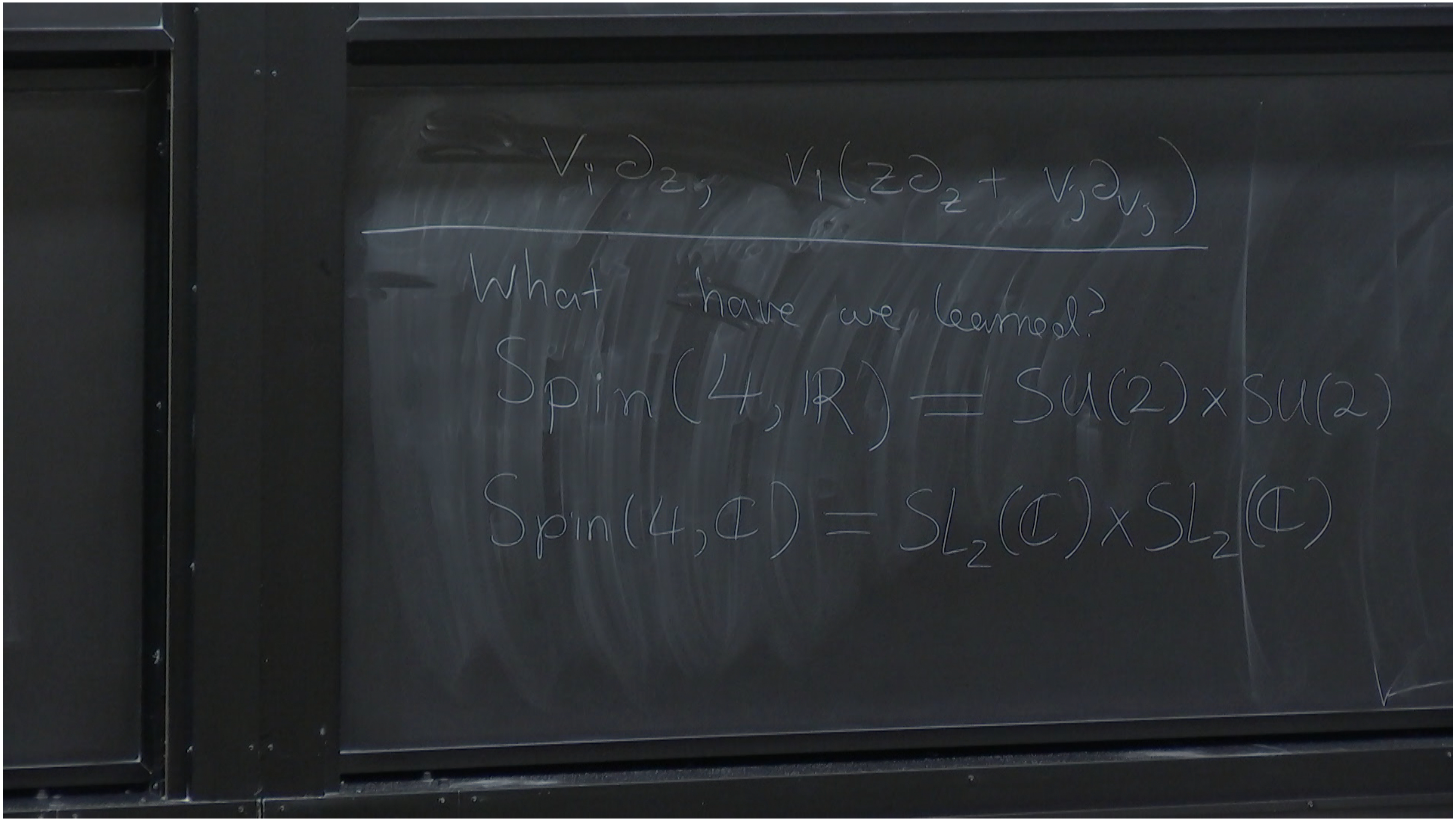
Other 3:

$\partial_z, \quad z\partial_z + v_i\partial_{v_i}, \quad z^2\partial_z + z\, v_i\partial_{v_i}$

3) Dilation:

$v_i\partial_{v_i}$

4) 4 special conformal transformations


$v_i \partial_z, \quad v_i(z\partial_z + v_j \partial_{v_j})$
What have we learned?
$Spin(4, \mathbb{R}) = SU(2) \times SU(2)$
$Spin(4, \mathbb{C}) = SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$

$$\mathbb{CP}^3 = \{ (u_1 : u_2 : u_3 : u_4),\ u_i \text{ not all } 0 \} / (u_1 \ldots u_4) \sim$$

$$\cup|$$

$$\mathbb{PT} = \{ (u_1 \ldots u_4),\ u_3, u_4 \text{ not zero} \}$$

Removing the locus where $u_3, u_4 = 0$

this is a $\mathbb{CP}^1$

$$= \{ (u_1 : u_2 : u_3 : u_4),\ u_i \text{ not all } 0 \} \,/\, (u_1 \ldots u_4) \sim (\lambda u_1 \ldots \lambda u_4)$$

$$= \{ (u_1 \ldots u_4),\ u_3, u_4 \text{ not zero} \}$$

ing the locus where $u_3, u_4 = 0$

this is a $\mathbb{CP}^1$

$\sim (\lambda u_1 \ldots \lambda u_4)$

$\mathcal{U} \subseteq \mathbb{P}\mathbb{T}$ is the set where $u_3 \neq 0$

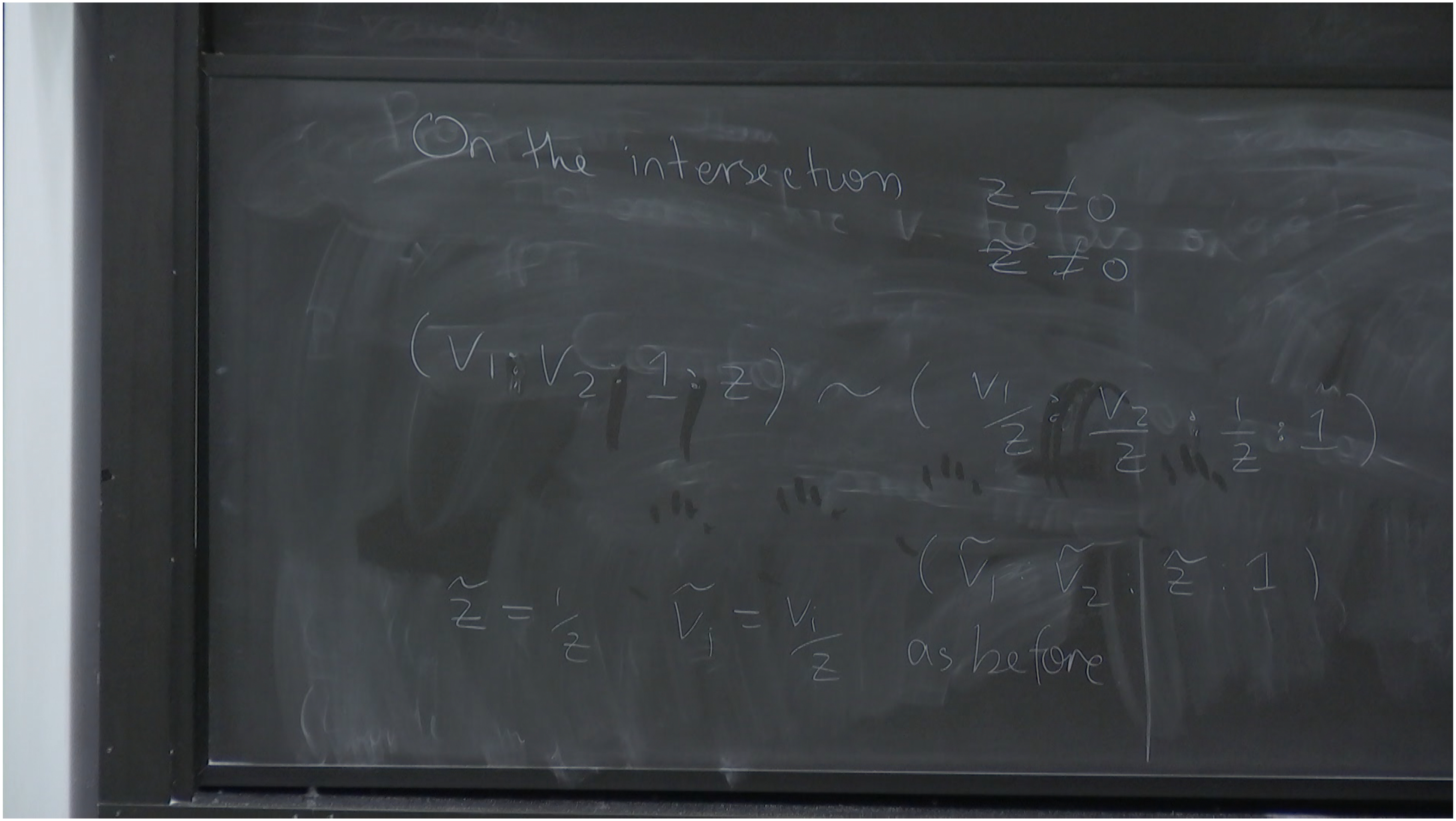
$\tilde{\mathcal{U}} \subseteq \mathbb{P}\mathbb{T}$ is the set $u_4 \neq 0$

On $\mathcal{U}$, we can rescale a vector to be

$$(u_1, u_2, 1, u_4) =: (v_1, v_2, 1, z)$$

$\tilde{\mathcal{U}}$ we have

$$(u_1, u_2, u_3, 1) =: (\tilde{v}_1, \tilde{v}_2, \tilde{z}, 1)$$


On the intersection
z ≠ 0
z̃ ≠ 0
(V1 : V2 : 1 : z) ~ (V1/z : V2/z : 1/z : 1)
~(Ṽ1 : Ṽ2 : z̃ : 1)
z̃ = 1/z
Ṽi = Vi/z
as before

The group $SL_4(\mathbb{C})$ acts on $\mathbb{C}\mathbb{P}^3$

by sending $(u_1 : \quad : u_4) \to A(u_1 : \quad : u_4)$

$A \in SL_4(\mathbb{C}) = \{ 4 \times 4$ complex matrices det $A = 1 \}$

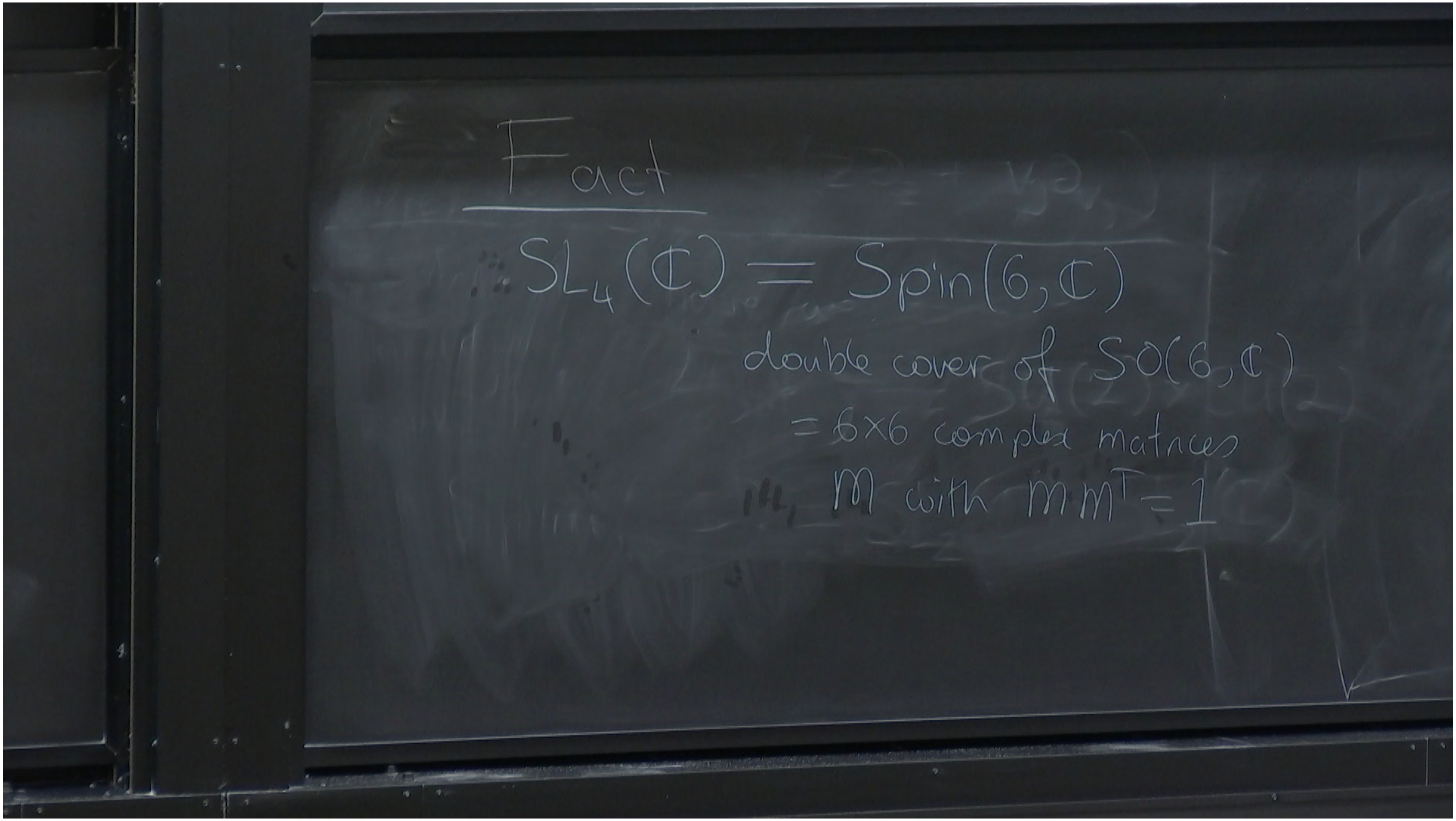
The group $SL_4(\mathbb{C})$ acts on $\mathbb{CP}^3$

by sending $(u_1 : \quad : u_4) \to A(u_1 : \quad : u_4)$

$A \in SL_4(\mathbb{C}) = \{ 4 \times 4$ complex matrices $\det A = 1 \}$

$\dim SL_4(\mathbb{C}) = 16 - 1 = 15$

$SL_4(\mathbb{C}) =$ conformal group (complexified) in 4 dimensions.


Fact
SL4(ℂ) = Spin(6,ℂ)
double cover of SO(6,ℂ)
= 6×6 complex matrices
M with MMᵀ = 1

$V =$ vector rep of $SL_4(\mathbb{C})$

$\Lambda^2 V$ is of dimension 6

$v_i \wedge v_j \quad 1 \le i < j \le 4$ is a basis

$\langle v_i \wedge v_j , v_k \wedge v_l \rangle = \varepsilon_{ijkl}$

is an inner product

SL_4 preserves the inner product.

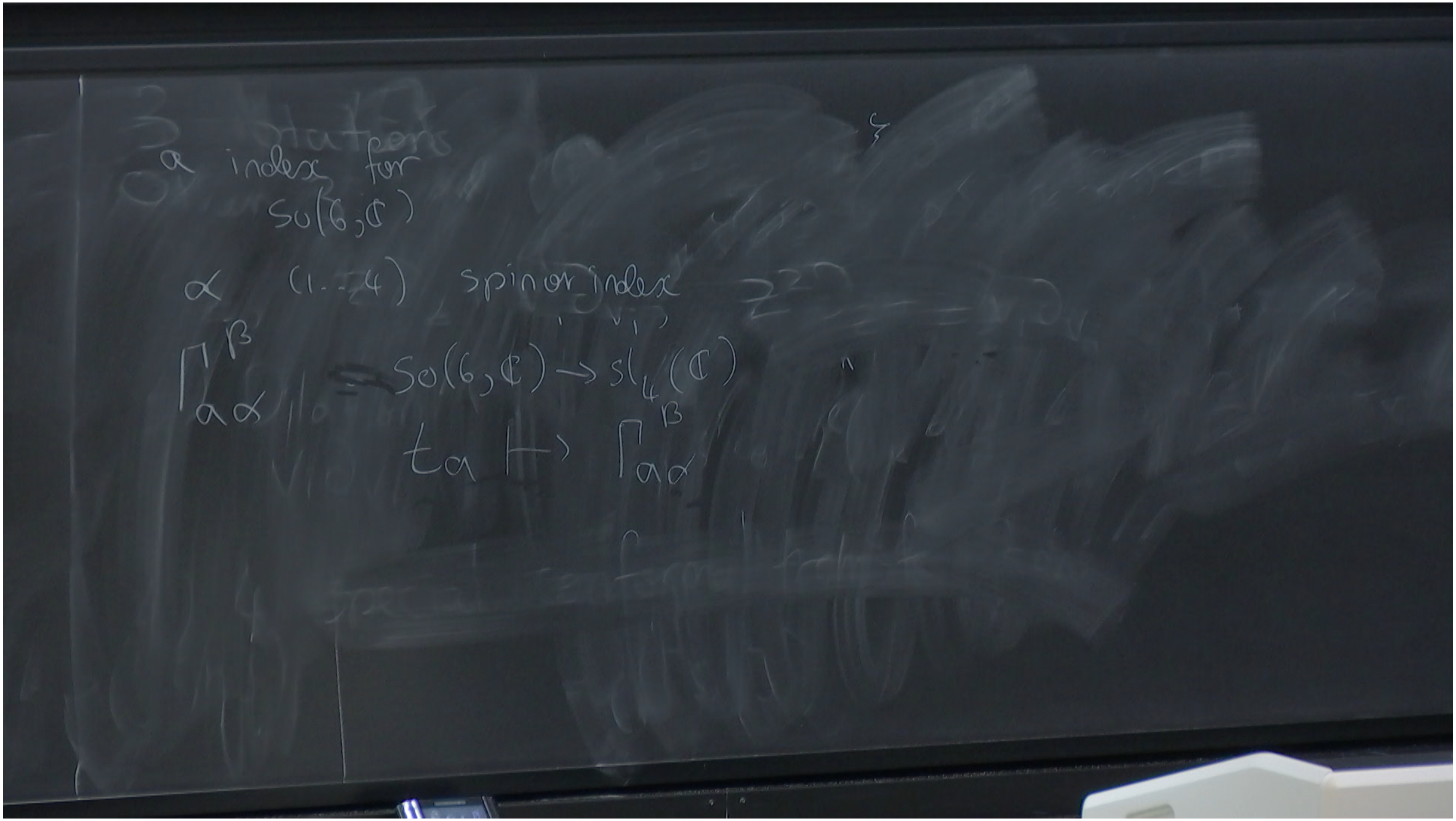
Solutions

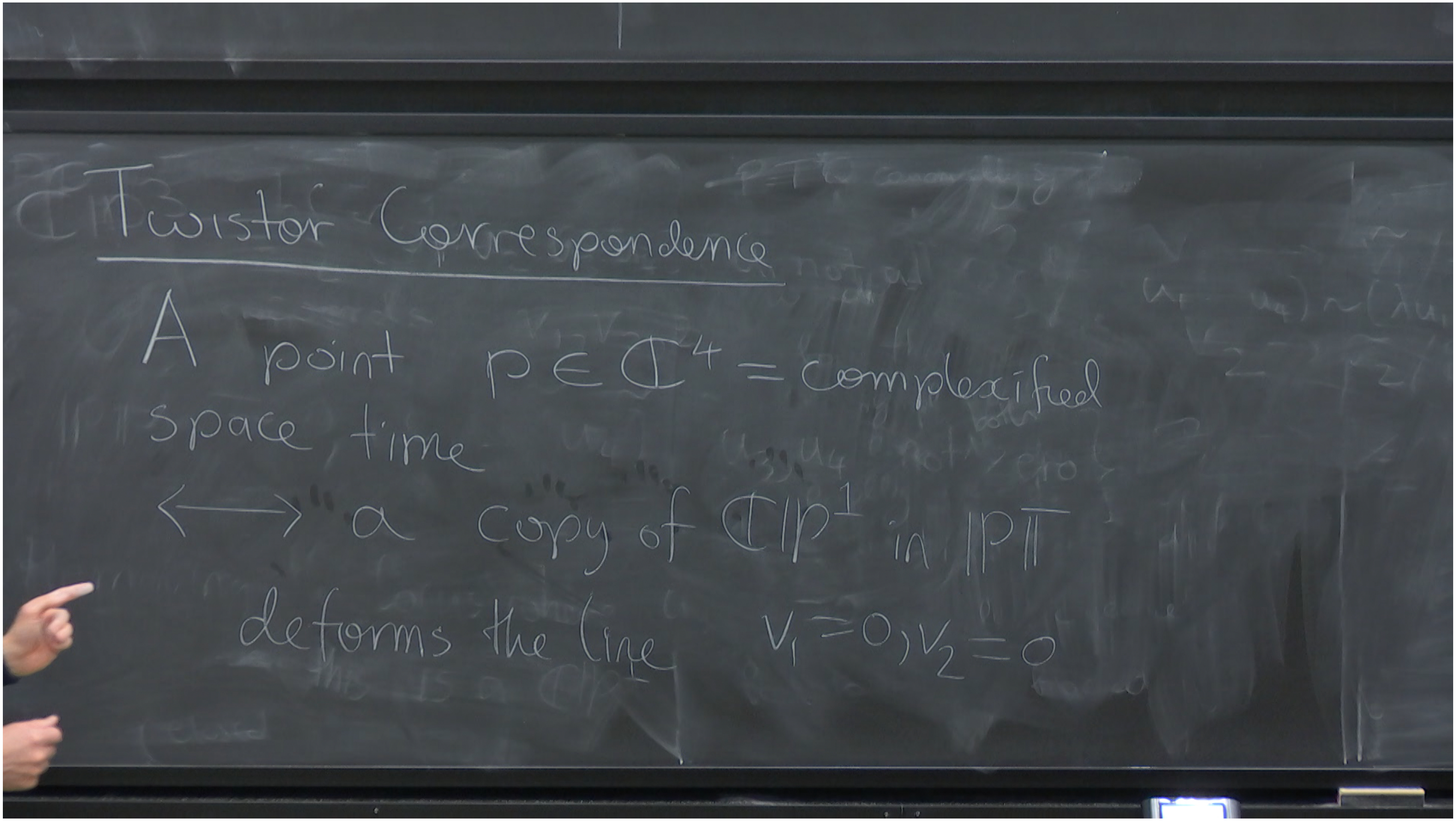
$$SL_4(\mathbb{C}) \longrightarrow SO(6,\mathbb{C})$$

which is a double cover.

$Spin(6,\mathbb{C})$ has a 4d spin rep.

$$Spin(6,\mathbb{C}) \xrightarrow[\simeq]{\Gamma \text{ matrices}} SL_4(\mathbb{C}).$$


a index for
so(6,ℂ)
α (1..4) spinor index
Γ aα β = so(6,ℂ) → sl4(ℂ)
ta ↦ Γ aα β


Twistor Correspondence
A point p ∈ ℂ⁴ = complexified space time
⟷ "a copy of ℂℙ¹ in ℙ𝕋"
deforms the line v₁ = 0, v₂ = 0

$p, q \in \mathbb{C}^4$ are light-like separated

$\iff \mathbb{CP}^1_p, \mathbb{CP}^1_q$ intersect.

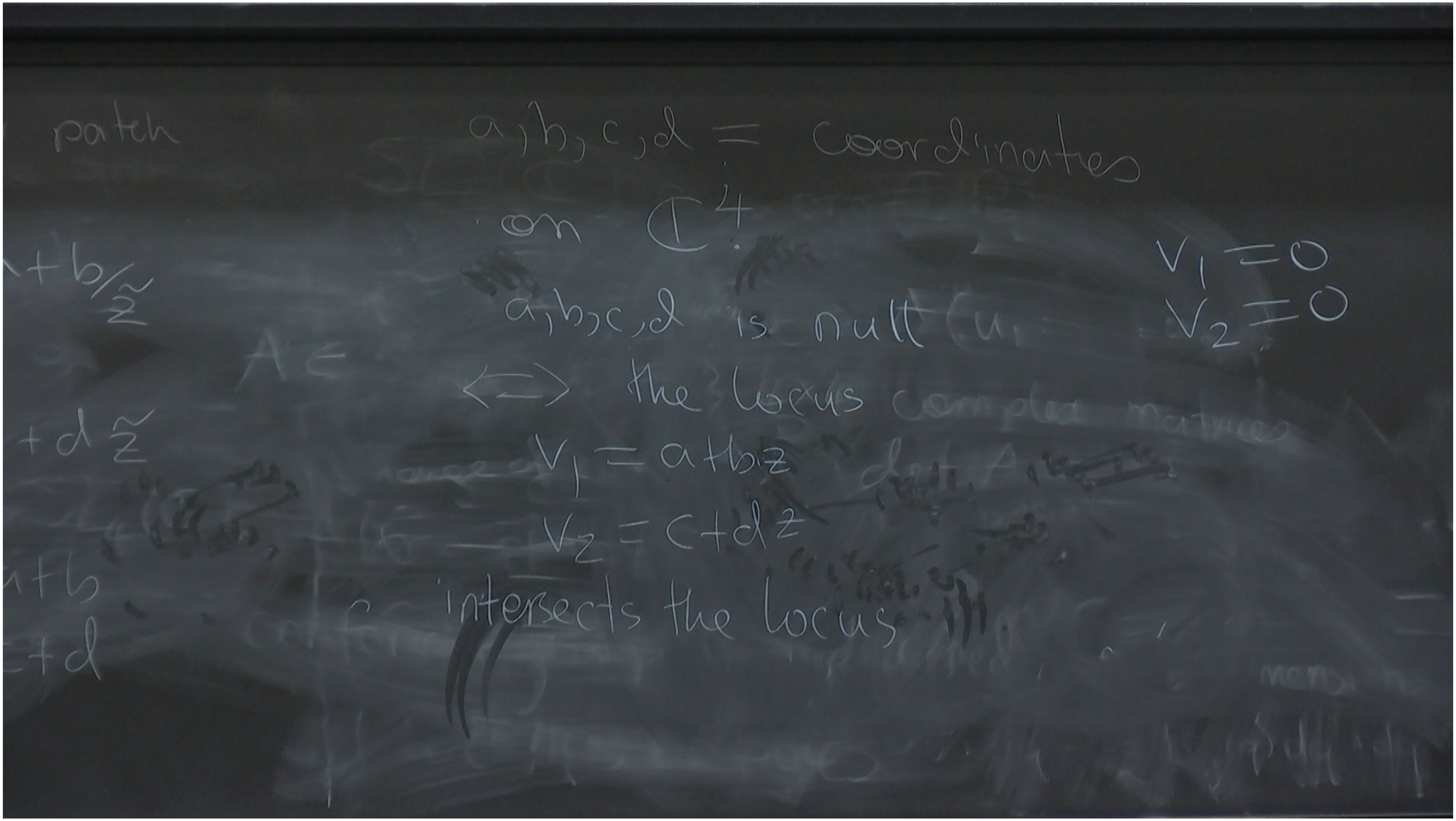
If we deform the eq^{n}s

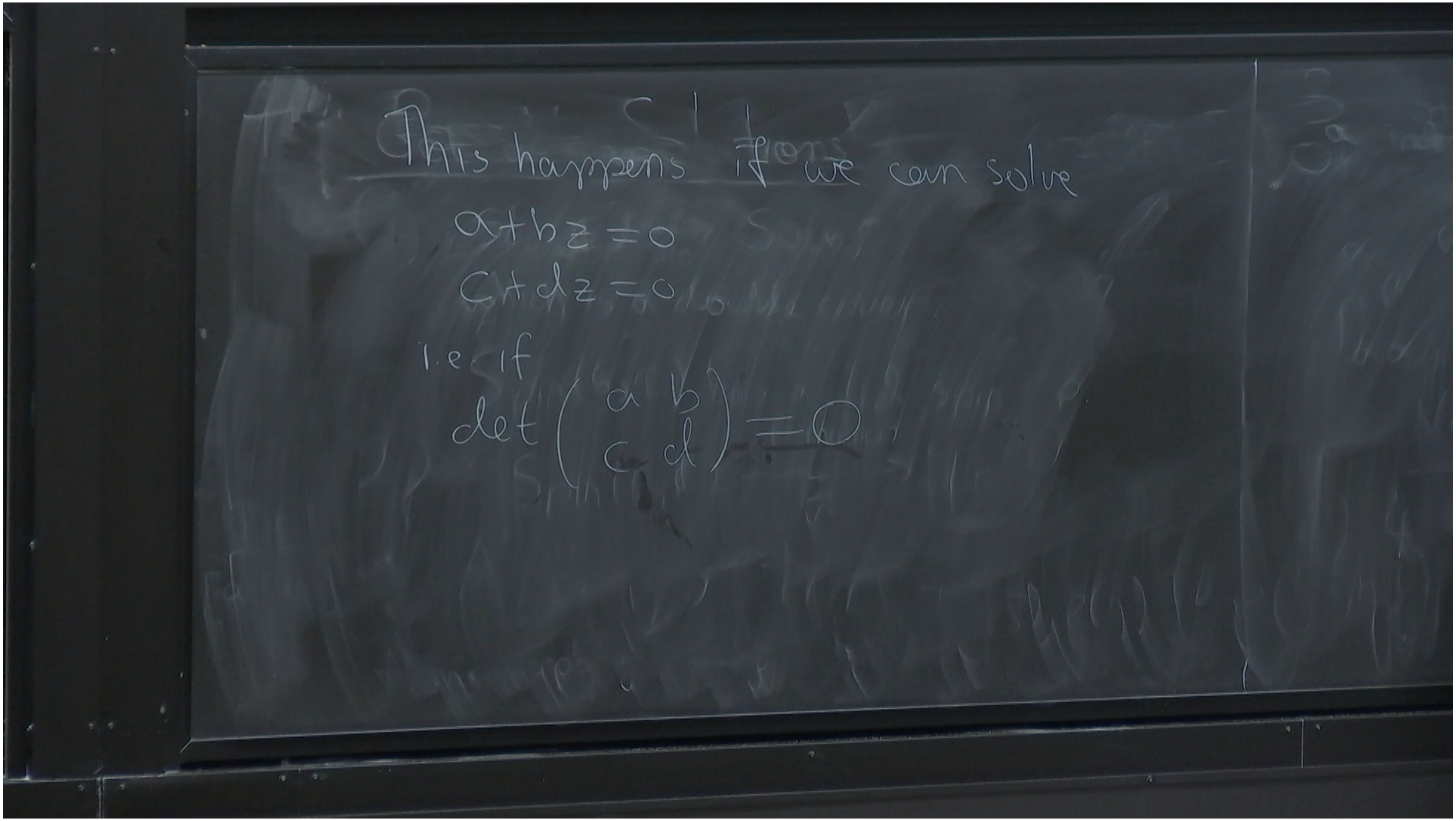
$V_1 = 0, \ V_2 = 0$

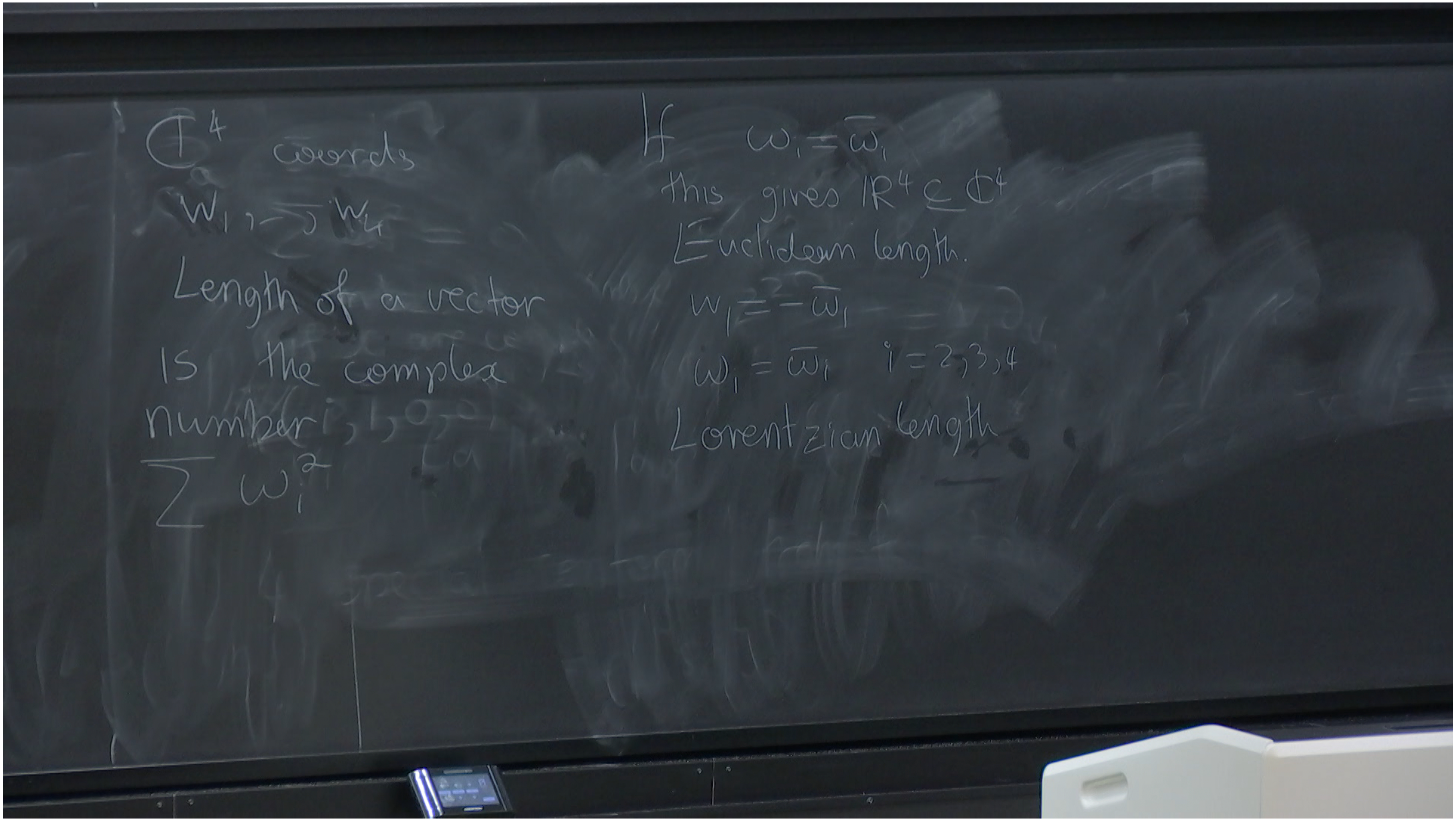
The only way to do this
working well on other patch
is to take a linear equation:

$$V_1 = a + bz$$

$$V_2 = c + dz$$


patch
a, b, c, d = coordinates
on ℂ⁴
a, b, c, d is null
⟺ the locus
V₁ = a + bz
V₂ = c + dz
intersects the locus
V₁ = 0
V₂ = 0


This happens if we can solve
a+bz=0
c+dz=0
i.e. if
det(a b / c d) = 0

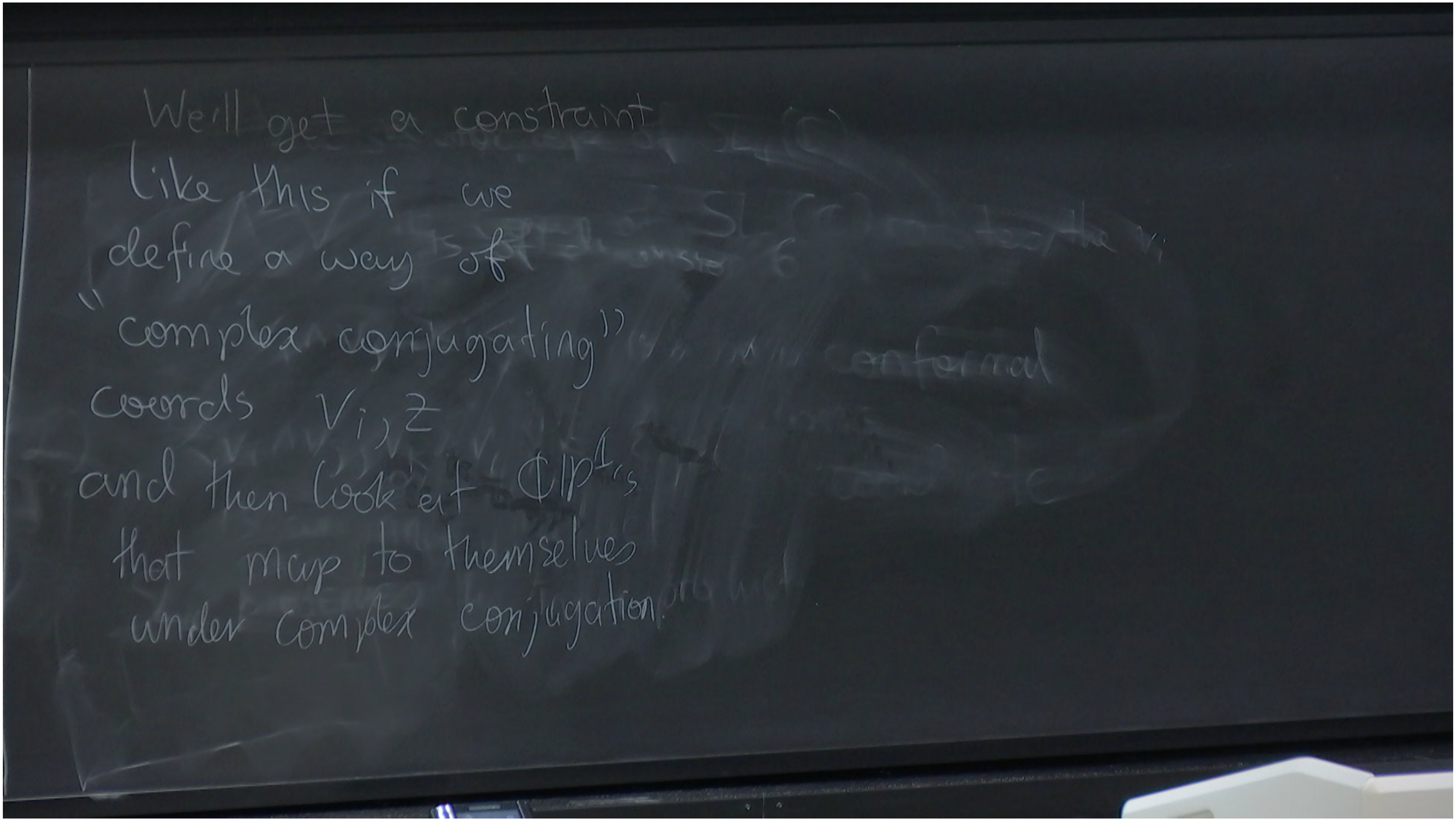

ℂ4 coords
w1, ..., w4
Length of a vector
is the complex
number
Σ wi2
If w1 = w̄1
this gives ℝ4 ⊂ ℂ4
Euclidean length.
w1 = −w̄1
wi = w̄i i = 2,3,4
Lorentzian length

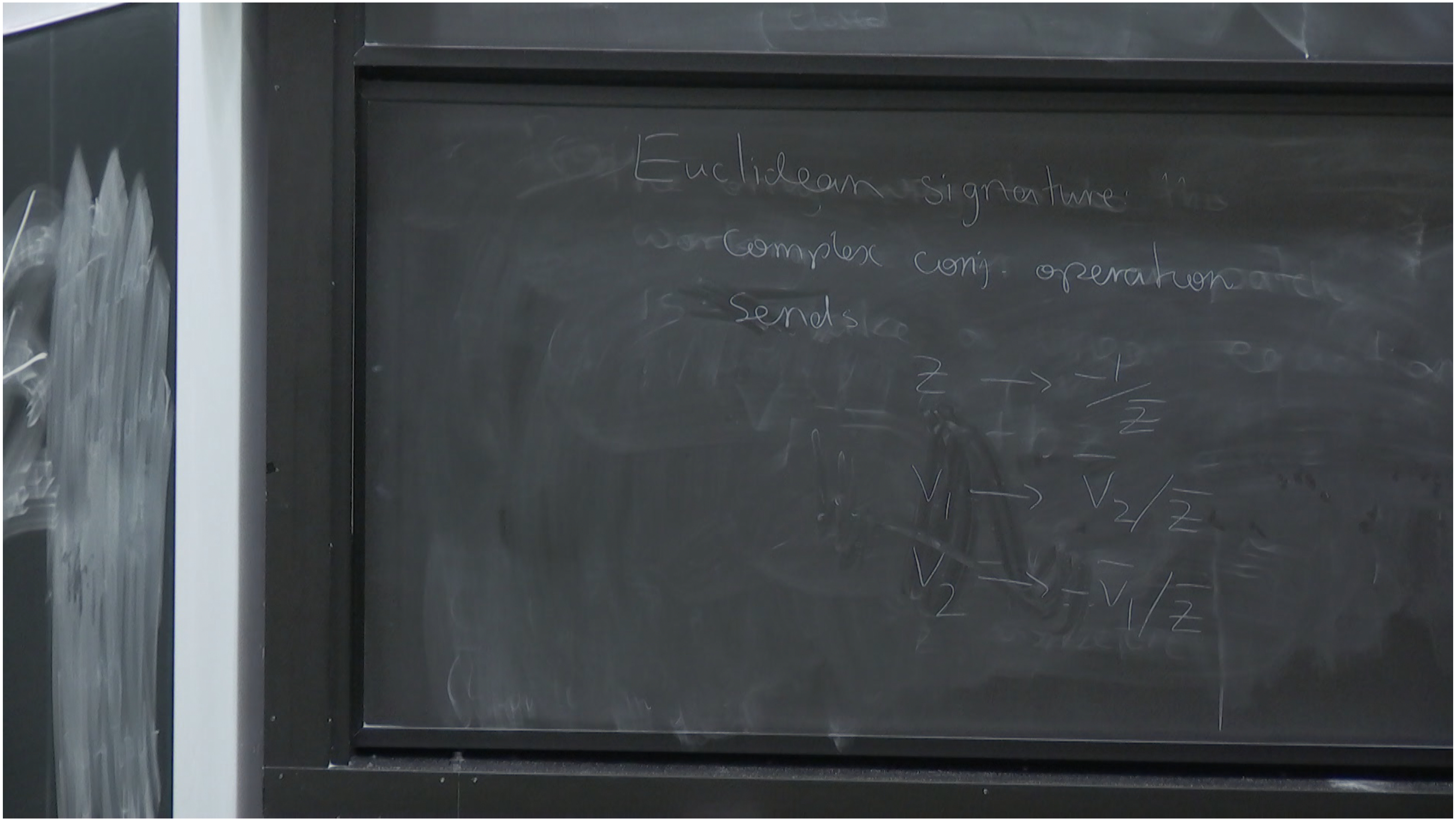
To get Euclidean metric from $\mathbb{C}^4$ with complex metric

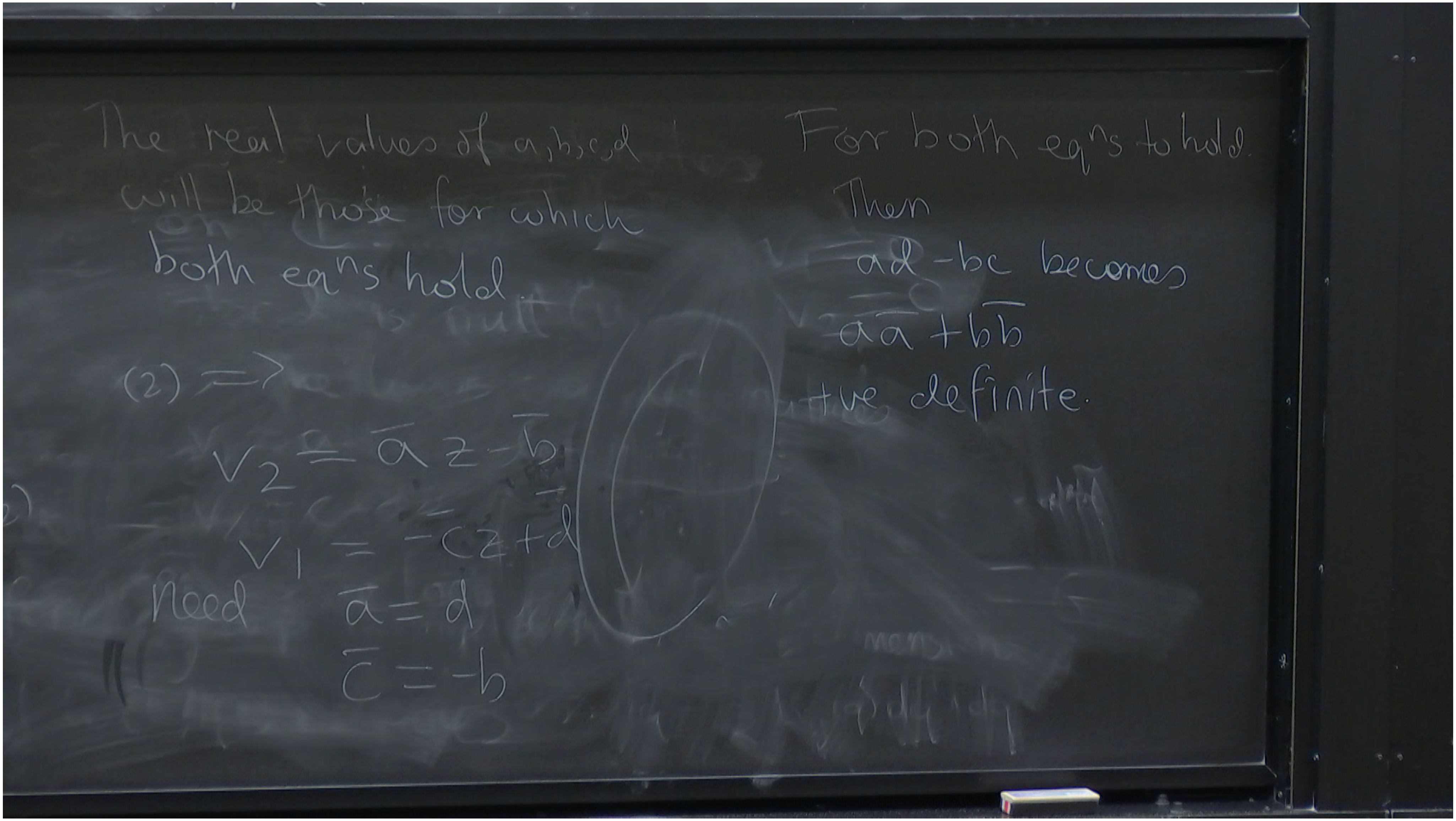
$$ad - bc$$

we need

$$d = \bar{a}$$
$$c = -\bar{b}$$


We'll get a constraint
like this if we
define a way of
"complex conjugating"
coords V_i, z
and then look at ℂℙ¹'s
that map to themselves
under complex conjugation.


Euclidean signature: the
complex conj. operation
sends
z → -1/z̄
V1 → V̄2/z̄
V2 → V̄1/z̄


The real values of a,b,c,d
will be those for which
both eqns hold
(2) ⟹
V2 = ā z − b̄
V1 = −c̄ z + d̄
Need ā = d
c̄ = −b
For both eqns to hold.
Then
ad − bc becomes
aā + bb̄
+ve definite.

$\rho : \mathbb{PT} \to \mathbb{PT}$

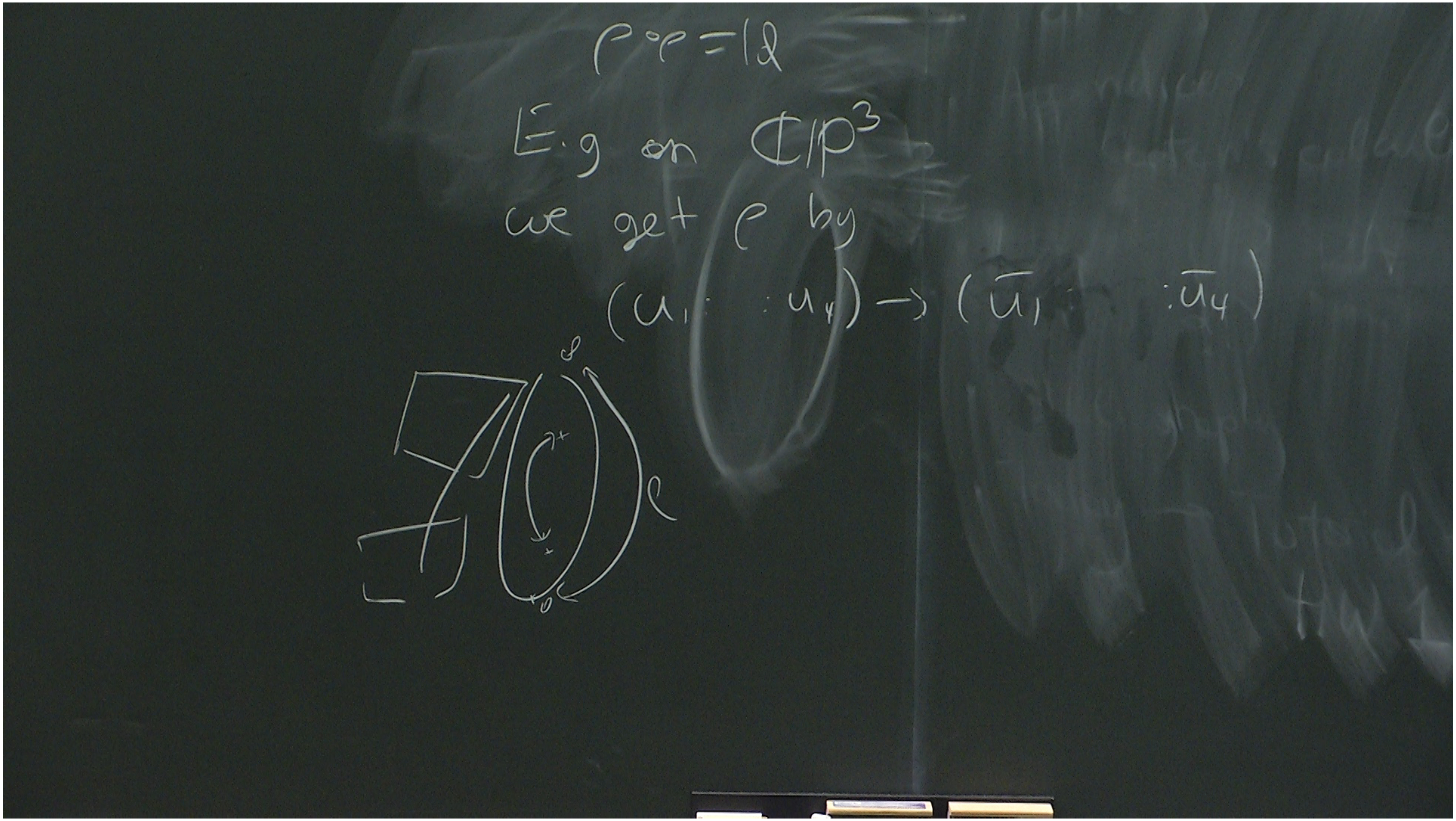
anti-holomorphic

$\rho \circ \rho = \mathrm{Id}$

E.g. on $\mathbb{CP}^3$

we get ρ by

$(u_1 : \cdots : u_4) \to (\bar{u}_1 : \cdots : \bar{u}_4)$


ρ∘ρ = Id
E.g on ℂP³
we get ρ by
(u₁ : : u₄) → (ū₁ : : ū₄)