Title: Antipodal (Self-)Duality in Planar N=4 Super-Yang-Mills Theory

Speakers: Lance Dixon

Series: Quantum Fields and Strings

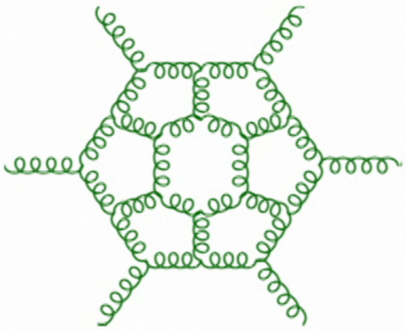
Date: April 04, 2023 - 2:00 PM

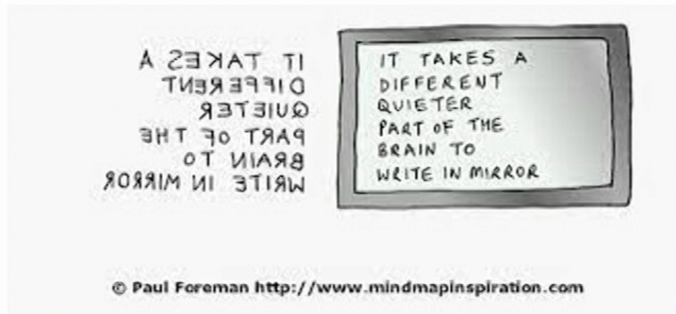
URL: https://pirsa.org/23040071

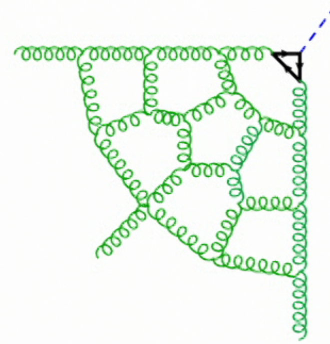
Abstract: Scattering amplitudes are where quantum field theory directly meets collider experiments. An excellent model for scattering in QCD is provided by N=4 super-Yang-Mills theory, particularly in the planar limit of a large number of colors, where the theory becomes integrable. The first nontrivial amplitude in this theory is for 6 gluons. It can be computed to 7 loops using a bootstrap based on the rigidity of the function space of multiple polylogarithms, together with a few other conditions. One can also bootstrap a particular form factor, for the chiral stress-tensor operator to produce 3 gluons, through 8 loops. This form factor is the N=4 analog of the LHC process, gluon gluon --> Higgs + gluon. Remarkably, the two sets of results are related by a mysterious `antipodal' duality, which exchanges the role of branch cuts and derivatives. Furthermore, this duality is `explained' by an antipodal self-duality of the 4 gluon form factor of the same operator; although it is still fair to say of the self-duality, `who ordered that?'

Zoom link: https://pitp.zoom.us/j/96265005656?pwd=Qndza3pIKzdmZVJGL0s1ZUZkRmp4QT09

Antipodal (Self-)Duality in Planar N=4 Super-Yang-Mills Theory







Lance Dixon (SLAC)

LD, Ö. Gürdoğan, Y.-T.Liu, A. McLeod, M. Wilhelm

2112.06243, 2204.11901, 2212.02410

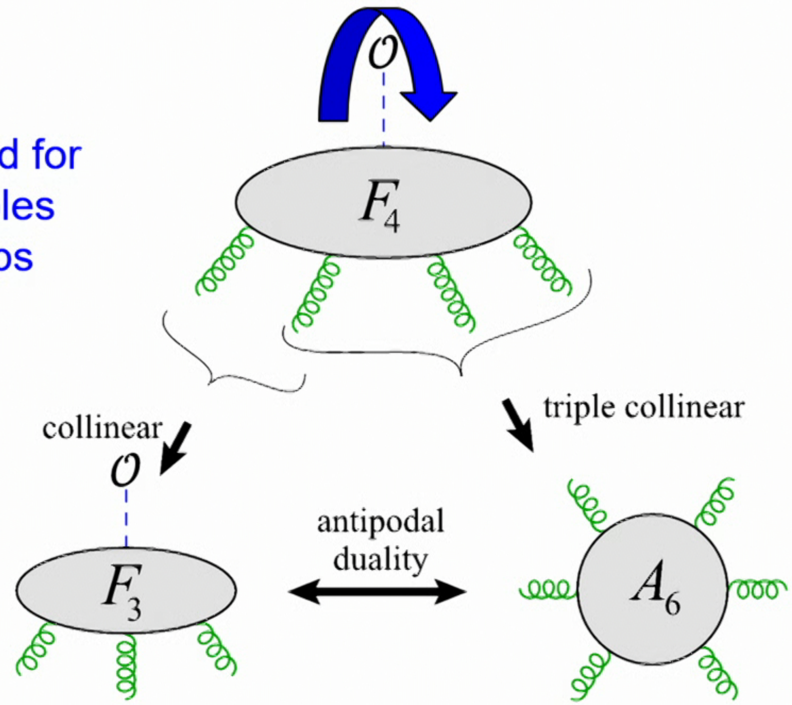
Quantum Fields and Strings Seminar
Perimeter Institute
4 April 2023

NATIONAL ACCELERATOR LABORATORY

This talk in one picture:

Antipodal Self Duality

validated for
4 variables
to 3 loops

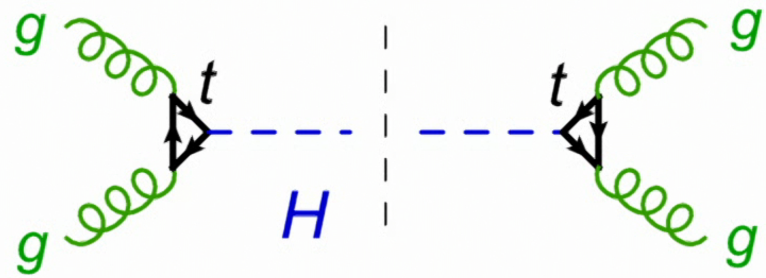


validated for
2 variables
to 8 loops!

Producing Higgs bosons at LHC

Higgs boson dominantly produced by **gluon fusion**, a quantum process at “one loop”, mediated by **top quark,** because ***t*** couples strongly to both gluons and Higgs

Leading Order (LO) cross section $= |\text{one} - \text{loop amplitude}|^2$



- Since $\boldsymbol{2m_{top}} = 350\ \text{GeV}$
  $\gg \boldsymbol{m_{Higgs}} = 125\ \text{GeV}$,

interaction between gluons and Higgs is approximately local (mediated by a leading dim 5 operator $\mathcal{O} = HG_{\mu\nu}^{a}G^{\mu\nu\, a}$)

Perturbative Short-Distance Cross Section

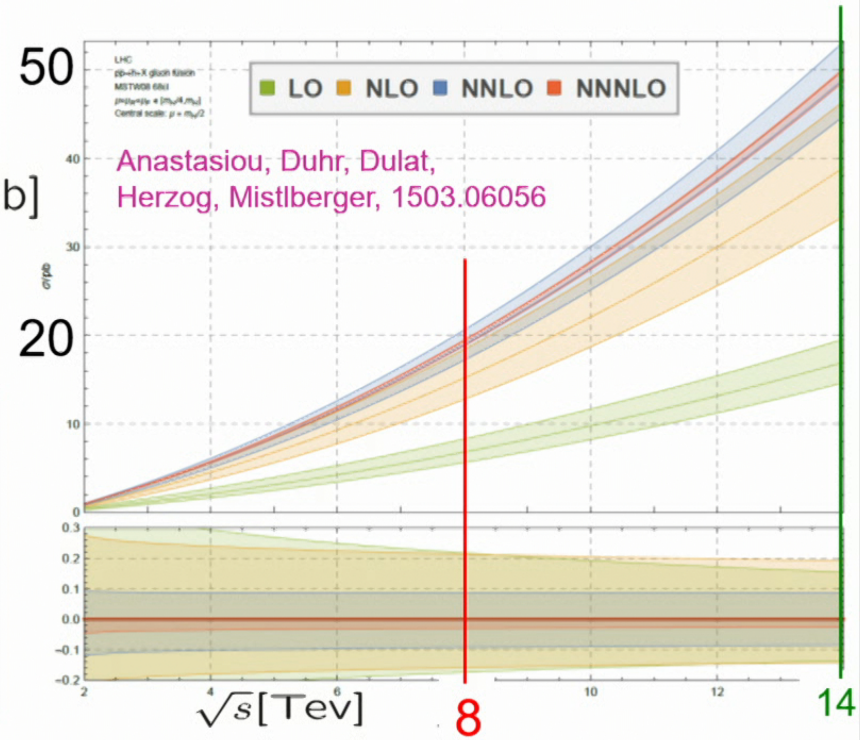
$$\hat{\sigma}(\alpha_s) = {\alpha_s}^n [\, \hat{\sigma}^{(0)} + \alpha_s\, \hat{\sigma}^{(1)} + \alpha_s^2\, \hat{\sigma}^{(2)} + \alpha_s^3\, \hat{\sigma}^{(3)} + \cdots]$$

LO NLO NNLO NNNLO

Higgs gluon fusion cross section at LHC vs. CM energy $\sqrt{s}$

LO approx. is terrible!
LO → NNNLO
→ factor of 2 or 3 increase!

Poor convergence
of expansion in $\alpha_s(\mu)$
Uncertainty bands from varying
$\mu_R = \mu_F = \mu$
Necessitates high orders!



Perturbative Short-Distance Cross Section

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LO NLO NNLO NNNLO

Higgs gluon
cross sectio
vs. CM ener

LO approx. is terrible!
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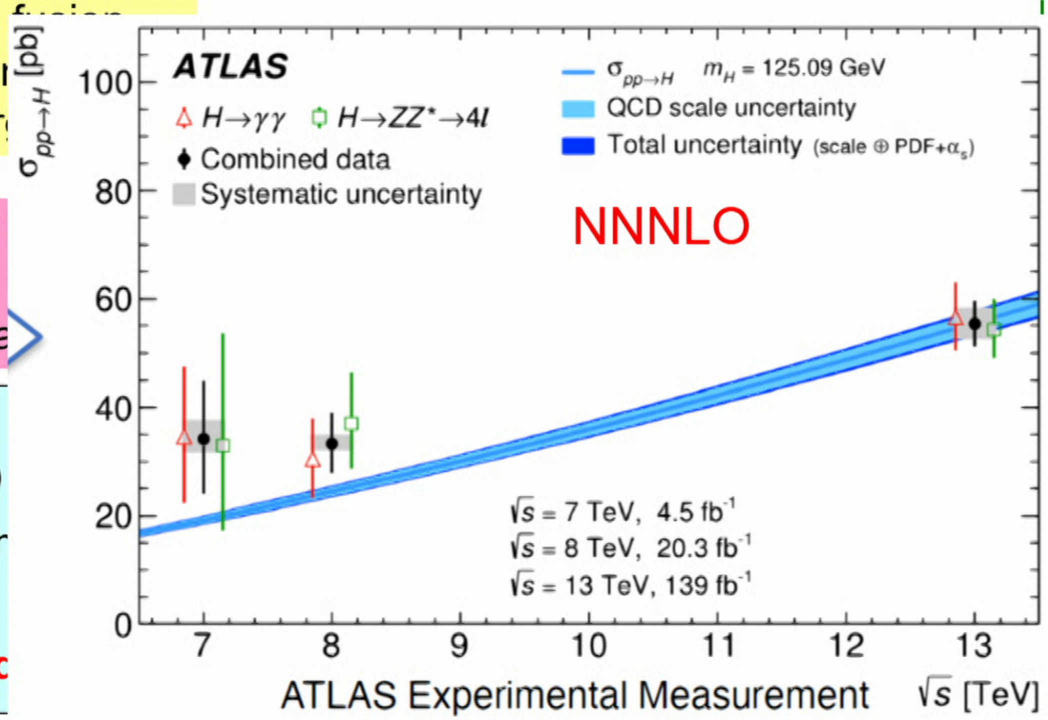
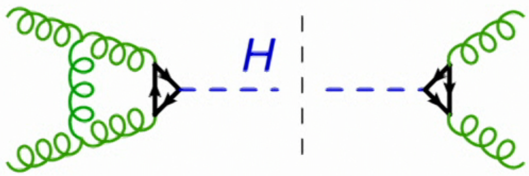
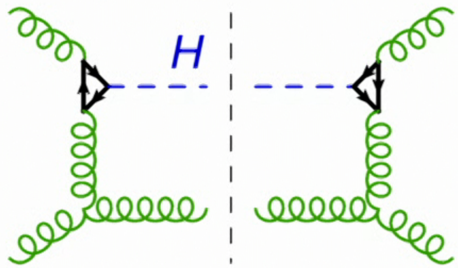


Some NLO QCD Feynman diagrams

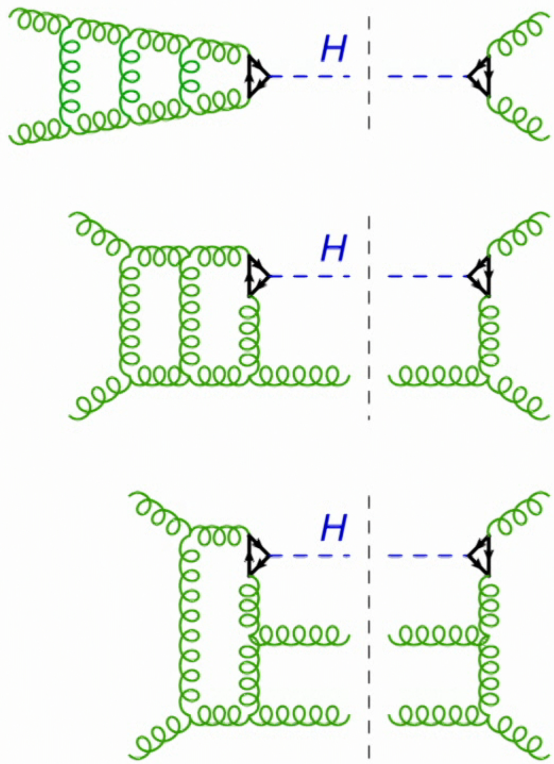


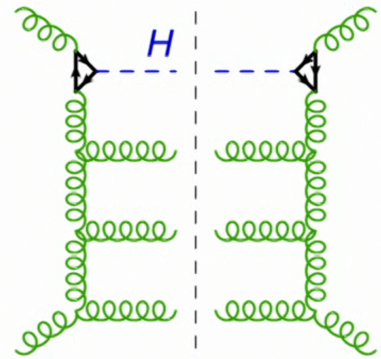
virtual $gg \to H$



real, $gg \to Hg$

Some NNNLO QCD topologies

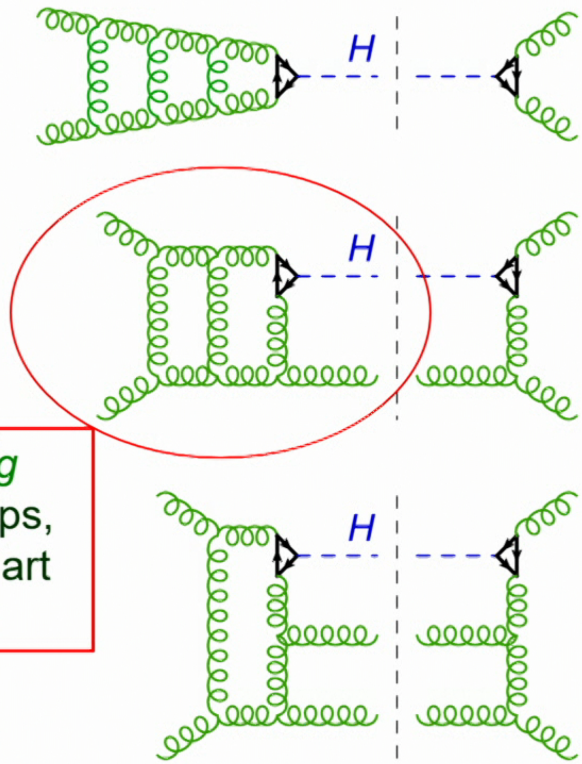


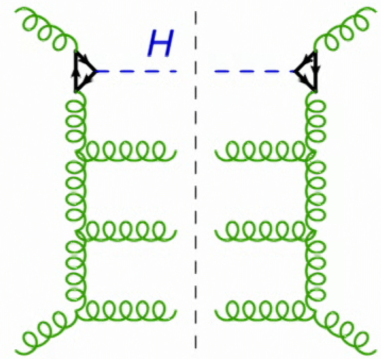


+ …
+ quarks
+ operator renormalization
+ $1/m_t^2$ corrections
+ parton distributions

Scattering amplitudes are the underlying building blocks

Some NNNLO QCD topologies





$gg \to Hg$
@ 2 loops,
state of art
in QCD

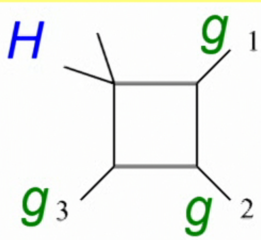
+ …
+ quarks
+ operator renormalization
+ $1/m_t^2$ corrections
+ parton distributions

Scattering amplitudes are the underlying building blocks

Beyond tree level

- Loop level Feynman diagrams come with an instruction to integrate over all loop momenta

- For example, at one loop the amplitude for $gg \to Hg$ involves the "scalar box" integral



$$= \int \frac{d^4 p}{p^2 (p-p_1)^2 (p-p_1-p_2)^2 (p-p_1-p_2-p_3)^2}$$

$$= \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{12}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{23}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right) + \cdots$$

where the dilogarithm is $\mathrm{Li}_2(x) \equiv -\int_0^x \frac{dt}{t} \ln(1-t)$

One loop not too bad

- For any number of external particles, all one-loop integrals (even in dimensional regularization, $D = 4 - 2\epsilon$) can be reduced to scalar box integrals + simpler

Brown-Feynman (1952), Melrose (1965), ‘t Hooft-Veltman (1974), Passarino-Veltman (1979), van Neerven-Vermaseren (1984), Bern, LD, Kosower (1992)

→ combinations of $\mathrm{Li}_2(x) \equiv -\int_0^x \frac{dt}{t} \ln(1-t)$

where x is (many different) functions of the kinematic variables (Mandelstam invariants),

plus logarithms

Multi-loop much more complex

- At L loops, instead of just Li_2's, get special functions with up to $2L$ integrations
- Weight $2L$ "iterated integrals"
- **Best case**: generalized polylogarithms, defined iteratively by

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

and $G(\vec{0}_n, x) = \frac{(\ln x)^n}{n!}$

- Still very intricate multi-variate functions

Planar N=4 SYM, "hydrogen atom" of amplitudes

- QCD's maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group $SU(N_c)$, in large N_c (planar) limit
- Structure very rigid:

$$\text{Amplitudes} = \sum_i rational_i \times transcendental_i$$

- For planar N=4 SYM, rational structure well understood, focus on transcendental functions.
- Furthermore, at least three dualities hold:

1. AdS/CFT
2. Amplitudes dual to Wilson loops

Transcendental Structure

- N=4 SYM amplitudes have "uniform weight" (transcendentality) $2L$ at loop order L
- Weight ~ number of integrations, e.g.

$$\ln(s) = \int_1^s \frac{dt}{t} = \int_1^s d\ln t \qquad 1$$

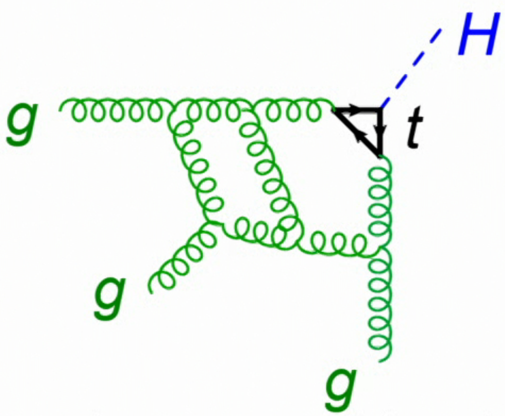
$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t) = \int_0^x d\ln t \cdot [-\ln(1-t)] \qquad 2$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) \qquad n$$

- QCD amps typically all weights from 0 to $2L$

"Goldilocks Process": $gg \rightarrow Hg$

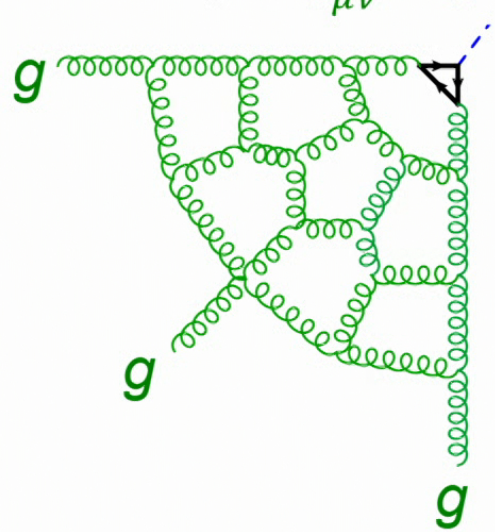
QCD state of art is two loops (not counting top quark loop)
Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554



Recent progress on integrals for 3 loops
DiVita, Mastrolia, Schubert, Yundin, 1408.3107; Canko, Syrrakos, 2112.14275; Henn, Lim, Torres Bobadilla, 2302.12776

Can get to **eight** loops in planar N=4 SYM
LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm 2012.12286, 2112.06243, 2204.11901

$$\mathcal{O} = G_{\mu\nu}^{a\,[SD]} G^{\mu\nu\,a\,[SD]}$$



Harmonic Polylogarithms in one variable (HPL{0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Generalize classical polylogs
- Define HPLs by iterated integration:

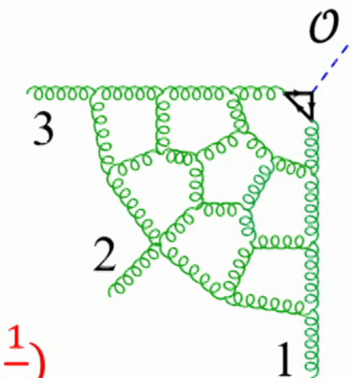
$$H_{0,\vec{w}}(x) = \int_0^x \frac{dt}{t} H_{\vec{w}}(t), \qquad H_{1,\vec{w}}(x) = \int_0^x \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives:

$$dH_{0,\vec{w}}(x) = H_{\vec{w}}(x) d\ln x \qquad dH_{1,\vec{w}}(x) = -H_{\vec{w}}(x) d\ln(1-x)$$

- [Symbol alphabet: $\mathcal{L} = \{x, 1-x\}$]
- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $n = 2L$ is number of binary strings: 2^{2L}
- Branch cuts dictated by first integration/entry in symbol
- Derivatives dictated by last integration/entry in symbol

A three-gluon form factor in planar N=4 SYM



$+ \sim 10^9$ more

$u = \frac{s_{12}}{s_{123}}, \quad v = \frac{s_{23}}{s_{123}}, \quad w = \frac{s_{31}}{s_{123}} = 1 - u - v$

$v \to \infty$ → Harmonic polylogarithms $H_{\vec{w}} \equiv H_{\vec{w}}(1 - \frac{1}{u})$

$$F_3^{(1)}(v \to \infty) = 2H_{0,1} + 6\zeta_2$$

$$F_3^{(2)}(v \to \infty) = -8H_{0,0,0,1} - 4H_{0,1,1,1} + 12\zeta_2 H_{0,1} + 13\zeta_4$$

$$\begin{aligned} F_3^{(3)}(v \to \infty) = {} & 96H_{0,0,0,0,0,1} + 16H_{0,0,0,1,0,1} + 16H_{0,0,0,1,1,1} + 16H_{0,0,1,0,0,1} + 8H_{0,0,1,0,1,1} \\ & + 8H_{0,0,1,1,0,1} + 16H_{0,1,0,0,0,1} + 8H_{0,1,0,0,1,1} + 12H_{0,1,0,1,0,1} + 4H_{0,1,0,1,1,1} \\ & + 8H_{0,1,1,0,0,1} + 4H_{0,1,1,0,1,1} + 4H_{0,1,1,1,0,1} + 24H_{0,1,1,1,1,1} \\ & - \zeta_2(32H_{0,0,0,1} + 8H_{0,0,1,1} + 4H_{0,1,0,1} + 52H_{0,1,1,1}) \\ & - \zeta_3(8H_{0,0,1} - 4H_{0,1,1}) - 53\zeta_4 H_{0,1} - \frac{167}{4}\zeta_6 + 2(\zeta_3)^2 \end{aligned}$$

8 loop result has $\sim 2^{2\times 8-2} = 16{,}384$ terms

6-gluon amplitude in planar N=4 SYM

$+ \sim 10^9$ more

Depends on 3 "dual-conformal cross-ratios, $(\hat{u}, \hat{v}, \hat{w})$

Simplest for $(\hat{u}, \hat{v}, \hat{w}) = (1, \hat{v}, \hat{v})$

Let $H_{\vec{w}} \equiv H_{\vec{w}}(1 - \frac{1}{\hat{v}})$

$$A_6^{(1)}(1,\hat{v},\hat{v}) = 2H_{0,1}$$

$$A_6^{(2)}(1,\hat{v},\hat{v}) = -8H_{0,1,1,1} - 4H_{0,0,0,1} - 4\zeta_2 H_{0,1} - 9\zeta_4$$

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Exact map at symbol level ($\zeta_n \to 0$), with $\frac{1}{\hat{v}} = 1 - \frac{1}{u}$, $0 \leftrightarrow 1$,
if you also reverse order of HPL indices!!!
Works to 7 loops, where $\sim 2^{2\times 7-2} = 4{,}096$ terms agree

6-gluon amplitude in planar N=4 SYM

$+ \sim 10^9$ more

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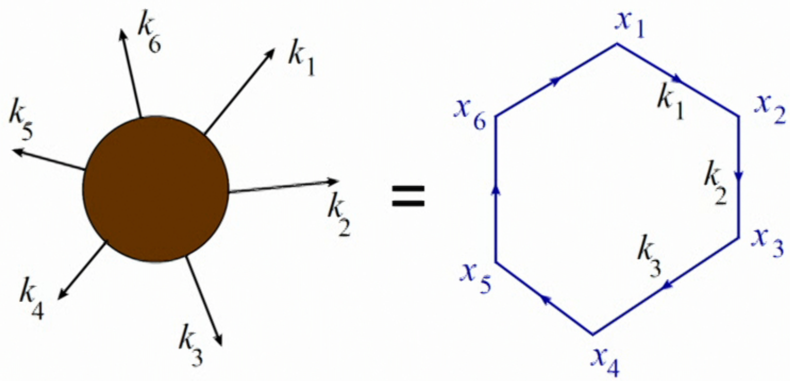
Amplitudes = Wilson loops



Spacetime Dual Spacetime

- Polygon vertices x_i are not positions but dual momenta,

  $x_i - x_{i+1} = k_i$

- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

Duality holds at both strong and weak coupling

weak-weak duality, holds order-by-order

Dual conformal invariance

- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2\, x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

- Fixed, up to functions of invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

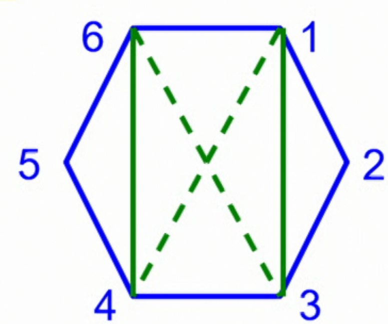
- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for $n = 4,5$

$n = 6$ → precisely 3 ratios:

$$\hat{u} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

$$\hat{v} = \frac{s_{23}s_{56}}{s_{234}s_{123}}$$

$$\hat{w} = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$



Dual conformal invariance

- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 \, x_j^2}$

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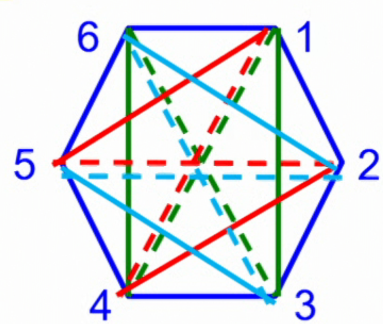
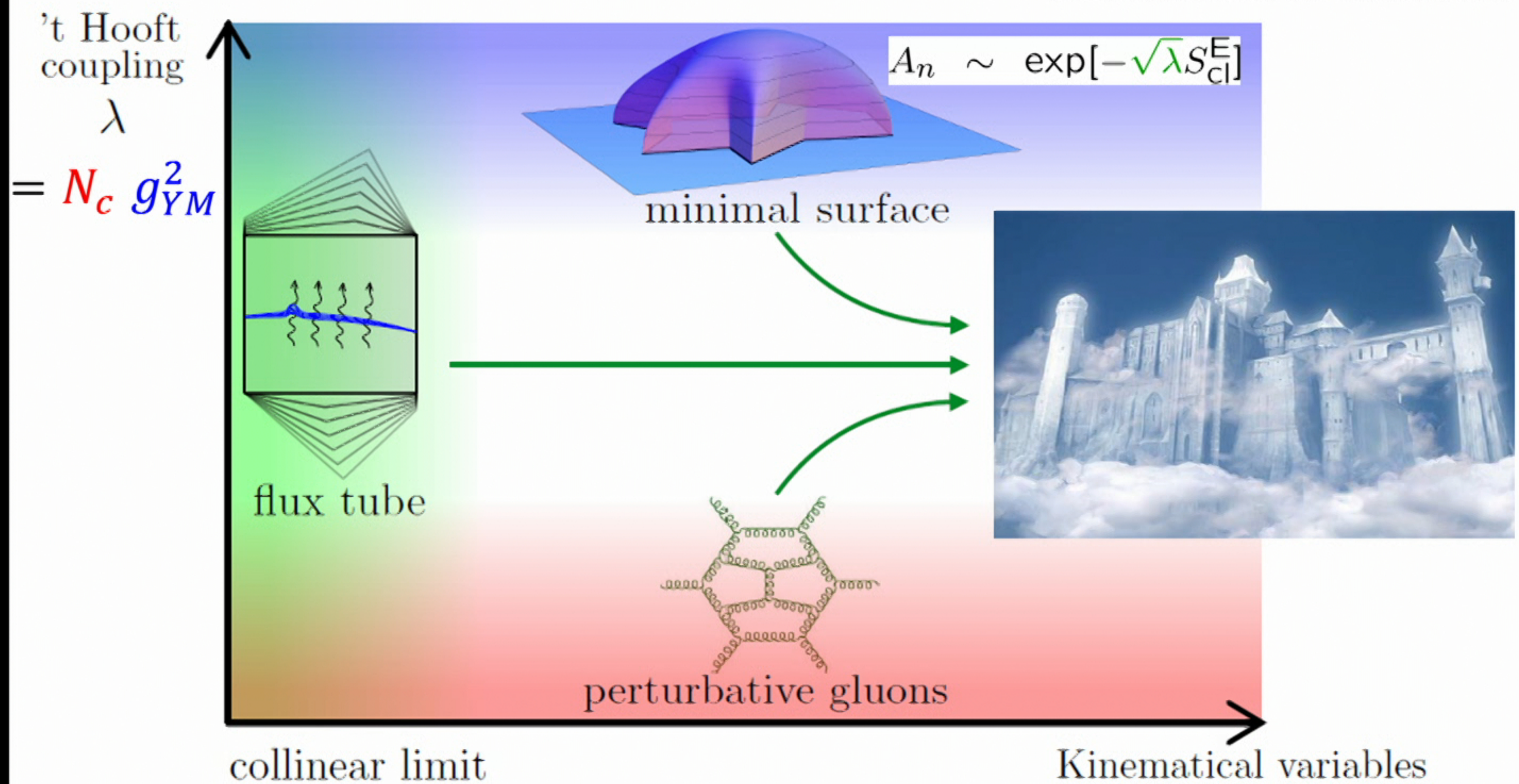


Solving Planar N=4 SYM Scattering

Images: A. Sever, N. Arkani-Hamed



Different routes to perturbative amplitudes

Feynman diagrams	Unitarity	Bootstrap
Draw all Feynman graphs G_i	Evaluate all unitarity cuts C_α	Bootstrap: Guess $A = \sum_m c_m F_m$ F_m known functions $c_m \in \mathbb{Q}$ unknown constants
↓	↓	↓
Evaluate all Feynman rules: I_i	Construct **local**: integrand I	Solve constraints, linear equations for $c_m \to r_m$
↓	↓	↓
Perform all loop integrations: A_i	Perform all loop integrations: A_α	
↓	↓	
$A = \sum_i A_i$	$A = \sum_\alpha A_\alpha$	$A = \sum_m r_m F_m$

Hexagon function bootstrap

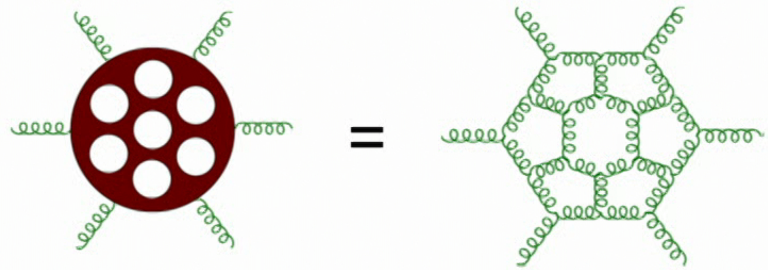
Loops

3 LD, Drummond, Henn, 1108.4461, 1111.1704;

4,5 Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;

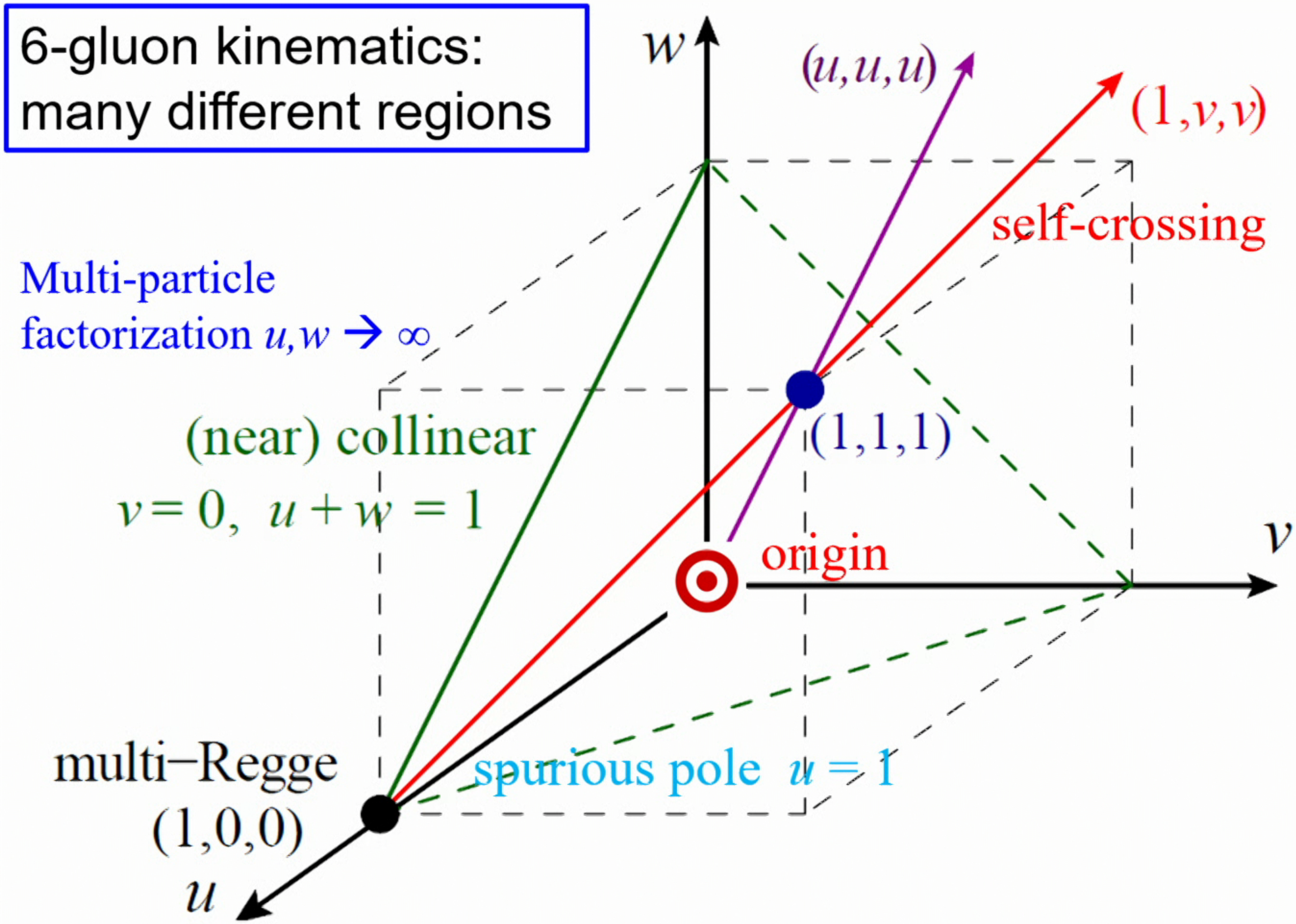
6,7 Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890, 1906.07116; LD, Dulat, 23mm.nnnnn (NMHV 7 loop)

- Planar N=4 SYM: Make ansatz for space of multiple polylogarithms (MPLs) in which 6-point amplitude resides to determine it directly. No explicit Feynman integrands or integrals.
- Use boundary data from flux tube (pentagon) OPE
  Basso, Sever, Vieira, 1303.1396, 1306.2058,…
- Same general method used for “Higgs” form factor



+ $\sim 10^9$ more

6-gluon kinematics:
many different regions



Iterate to get **symbol**

- $\{n\text{-}1,1\}$ coaction can be applied iteratively
- Define $\{n\text{-}2,1,1\}$ double coproducts, F^{s_k,s_j}, via derivatives of $\{n\text{-}1,1\}$ single coproducts F^{s_j}:

$$dF^{s_j} \equiv \sum_{s_k \in \mathcal{L}} F^{s_k,s_j}\, d\ln s_k$$

- And so on for $\{n\text{-}m,1,\ldots,1\}$ $m^{\rm th}$ coproducts of F.
- Maximal iteration, n times for weight n function, is the symbol,

["ln" is implicit in s_{i_k}]

$$\mathcal{S}[F] \equiv \sum_{s_{i_1},\ldots,s_{i_n} \in \mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}}\, s_{i_1} \otimes \ldots \otimes s_{i_n}$$

where now $F^{s_{i_1},\ldots,s_{i_n}}$ are just rational numbers

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Example: Classical polylogarithms

$$\mathrm{Li}_1(x) = -\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

- Regular at $x = 0$, branch cut starts at $x = 1$.
- Iterated differentiation gives the symbol:

$$\mathcal{S}[\mathrm{Li}_n(x)] = \mathcal{S}[\mathrm{Li}_{n-1}(x)] \otimes x$$
$$= \cdots = -(1-x) \otimes x \otimes x \ \ldots \otimes x$$

- Branch cut discontinuities displayed in first entry of symbol, e.g. clip off leading $(1-x)$ to compute discontinuity at $x = 1$.
- Derivatives computed from symbol by clipping last entry, multiplying by that $d\ln(\ldots)$. **Alphabet** $\mathcal{L} = \{x, 1-x\}$

Symbol alphabets for n-gluon amplitudes

parity-odd letters, algebraic in $\hat{u}, \hat{v}, \hat{w}$

$n = 6$ has 9 letters: $\mathcal{L}_6 = \{\hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w\}$

Goncharov, Spradlin, Vergu, Volovich, 1006.5703; LD, Drummond, Henn, 1108.4461; Caron-Huot, LD, von Hippel, McLeod, 1609.00669

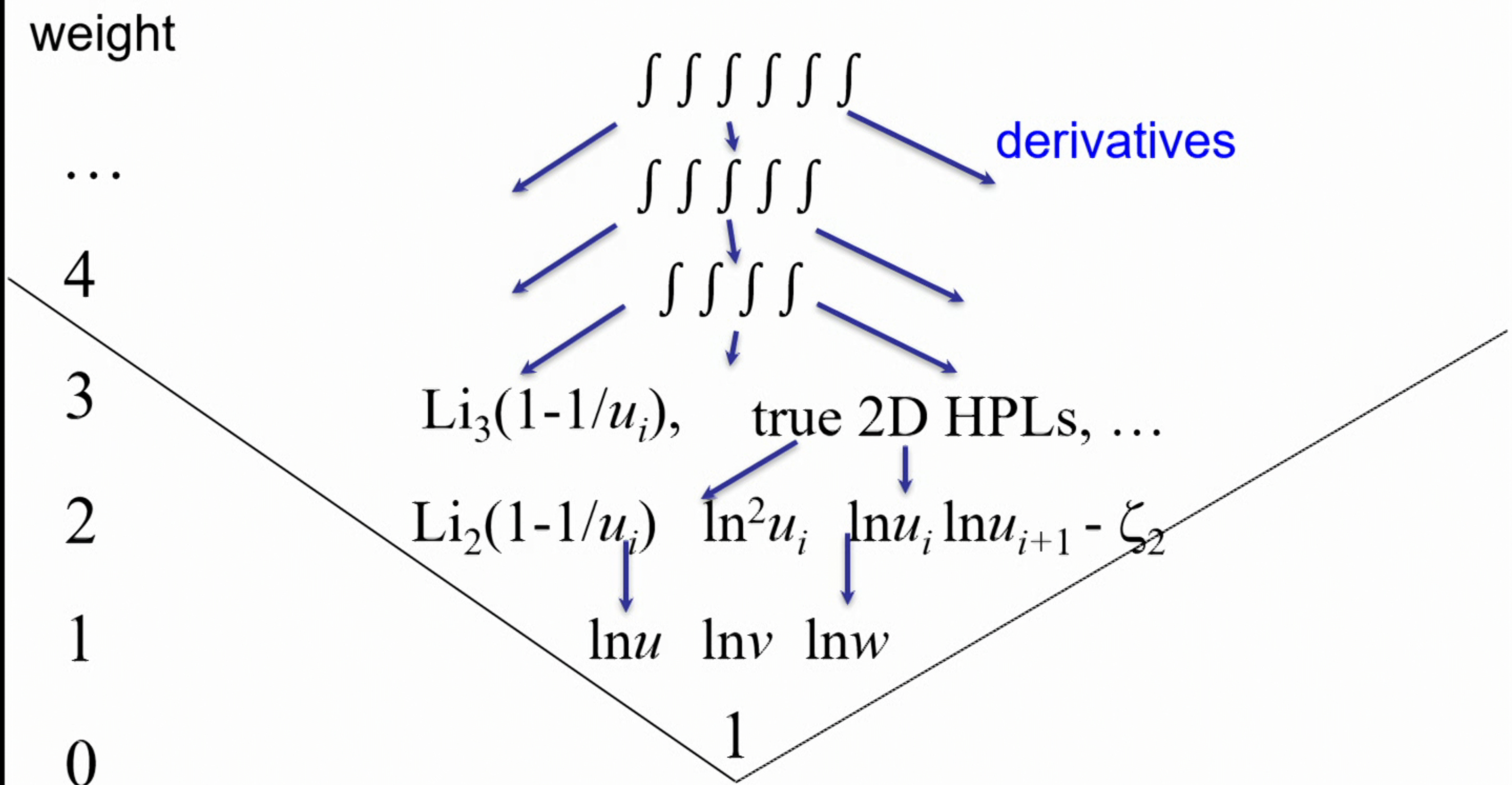
$n = 7$ has 42 letters

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Drummond, Papathanasiou, Spradlin 1412.3763

$n = 8$ has at least 222 letters, could even be infinite as $L \to \infty$

Arkani-Hamed, Lam, Spradlin, 1912.08222;
Drummond, Foster, Gürdoğan, Kalousios, 1912.08217, 2002.04624;
Henke, Papathanasiou 1912.08254, 2106.01392;
Z. Li, C. Zhang, 2110.00350

Heuristic view of function space



Hggg kinematics is two-dimensional

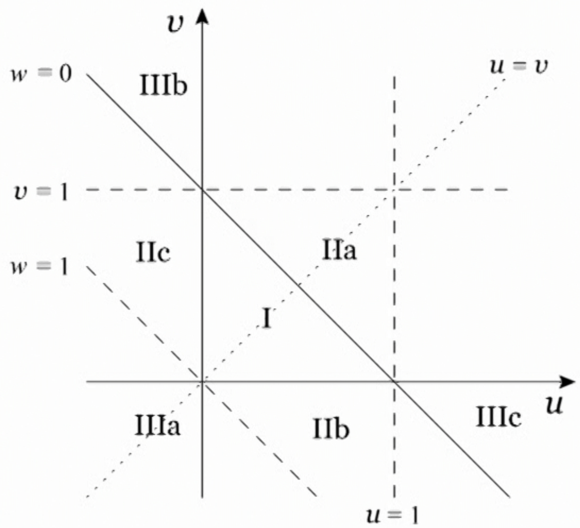
$$k_1 + k_2 + k_3 = -k_H$$

$$s_{123} = s_{12} + s_{23} + s_{31} = m_H^2$$

$$s_{ij} = (k_i + k_j)^2 \qquad k_i^2 = 0$$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}}$$

$$u + v + w = 1$$



I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

N=4 amplitude is S_3 invariant

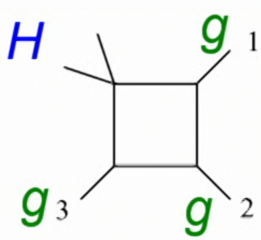
$D_3 \equiv S_3$ dihedral symmetry generated by:

a. cycle: $i \to i + 1 \pmod 3$, or $u \to v \to w \to u$

b. flip: $u \leftrightarrow v$

One loop

scalar box integral:



$$= \int \frac{d^4p}{p^2(p-p_1)^2(p-p_1-p_2)^2(p-p_1-p_2-p_3)^2}$$

$$= \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{12}}\right) + \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{23}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right) + \cdots$$

$$= \mathrm{Li}_2\left(1-\frac{1}{u}\right) + \mathrm{Li}_2\left(1-\frac{1}{v}\right) + \frac{1}{2}\ln^2\left(\frac{u}{v}\right) + \cdots$$

→ Symbol is $u \otimes (1-u) + v \otimes (1-v) - u \otimes v - v \otimes u$

Including cycles, $u \to v \to w \to u$, symbol alphabet at one loop is

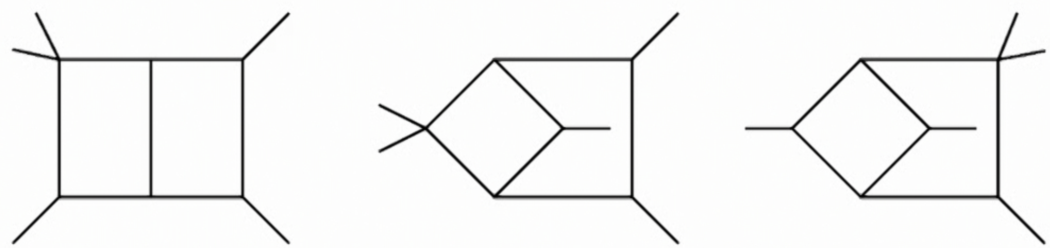
$$\mathcal{L} = \{u, v, w, 1-u, 1-v, 1-w\}$$

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Two loops

- *Hggg* computed in QCD at 2 loops
  Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554
- Chiral stress tensor 3-point form factor $\mathcal{F}_3$ in N=4 SYM computed next Brandhuber, Travaglini, Yang, 1201.4170
- Symbol alphabet **still**

$$\mathcal{L} = \{u, v, w, 1-u, 1-v, 1-w\}$$



2d HPLs

Gehrmann, Remiddi, hep-ph/0008287

$\mathcal{L} = \{u, v, w, 1-u, 1-v, 1-w\}$

Space graded by weight. Every weight n function F obeys:

$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w}$$

$$\frac{\partial F(u,v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{w} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{1-w}$$

$w = 1 - u - v$

where $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$ are weight n-1 2d HPLs.

To **bootstrap** *Hggg* amplitude beyond 2 loops, find as small a subspace of 2d HPLs as possible, construct it to high weight, impose various constraints to get a unique answer

Better (equivalent) alphabet

- Motivated by 6 gluon experience, we switch to an equivalent symbol alphabet

$$\mathcal{L}' = \{ a = \frac{u}{vw},\ b = \frac{v}{wu},\ c = \frac{w}{uv},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w} \}$$

- Symbols of form factor $F_3^{(L)}$ at 1 and 2 loops: just 1 and 2 terms, plus D_3 dihedral images(!!!):

$$\mathcal{S}\left[F_3^{(1)}\right] = (-1)\ b\otimes d + \text{dihedral}$$

$$\mathcal{S}\left[F_3^{(2)}\right] = 4\ b\otimes d\otimes d\otimes d + 2\ b\otimes b\otimes b\otimes d + \text{dihedral}$$

Brandhuber, Travaglini, Yang, 1201.4170

dihedral cycle: $a \to b \to c \to a, \quad d \to e \to f \to d$

dihedral flip: $a \leftrightarrow b, \quad d \leftrightarrow \mathrm{e}$

Examples of patterns

- Every term in the symbol starts with a, b, c; never d, e, f
- Physical reason related to causality, which dictates where branch cuts can appear: only for $(p_i + p_j)^2 \sim 0$
- Empirically, 12 pairs of adjacent letters are forbidden:

$$\ldots a \otimes d \ldots , \qquad \ldots b \otimes e \ldots , \qquad \ldots c \otimes f \ldots$$
$$\ldots d \otimes a \ldots , \qquad \ldots e \otimes b \ldots , \qquad \ldots f \otimes c \ldots$$
$$\ldots d \otimes e \ldots , \qquad \ldots e \otimes f \ldots , \qquad \ldots f \otimes d \ldots$$
$$\ldots e \otimes d \ldots , \qquad \ldots f \otimes e \ldots , \qquad \ldots d \otimes f \ldots$$

- Resemble constraints from causality:
  Steinmann relations   Steinmann, Helv. Phys. Acta (1960)
- But not quite, which mystified us for a while…

Antipodal duality in full 2d

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

$$F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w})\right)$$

Antipode map S, at symbol level, reverses order of all letters:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \, x_m \otimes \cdots \otimes x_2 \otimes x_1$$

Kinematic map is

$$\hat{u} = \frac{vw}{(1-v)(1-w)}, \qquad \hat{v} = \frac{wu}{(1-w)(1-u)}, \qquad \hat{w} = \frac{uv}{(1-u)(1-v)}$$

Maps $u + v + w = 1$ to parity-preserving surface

$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

$Hggg$ kinematics is two-dimensional

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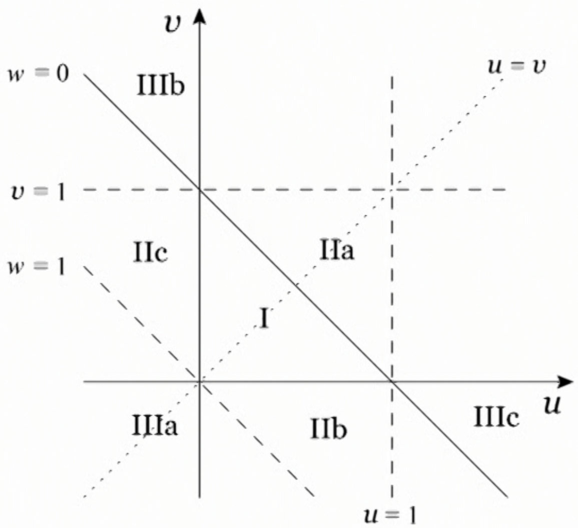

$$u + v + w = 1$$

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$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

6-gluon alphabet and symbol map

- $\mathcal{L}_6 = \{\, \hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w \}$ → 1 for $\Delta = 0$

→ $\mathcal{L}_6' = \{\, \hat{a} = \frac{\hat{u}}{\hat{v}\hat{w}},\ \hat{b} = \frac{\hat{v}}{\hat{w}u},\ \hat{c} = \frac{\hat{w}}{\hat{u}\hat{v}},\ \hat{d} = \frac{1-\hat{u}}{\hat{u}},\ \hat{e} = \frac{1-\hat{v}}{\hat{v}},\ \hat{f} = \frac{1-\hat{w}}{\hat{w}} \}$

- Kinematic map on letters:

$\sqrt{\hat{a}} = d, \quad \hat{d} = a,$ *plus cyclic relations*

$$\mathcal{S}\left[A_6^{(1)}\right] = (-\frac{1}{2})\hat{b}\otimes\hat{d} + \text{dihedral}$$

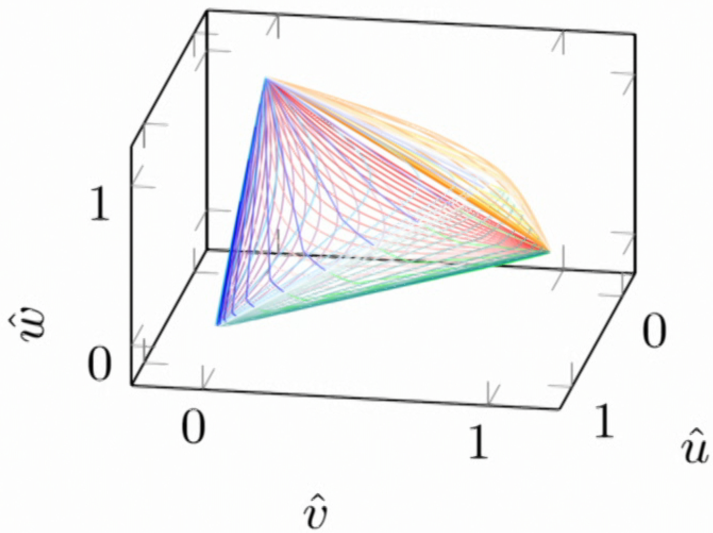
$$\mathcal{S}\left[A_6^{(2)}\right] = \hat{b}\otimes\hat{d}\otimes\hat{d}\otimes\hat{d} + \frac{1}{2}\hat{b}\otimes\hat{b}\otimes\hat{b}\otimes\hat{d} + \text{dihedral}$$

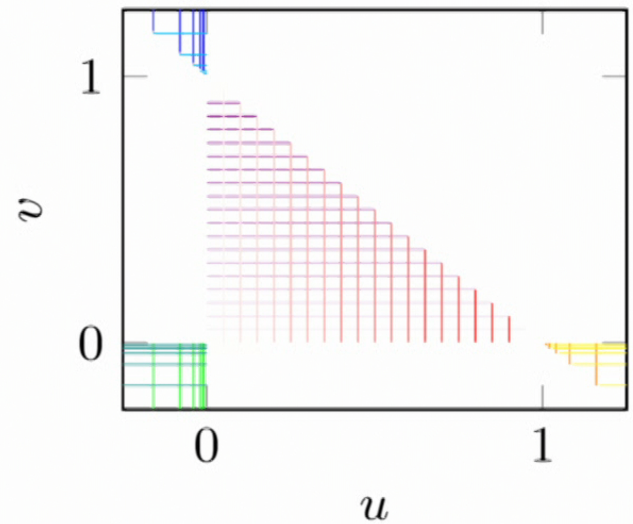
...

L	number of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

- Works through 7 loops!

Map covers entire phase space for 3-gluon form factor





- Soft is dual to collinear; collinear is dual to soft
- White regions in (u, v) map to some of $\hat{u}, \hat{v}, \hat{w} > 1$

"OPE" coordinates simplify kinematic map

Basso, Sever, Vieira, 1303.1396, 1306.2058,…;
Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

- Amplitude:

$$\hat{u} = \frac{1}{1 + (\hat{T} + \hat{S}\hat{F})(\hat{T} + \hat{S}/\hat{F})},$$

$$\hat{v} = \hat{u}\hat{w}\hat{S}^2/\hat{T}^2, \qquad \hat{w} = \frac{\hat{T}^2}{1 + \hat{T}^2}$$

($\hat{F} = 1$ for $\Delta = 0$)

- Form factor:

$$u = \frac{1}{1 + S^2 + T^2}, \qquad v = \frac{T^2}{1 + T^2},$$

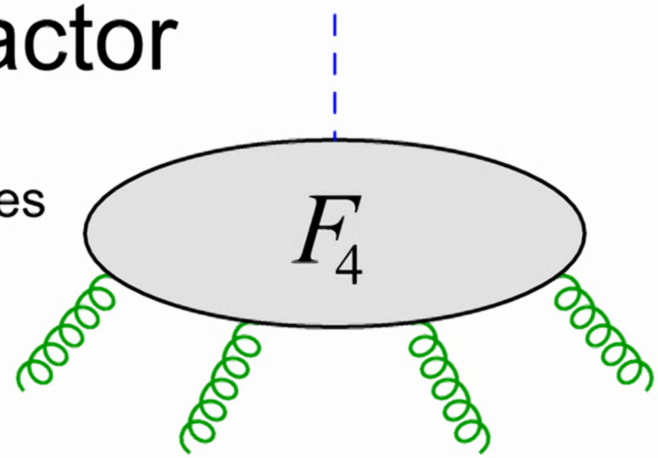
$$w = \frac{1}{(1 + T^2)(1 + S^{-2}(1 + T^2))},$$

- Kinematic map →

$$\hat{T} = \frac{T}{S}, \qquad \hat{S} = \frac{1}{TS}$$

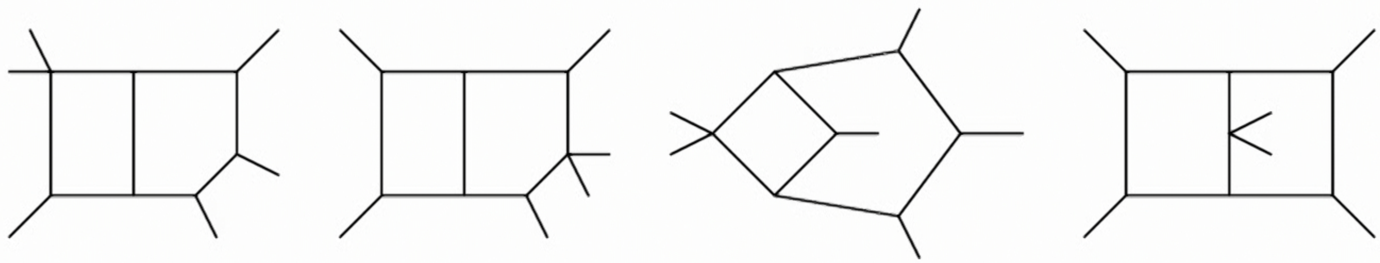
$$T = \sqrt{\frac{\hat{T}}{\hat{S}}}, \qquad S = \sqrt{\frac{1}{\hat{T}\hat{S}}}$$

Four-gluon form factor



Depends on 5 kinematical variables instead of 2.

Even just at two loops, contains state-of-the art loop integrals → 113 possible symbol letters!

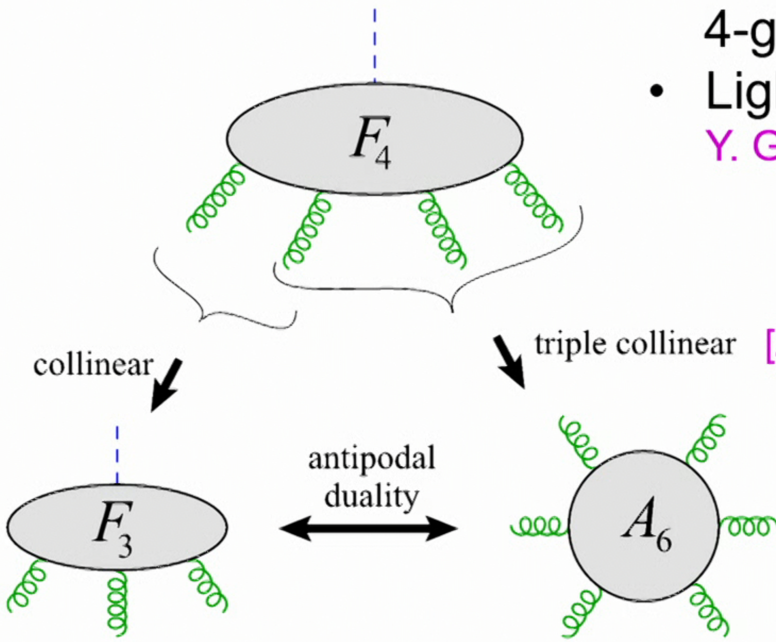


Abreu, Ita, Moriello, Page, Tschernow, 2005.04195;
Abreu, Ita, Page, Tschernow, 2107.14180;
Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, to appear

Antipodal Self Duality

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, 2212.02410

- Bootstrapped symbol of 4-gluon form factor F_4 at 2 loops
- Light-like limit matches
  Y. Guo, L. Wang, G. Yang, 2209.06816



[Z. Bern et al., 0803.1465, or FFOPE]

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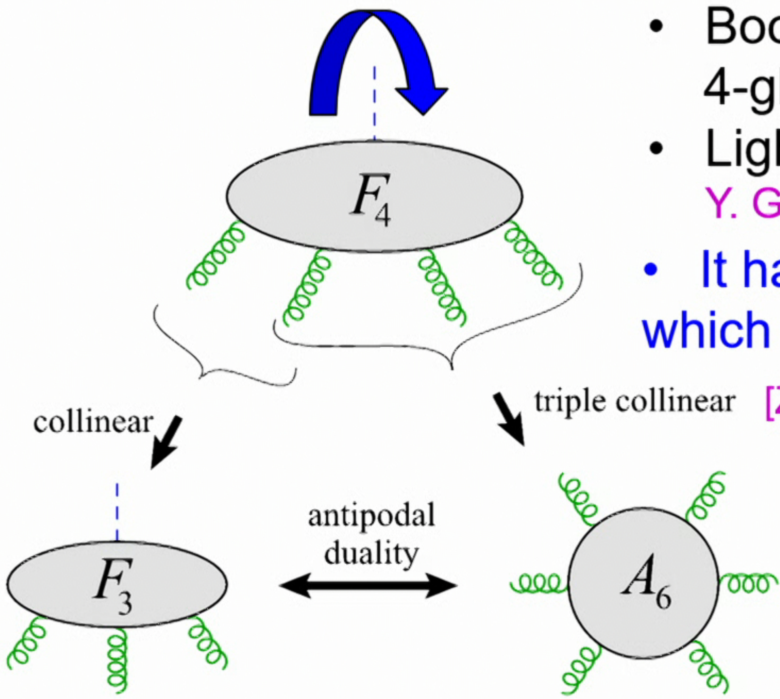
- Bootstrapped symbol of 4-gluon form factor F_4 at 2 loops
- Light-like limit matches Y. Guo, L. Wang, G. Yang, 2209.06816
- It has an antipodal self-duality which subsumes the $F_3 - A_6$ duality!



[Z. B

$$T_2 = \frac{T}{S}\,, \qquad S_2 = \frac{1}{TS}$$

$$T = \sqrt{\frac{T_2}{S_2}}\,, \qquad S = \sqrt{\frac{1}{T_2 S_2}}$$

$$F_2 = 1$$

“OPE” coordinates simplify kinematic map

Basso, Sever, Vieira, 1303.1396, 1306.2058,…;
Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

- Amplitude:

$$\hat{u} = \frac{1}{1 + (\hat{T} + \hat{S}\hat{F})(\hat{T} + \hat{S}/\hat{F})},$$

$$\hat{v} = \hat{u}\hat{w}\hat{S}^2/\hat{T}^2, \qquad \hat{w} = \frac{\hat{T}^2}{1 + \hat{T}^2}$$

$(\hat{F} = 1 \text{ for } \Delta = 0)$

- Form factor:

$$u = \frac{1}{1 + S^2 + T^2}, \qquad v = \frac{T^2}{1 + T^2},$$

$$w = \frac{1}{(1 + T^2)(1 + S^{-2}(1 + T^2))},$$

- Kinematic map →

$$\hat{T} = \frac{T}{S}, \qquad \hat{S} = \frac{1}{TS}$$

$$T = \sqrt{\frac{\hat{T}}{\hat{S}}}, \qquad S = \sqrt{\frac{1}{\hat{T}\hat{S}}}$$

ASD at 3 loops

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, 23mm.nnnnn

- Can bootstrap the symbol of F_4 at 3 loops too, using the same 113 letter alphabet.
- 2 loop symbol uses only 34 of the letters
- 3 loop symbol uses only 88 of the letters, and ASD holds!
- If we assume only 113 letters at higher loops too, and ASD, then there can only be 93 letters (the others transform outside the 113 under the ASD kinematic map).
- 4 loops in progress

Summary & Open Questions

- Form factors as well as scattering amplitudes in planar N=4 SYM can now be **bootstrapped** to high loop order
- 6-gluon amplitude and 3-gluon form factor are related by a strange new antipodal duality, swapping the role of branch cuts and derivatives
- Embedded in a 4-gluon form factor self-duality!

- Who ordered that?
- Underlying physical reason for this duality?
- (How) does it hold at strong coupling?
- What other theories might it hold in?
- How much more can we exploit it to learn more about both amplitudes and form factors?