Title: The 't Hooft equation as a quantum spectral curve

Speakers: David Vegh

Series: Quantum Fields and Strings

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Abstract: In this talk, I examine the massless 't Hooft equation. This integral equation governs meson bound state wavefunctions in 2D SU(N) gauge theory in the large-N limit, and it can also be obtained by quantizing a folded string in flat space. The folded string is a limiting case of a more general setup: a four-segmented string moving in three-dimensional anti-de Sitter (AdS) space. I compute its classical spectral curve using celestial variables and planar bipartite graphs, also known as on-shell diagrams or brane tilings. In this more general setup, the 't Hooft equation acquires an extra term, which has previously been proposed as an effective confining potential in QCD. After an integral transform, the equation can be inverted in terms of a finite difference equation. I show that this difference equation has a natural interpretation as the quantized (non-analytic) spectral curve of the string. The spectrum interpolates between equally spaced energy levels in the tensionless limit and 't Hooft's nearly linear Regge trajectory at infinite AdS radius.

Zoom link: https://pitp.zoom.us/j/97373882483?pwd=NmphN0N3ckdHbHdlcVpKcGkzMHY1Zz09

The 't Hooft equation as a quantum spectral curve

David Vegh
Perimeter Institute, April 18, 2023

Queen Mary University of London
Based on arXiv:2301.07154

Overview

Motivation

Segmented strings

Bipartite graphs

The simplest closed segmented string

The 't Hooft equation

Summary

Motivation

Strings in anti-de Sitter space:

- String worldsheet as a **toy model for quantum gravity**

  Dubovsky-Flauger-Gorbenko 2012

  Lyapunov exponent λ_L saturates the universal bound: $\lambda_L = 2\pi T$

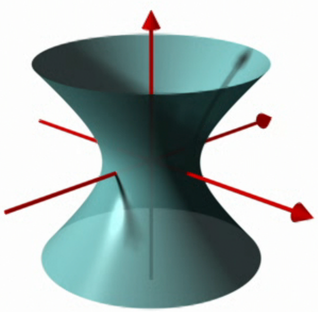
  de Boer-Llabres-Pedraza-DV 2017

- **AdS/CFT correspondence**
- Non-linear waves DV 2017

Quantum spectral curves:

- AdS/CFT in the planar limit is integrable
- Different way of quantizing

Geometry of anti-de Sitter space



- AdS_d in $\mathbb{R}^{2,d-1}$ ambient space

$$\vec{Y}\cdot\vec{Y} \equiv -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + \cdots + Y_{d-1}^2 = -1$$

- AdS_3 Poincaré patch metric

$$ds^2 = \frac{-dt^2 + dz^2 + dx^2}{z^2}$$

- string equation of motion and Virasoro constraints

$$\partial\bar{\partial}\vec{Y} - (\partial\vec{Y}\cdot\bar{\partial}\vec{Y})\vec{Y} = 0 \qquad \partial\vec{Y}\cdot\partial\vec{Y} = \bar{\partial}\vec{Y}\cdot\bar{\partial}\vec{Y} = 0$$

Can we simplify the system?

Idea: segmented strings

glue linear string segments in AdS_3

the segments have constant normal vectors:

$$N_a \propto \epsilon_{abcd} Y^b \partial Y^c \bar{\partial} Y^d \qquad Y \in \mathbb{R}^{2,2}$$

DV 2015 Callebaut-Gubser-Samberg-Toldo 2015

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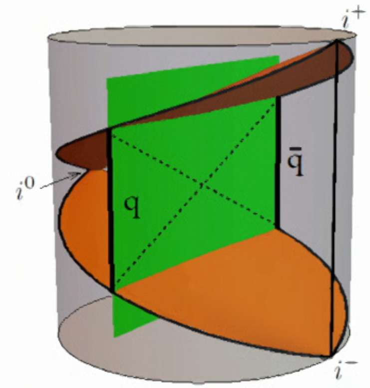
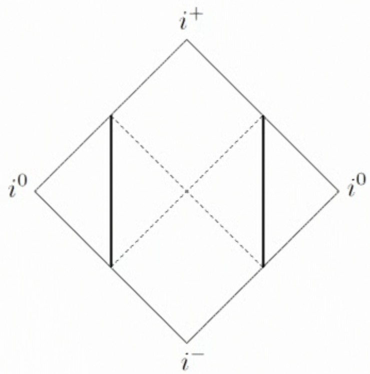
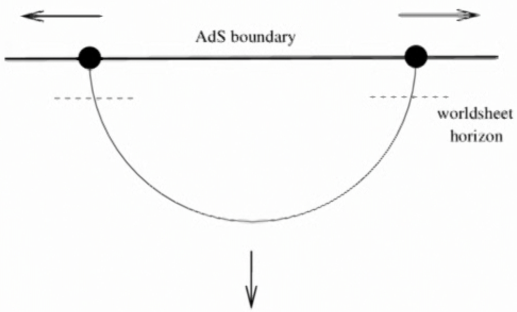
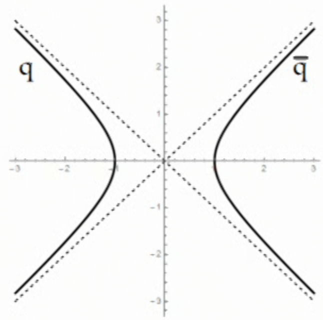
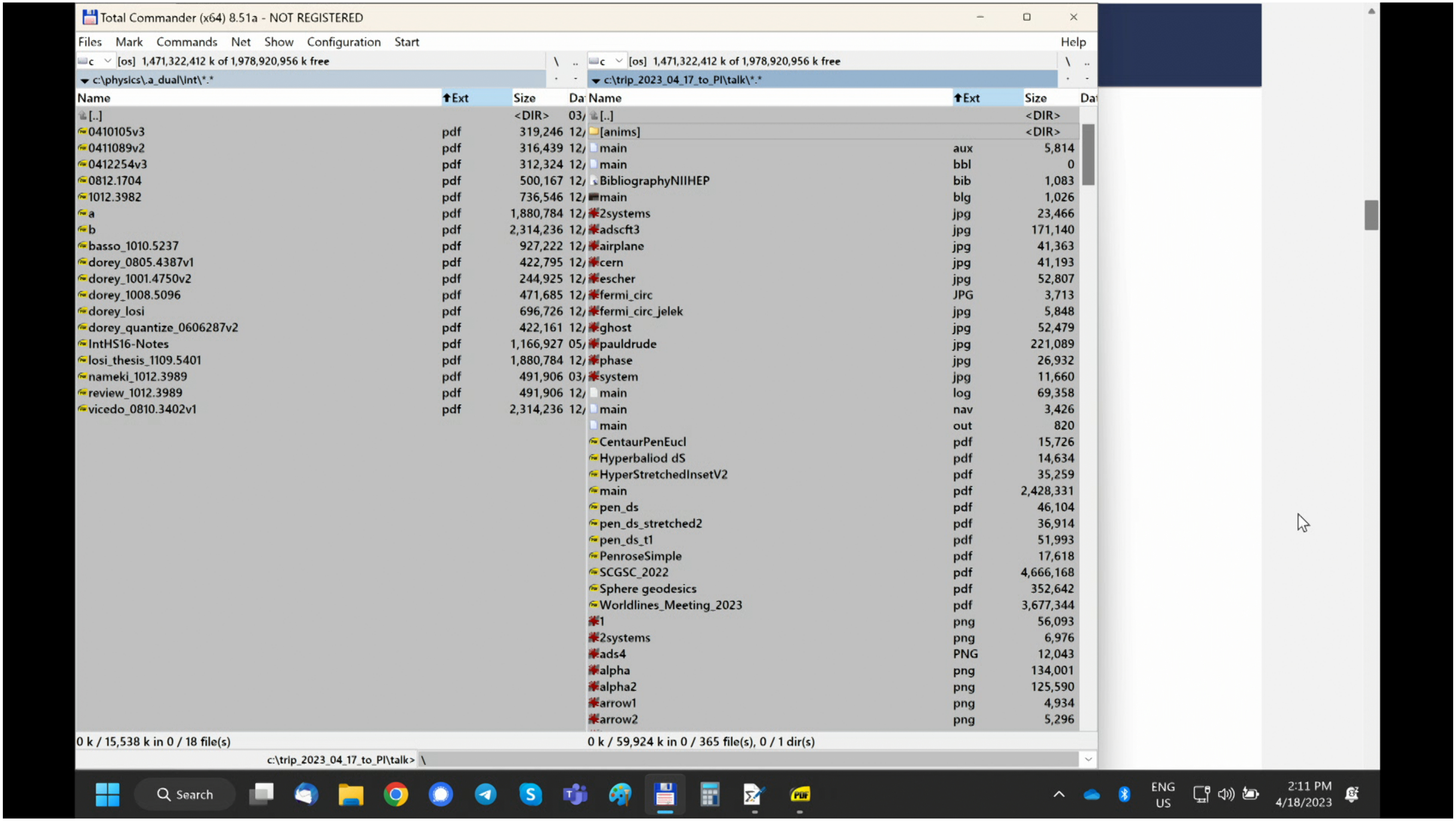
Elementary segment







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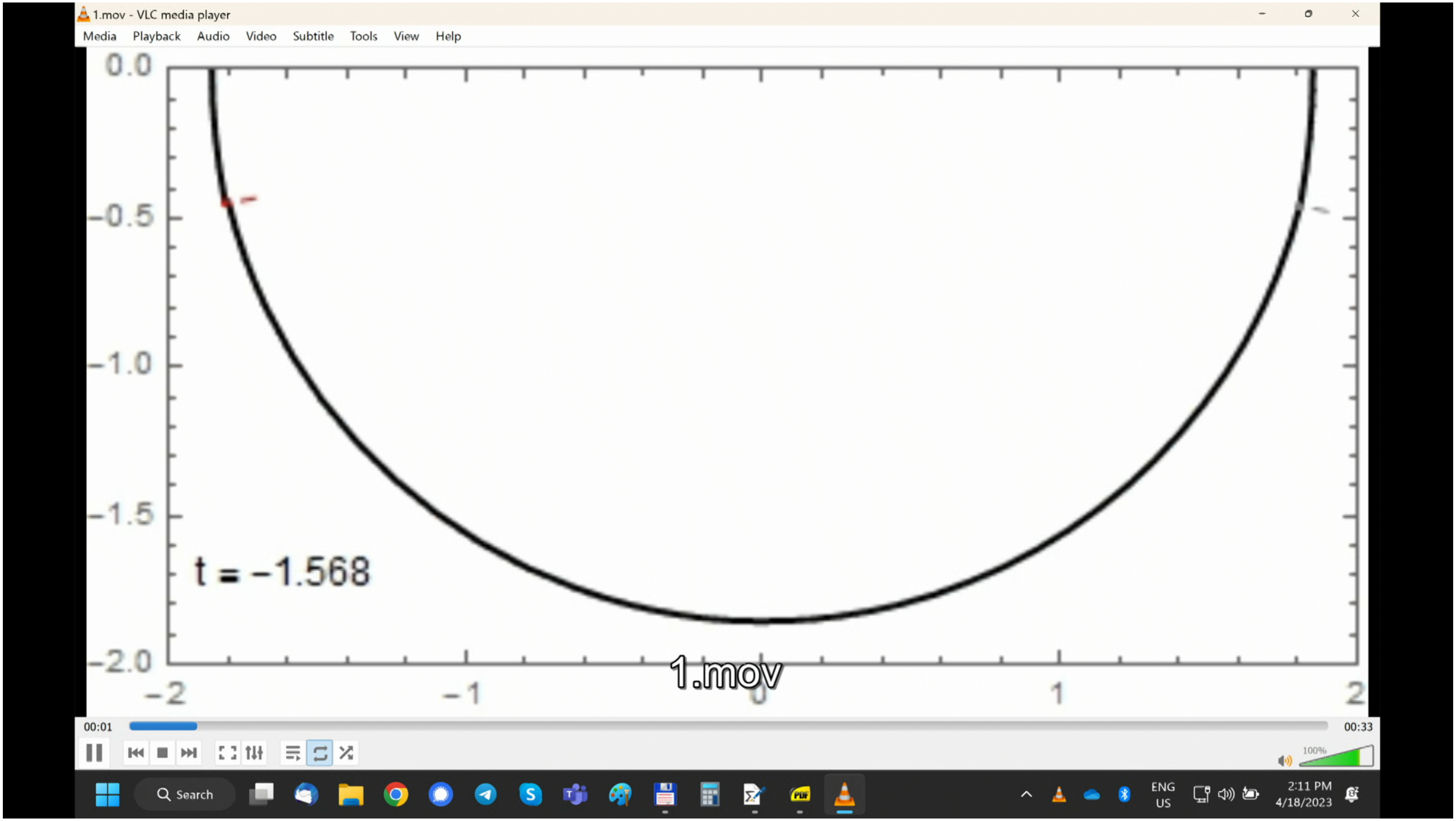
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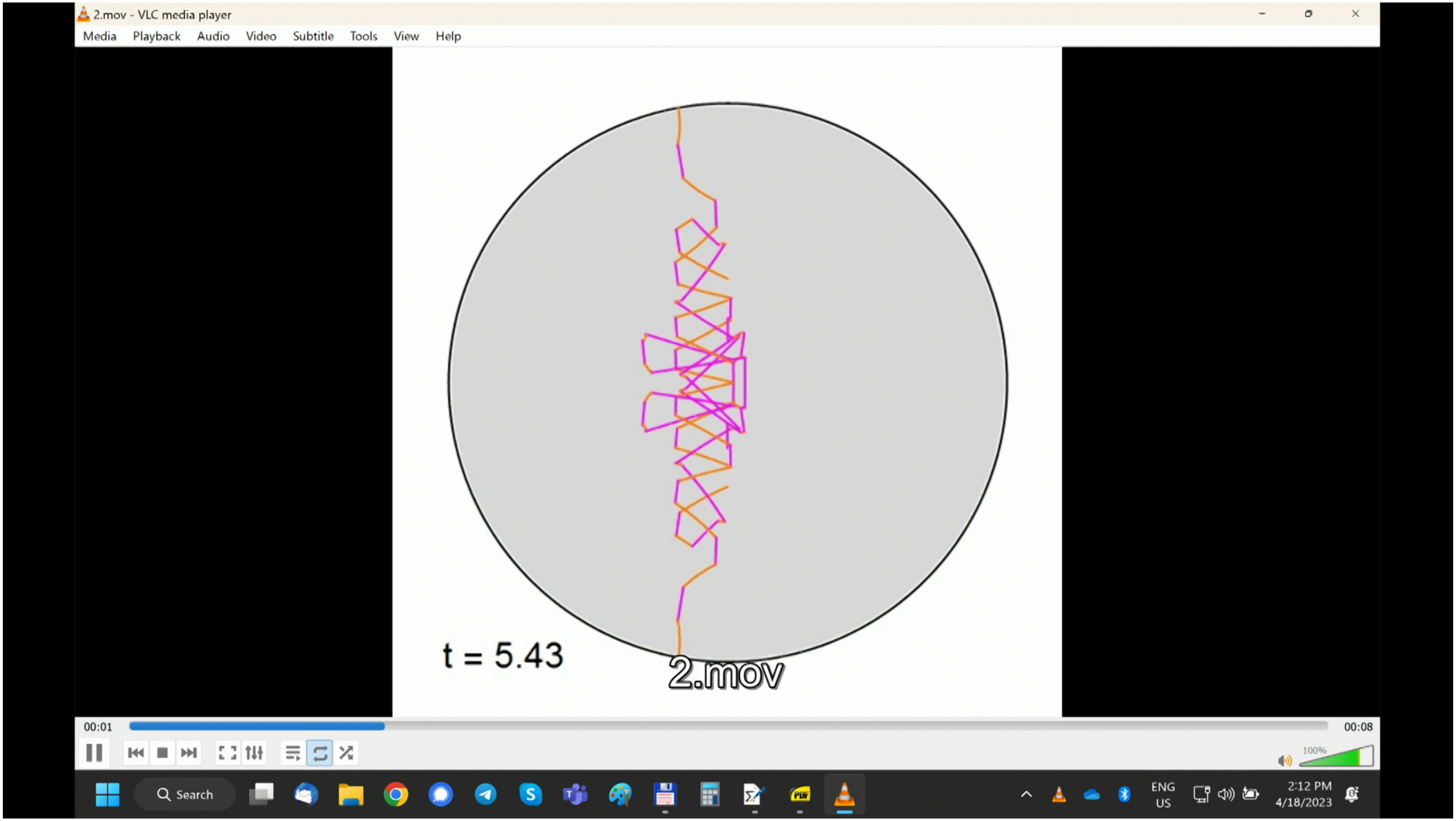
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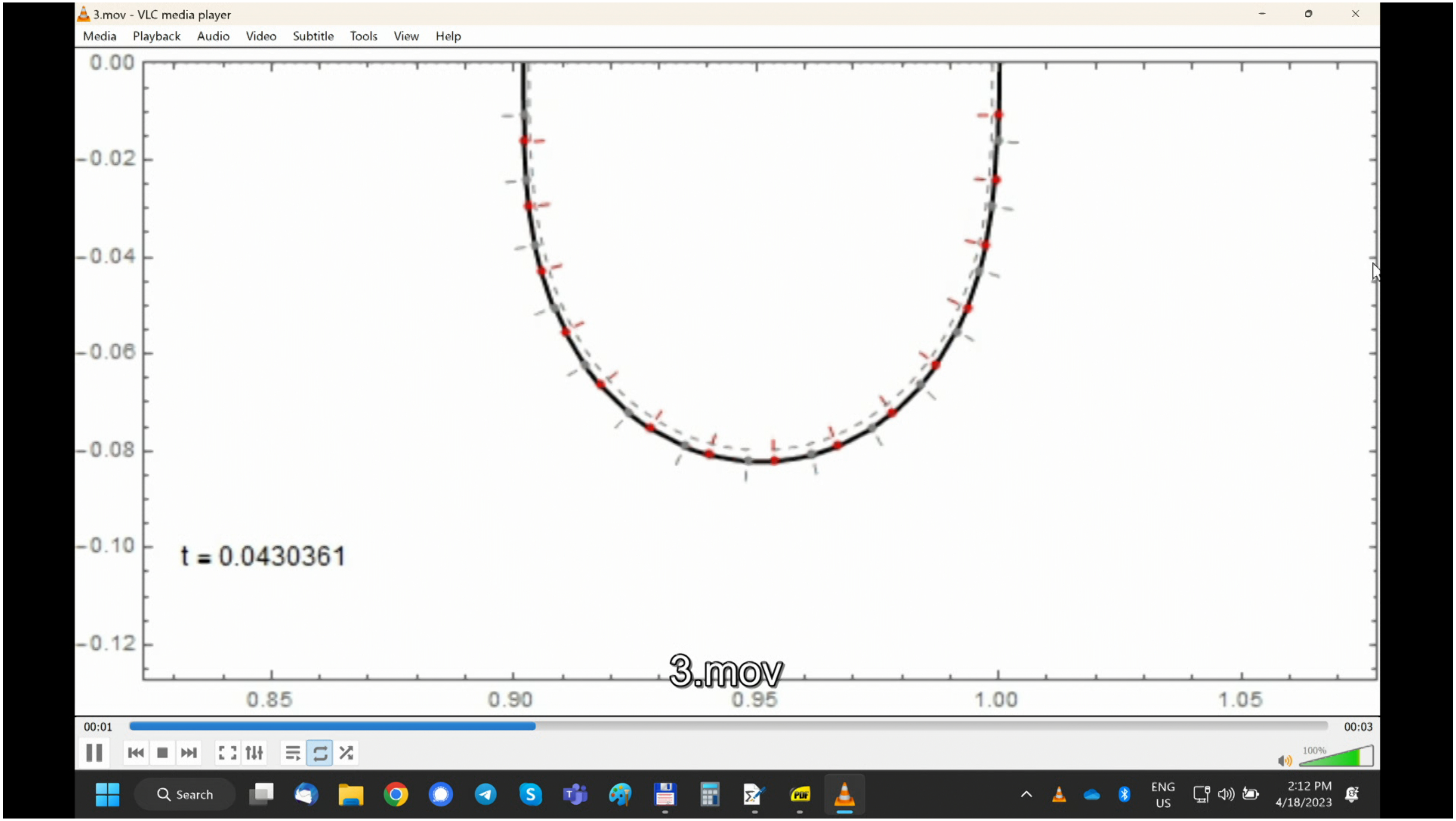
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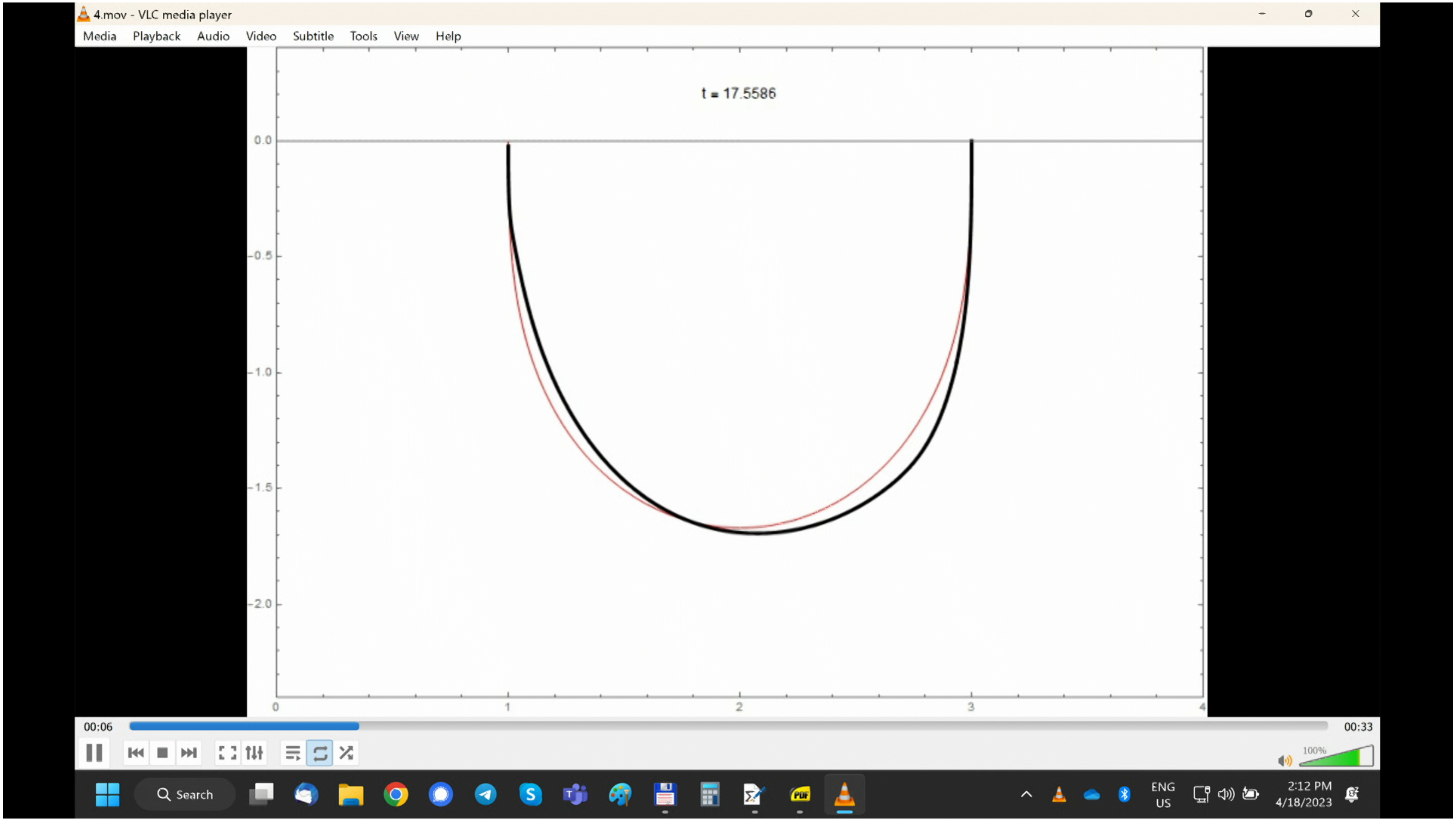

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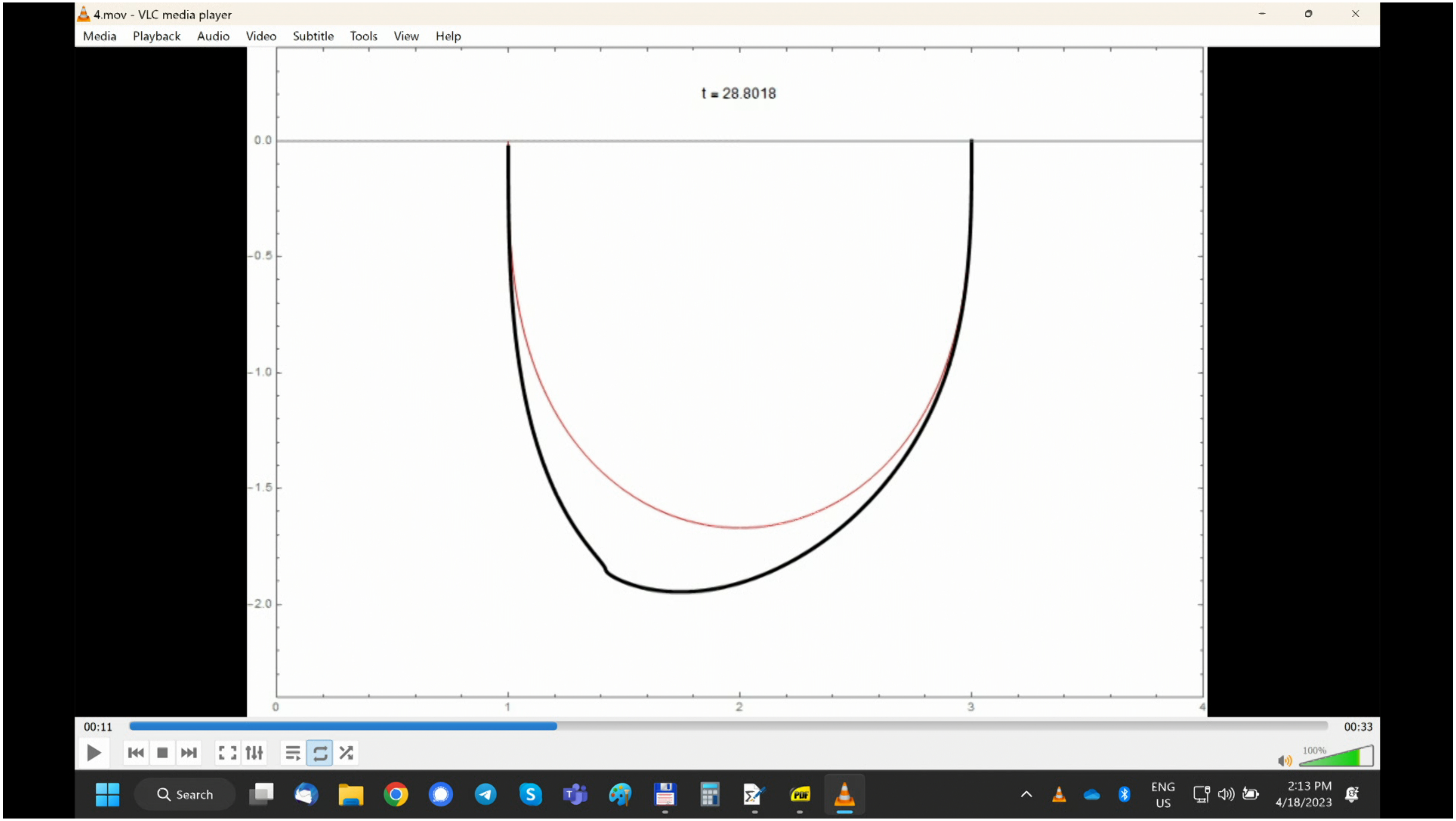

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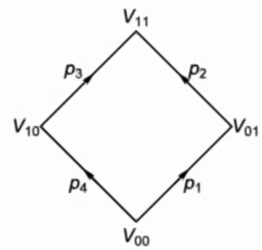
Celestial variables



- Mandelstam variables: $s = (p_1 + p_2)^2$ and $u = (p_1 - p_4)^2$

$$S = Area = \log\left[u/s\right]^2$$

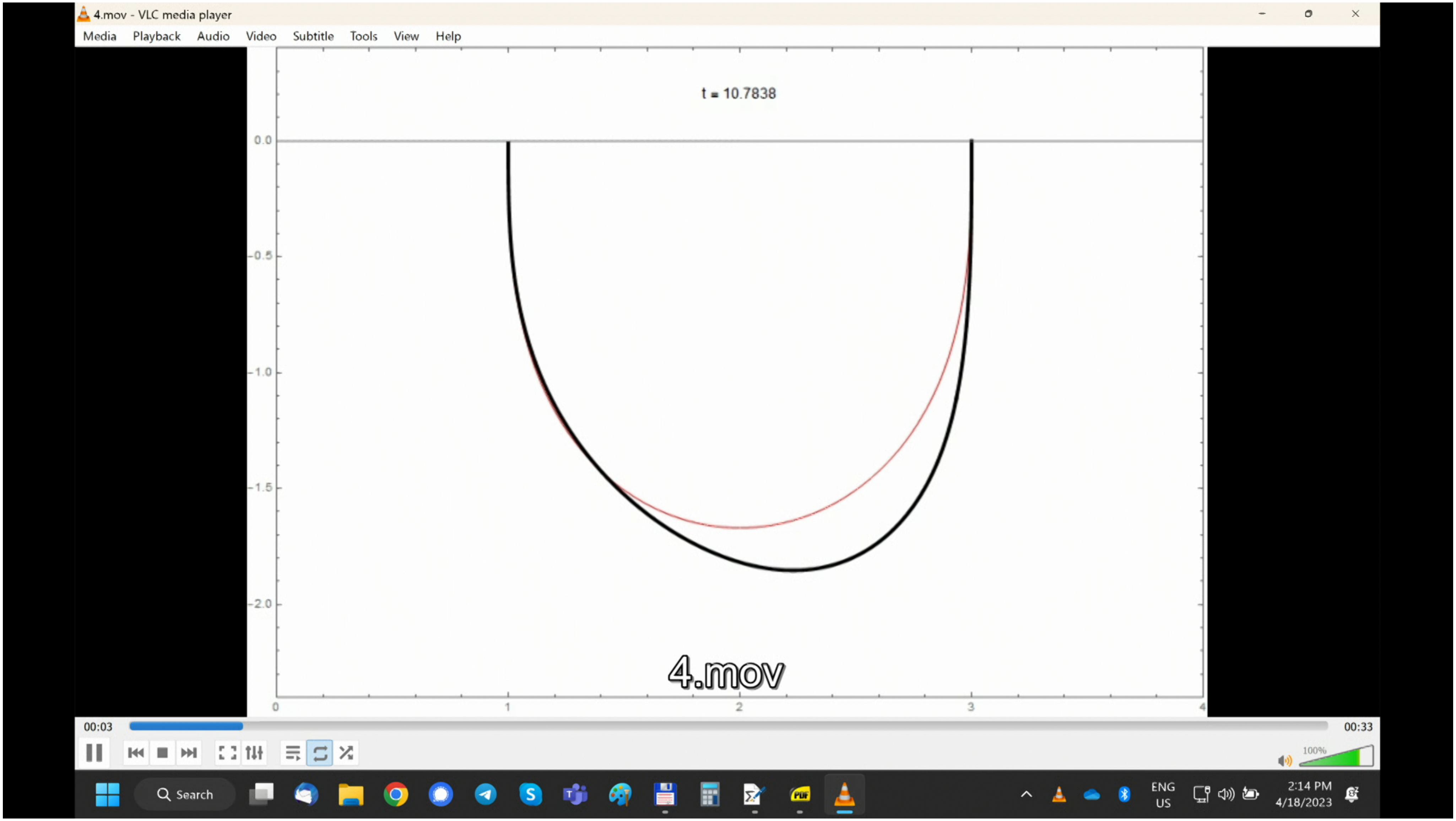
- Since $p^2 = \det(p_{a\dot{a}}) = 0$, we can write $\sigma^{\mu}_{a\dot{a}} p_{\mu} = \lambda_a \tilde{\lambda}_{\dot{a}}$, where $\sigma^{\mu} = (1, -i\sigma_2, \sigma_1, \sigma_3)$

$$S = 2\log\left|\frac{\langle\lambda_1, \lambda_4\rangle\langle\lambda_2, \lambda_3\rangle}{\langle\lambda_1, \lambda_2\rangle\langle\lambda_3, \lambda_4\rangle}\right|$$

Now define the **celestial variable** $a := \frac{\lambda_{\uparrow}}{\lambda_{\downarrow}} = \frac{p_{-1}+p_2}{p_0+p_1}$.

$$Area = 2\log\left|\frac{(a_1 - a_4)(a_2 - a_3)}{(a_1 - a_2)(a_3 - a_4)}\right|$$

- summary: **vertices** $\rightarrow$ **difference vectors** $\rightarrow$ **spinors** $\rightarrow$ **celestial variables** DV 2016



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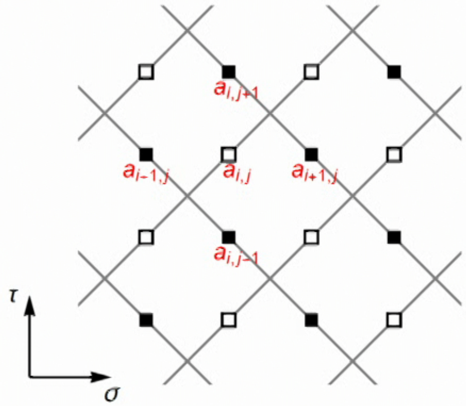
Discrete integrable system

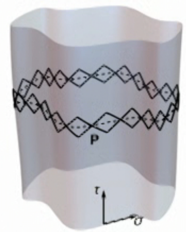


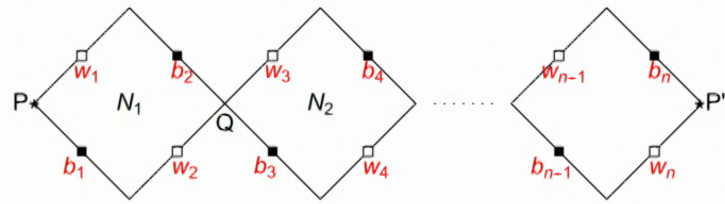
In terms of the celestial variables, the string EOM is simply

$$\frac{1}{a_{ij} - a_{i,j+1}} + \frac{1}{a_{ij} - a_{i,j-1}} = \frac{1}{a_{ij} - a_{i+1,j}} + \frac{1}{a_{ij} - a_{i-1,j}}$$

DV 2016

Reconstructing the string embedding





- Define the 'reflection matrix' DV 2019

$$\mathcal{R}_{b,w} = \frac{1}{b-w}\begin{pmatrix} 0 & bw+1 & bw-1 & -b-w \\ -1-bw & 0 & b+w & bw-1 \\ -1+bw & b+w & 0 & -1-bw \\ -b-w & bw-1 & bw+1 & 0 \end{pmatrix}$$

- Normal vectors and vertices can be computed from the celestial variables

$$N_1 = \mathcal{R}_{b_1,w_1} P$$

$$Q = \mathcal{R}_{b_2,w_2} N_1$$

$$etc.$$

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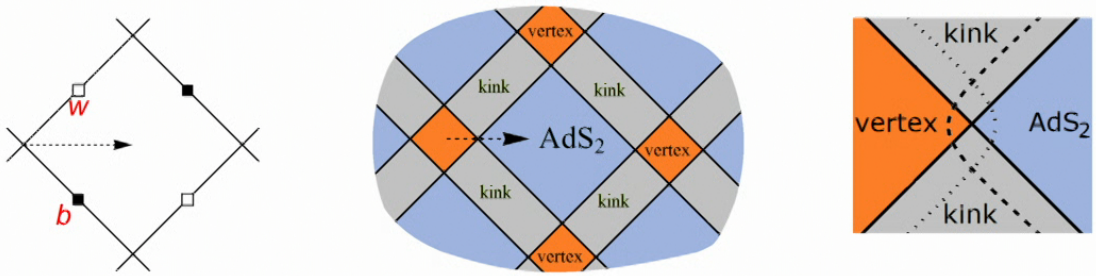
Matching spinor solutions

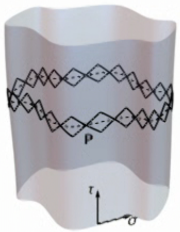


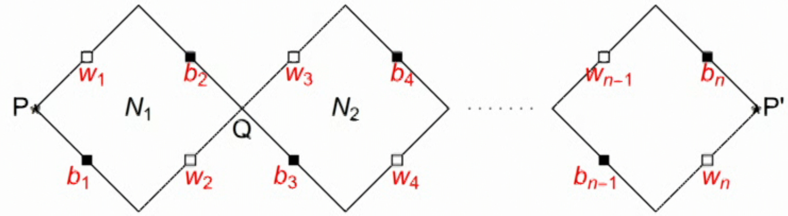
- Match solutions in the overlapping region (matched asymptotic expansion)
- Lax matrix DV 2021

$$\Omega_{b,w}(x) = \frac{1}{(b-w)\sqrt{x}} \begin{pmatrix} bx - w & bw(1-x) \\ x - 1 & b - wx \end{pmatrix}$$

$$\det \Omega_{b,w} = 1\,, \qquad \Omega_{b,w}(x=1) = \mathbb{1}$$

Monodromy





- Suppose there are N segments in a closed string. The monodromy is given by

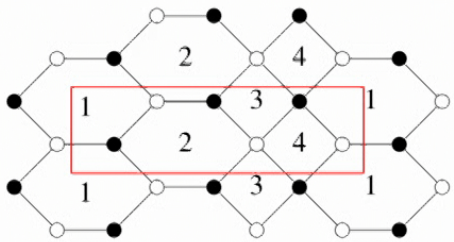
$$\Omega(x) = \Omega^{-1}_{b_N, w_N} \Omega_{b_{N-1}, w_{N-1}} \cdots \Omega^{-1}_{b_2, w_2} \Omega_{b_1, w_1}$$

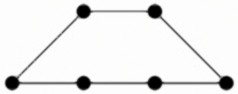
- For generic values of b_i, w_j, the string will not close in AdS_3.
  Closure is equivalent to demanding

$$\Omega(x = -1) = \mathbb{1}$$

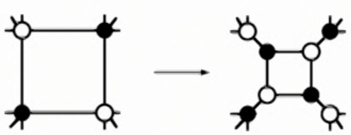
  This gives 3 constraints on the celestial variables.

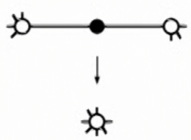
Brane tilings





- **brane tiling:** doubly-periodic bipartite graph
- its dual graph is the quiver
  (in the context of 4d $\mathcal{N} = 1$ theories, the tiling also encodes the superpotential Franco-Hanany-Kennaway-DV-Wecht 2005)
- related to on-shell diagrams Arkani-Hamed-Bourjaily-Cachazo-Goncharov-Postnikov-Trnka 2012
- an invariant: Newton polygon of a Laurent polynomial
  (toric diagram of CY threefold)
- invariant under the transformations





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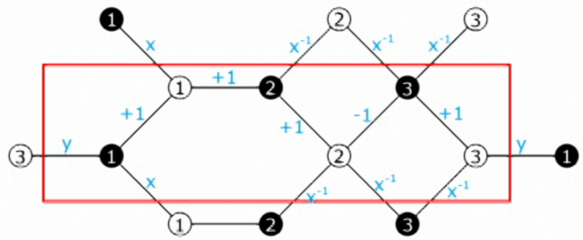
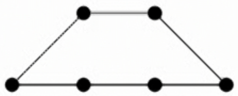
Kasteleyn matrix



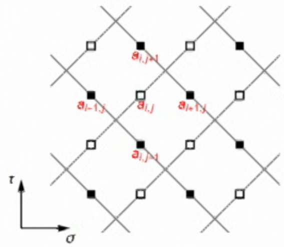
- brane tiling dressed with edge weights
- white and black vertices label the rows and columns of the Kasteleyn matrix

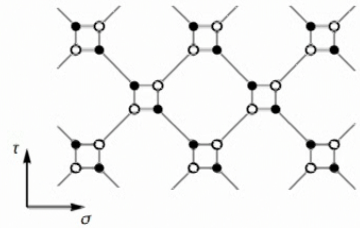
$$K = \begin{pmatrix} 1+x & 1 & 0 \\ 0 & 1+x^{-1} & -1+x^{-1} \\ y & 0 & 1+x^{-1} \end{pmatrix} \qquad (3.1)$$

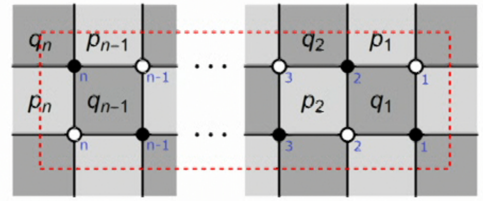
$$\det K = 3 + x^{-2} + 3x^{-1} + x - y + x^{-1}y \qquad (3.2)$$



$Y^{n,0}$ tilings







- cross-ratio

$$(a,\ b;\ c,\ d) \equiv \frac{(a-b)(c-d)}{(a-d)(b-c)}$$

$$p_{i,j} = (a_{i+1,j},\ a_{i+1,j+1};\ a_{i+2,j},\ a_{i+2,j+1})$$

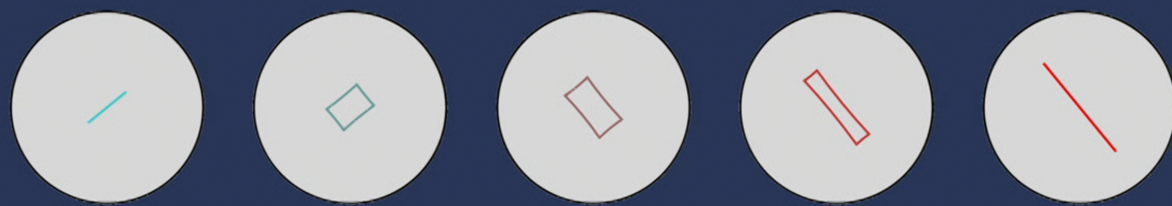
- performing cluster transformations on all faces gives DV 2021

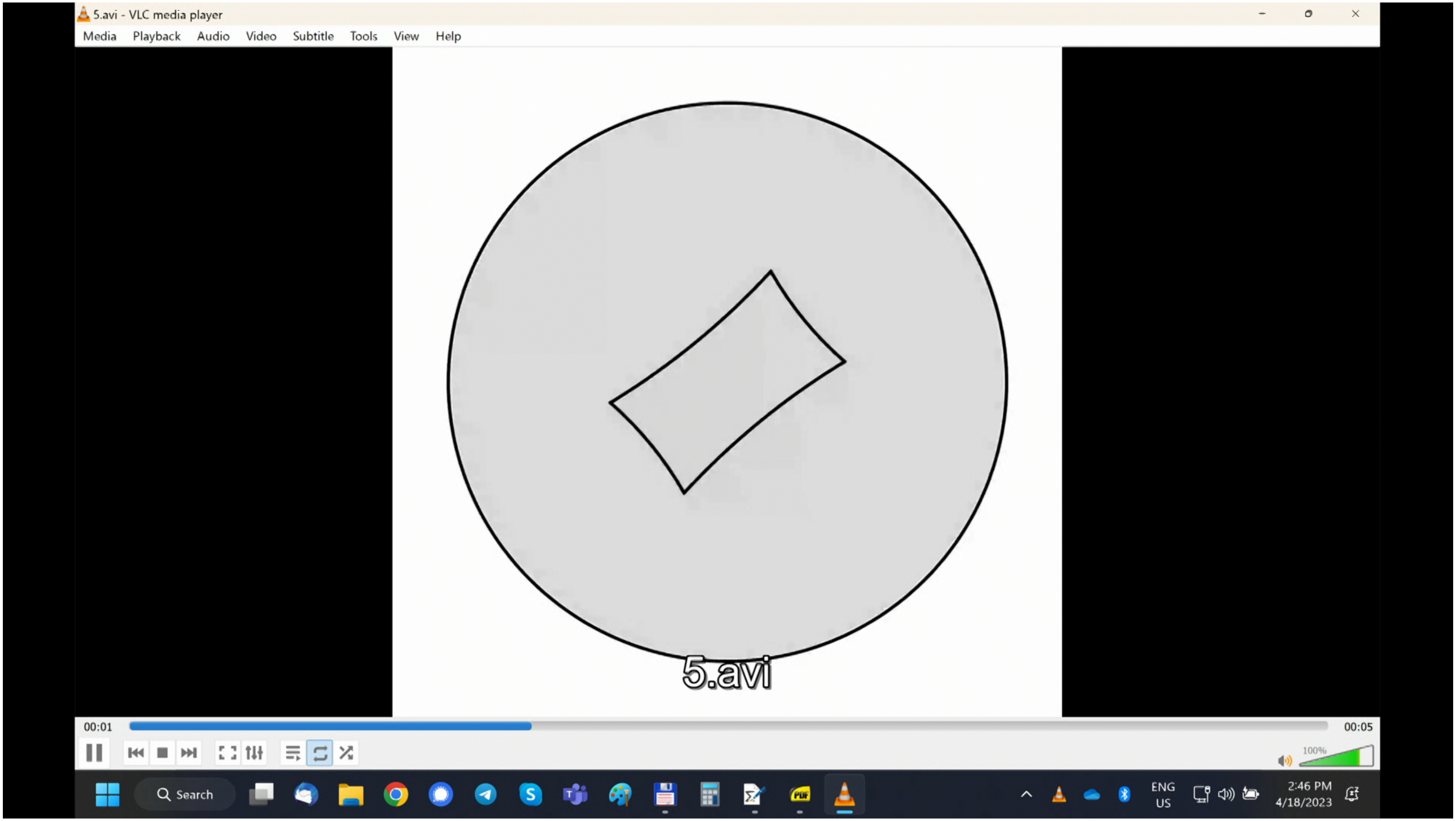
$$q_{i,j+1} = \frac{1}{p_{i,j}}\,, \qquad p_{i,j+1} = q_{i,j}p_{i,j}^2\frac{(1+p_{i-1,j})(1+p_{i+1,j})}{(1+p_{i,j})^2}$$

compatible with the EOM: $\frac{1}{a_{ij}-a_{i,j+1}} + \frac{1}{a_{ij}-a_{i,j-1}} = \frac{1}{a_{ij}-a_{i+1,j}} + \frac{1}{a_{ij}-a_{i-1,j}}$

- det $\tilde{K} = 0$ gives the spectral curve, where $\tilde{K}(x,y)$ is the dressed Kasteleyn matrix

An example: string with four segments

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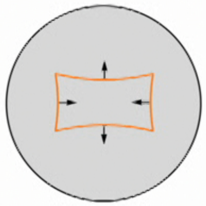
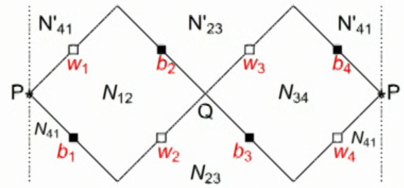
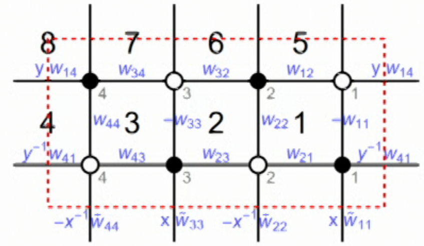
Example: string with four segments







- The spectral curve can be computed:

$$y + y^{-1} - \frac{\Delta^2}{16}(x^2 + x^{-2}) + 2 + \frac{\Delta^2}{8} = 0$$

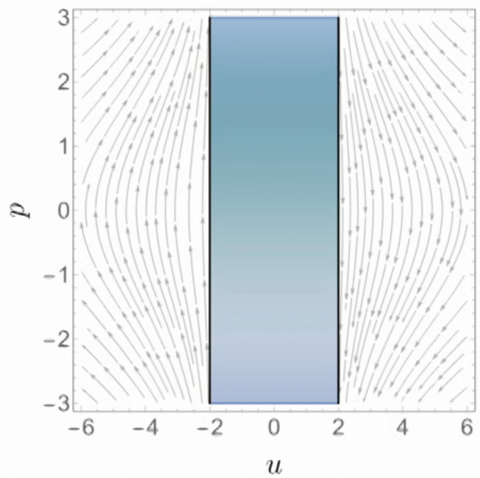
- Define new spectral parameters $x \to \check{x} \to u$

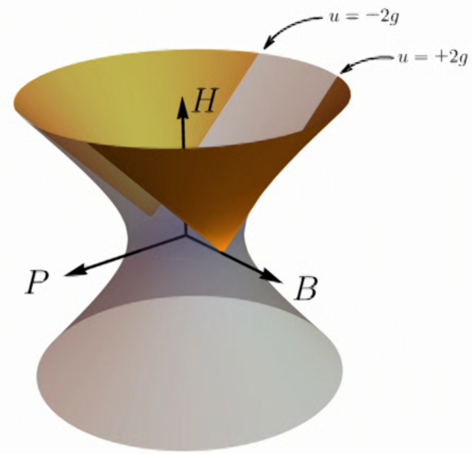
$$x = \frac{\check{x} - 1}{\check{x} + 1} \qquad u = (\check{x} + \check{x}^{-1})g \qquad y = e^p$$

$$e^p + e^{-p} + 2 - \frac{\Delta^2}{u^2 - 4g^2} = 0$$

where p and u are canonically conjugate variables and $g = \frac{L^2}{2\pi\alpha'}$

The phase space





- Ruijsenaars-Schneider-type Hamiltonian

$$H(p, u) = \Delta = 2\cosh\left(\frac{p}{2}\right)\sqrt{u^2 - 4g^2}$$

$$P = 2\sinh\left(\frac{p}{2}\right)\sqrt{u^2 - 4g^2} \qquad B = -2u$$

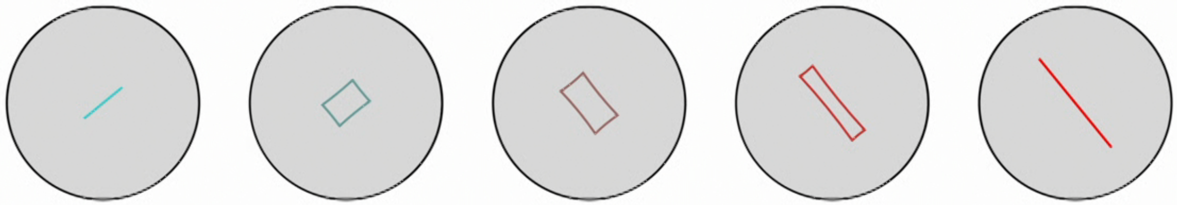
- Together they form an $\mathfrak{sl}(2)$ algebra,

$$\{H, P\} = -B \qquad \{H, B\} = P \qquad \{P, B\} = H$$

We also have

$$-H^2 + P^2 + B^2 = 16g^2$$

An observation



$$e^p + e^{-p} + 2 - \frac{\Delta^2}{u^2 - 4g^2} = 0$$

- Spectral curve does not depend on the precise shape of the string
- Squash the string & go to flat space limit $\to$ folded string

- In this limit, a quantization is given by the 't Hooft equation 't Hooft 1974

The 't Hooft equation

- Two-particle Hamiltonian

$$H_2 = (p_1^2 + m_1^2)^{\frac{1}{2}} + (p_2^2 + m_2^2)^{\frac{1}{2}} + \kappa|x_1 - x_2|\,.$$

- Canonical transformation

$$x_{1,2} = x_0 \pm \frac{x'}{2}\,, \qquad p_{1,2} = \frac{p_0}{2} \pm p'\,, \tag{5.3}$$

- Infinite momentum frame $p_0 \to \infty$

$$\Delta^2 = H_2^2 - p_0^2 \approx 2p_0(H_2 - p_0) = \frac{m_1^2}{z} + \frac{m_2^2}{1-z} + 2\kappa|s|$$

- where we the momentum fraction variable z and the conjugate (signed) action variable s,

$$z := \frac{p_1}{p_1 + p_2}\,, \qquad s := x'(p_1 + p_2)$$

- 't Hooft equation 't Hooft 1974

$$\Delta^2\varphi(z) = \left[\frac{m_1^2}{z} + \frac{m_2^2}{1-z}\right]\varphi(z) - \frac{2\kappa}{\pi}\int_0^1 dz'\,\frac{\varphi(z')}{(z'-z)^2}$$

The 't Hooft equation

- What does the 't Hooft equation have to do with the spectral curve?

$$e^p + e^{-p} + 2 - \frac{\Delta^2}{u^2 - 4g^2} = 0$$

- The particles collide at $u = \pm 2g$.
- Assuming u is real, we can define a new coordinate

$$q := \begin{cases} u - 2g & \text{for } u \geq +2g\,, \\ u + 2g & \text{for } u \leq -2g\,. \end{cases}$$

  This cuts out the $u \in (-2g, 2g)$ region, which is not part of the classical configuration space.
- Take the $g \to \infty$ limit while keeping $\mu \equiv \frac{\pi \Delta^2}{4g}$ fixed gives

$$e^p + e^{-p} + 2 - \frac{\mu^2}{\pi |q|} = 0$$

The 't Hooft equation

- The 't Hooft equation with zero renormalized masses

$$\mu^2 \varphi(z) = -\!\!\!\!\!\!\int_0^1 dz' \, \frac{\varphi(z')}{(z'-z)^2}$$

- Switch to rapidity coordinates $p = \log \frac{z}{1-z}$ and Fourier transform $\varphi(p) \to \varphi(q)$

  Fateev-Lukyanov-Zamolodchikov 2009 Brower-Spence-Weis 1979

- The resulting integral equation can be inverted

$$Q(q+i) + Q(q-i) - 2Q(q) = -\mu^2 \frac{\tanh(\pi q)}{\pi q} Q(q)$$

  where we defined the Q-function: $Q(q) \equiv q \cosh(\pi q) \, \varphi(q)$

- Compare this equation with the classical spectral curve

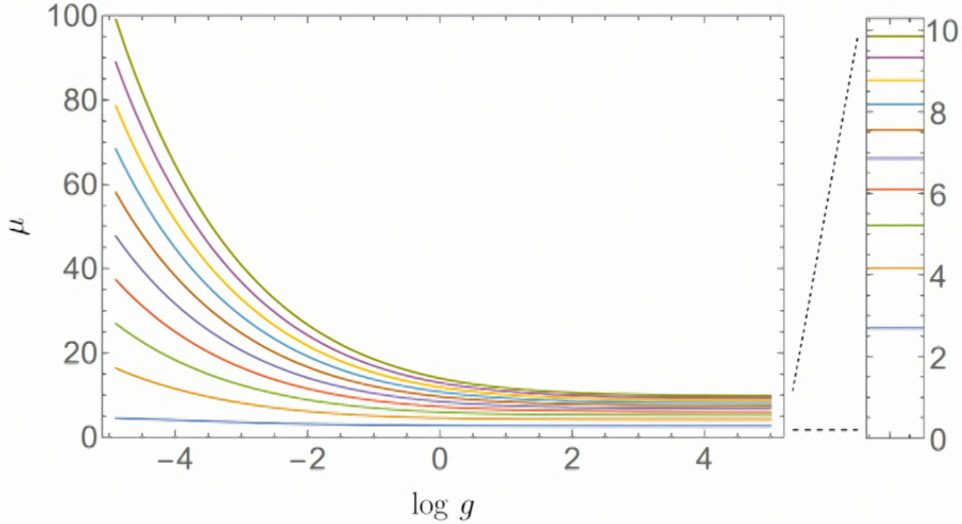
$$e^p + e^{-p} + 2 - \frac{\mu^2}{\pi |q|} = 0$$

AdS_3 generalization

- using brane tilings, the 't Hooft equation can be generalized to AdS_3

$$\mu^2\varphi(z) = -\!\!\!\!\!\!\int_0^1 dz' \frac{\varphi(z')}{(z'-z)^2} + \frac{\pi}{4g}(-i\partial_z)z(1-z)(-i\partial_z)\varphi(z)$$

- one can compute the spectrum numerically



Summary

- we studied segmented strings in AdS_3 & flat space
- system reformulated in terms of **celestial variables**
- further reformulation in terms of Möbius-invariant **tiling variables**
- identified canonical maps between different sets of coordinates
- this allowed us to interpret the transformed 't Hooft equation as a quantum spectral curve
- and also provided a generalization to AdS_3