Title: Mathematical Physics Lecture (230421)

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Collection: Mathematical Physics - Elective (2022/2023)

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Non-linear Graviton

$$\left\{ \begin{array}{l} \text{Self-dual (meaning } C_{-}=0) \\ \text{Einstein manifolds} \end{array} \right\} \Longleftarrow$$

$\Longleftrightarrow$ $\left\{\begin{array}{l}\text{3 complex dimensional manifold } X \\ \text{with a fibration } X \longrightarrow \mathbb{CP}^1 \\ \text{and a symplectic form } \omega \text{ on fibres,} \\ \text{Valued in } \mathcal{O}(2)\end{array}\right\}$

Additional conditions on X:

1) The fibration $X \longrightarrow \mathbb{CP}^1$ must admit a hol. section $\sigma : \mathbb{CP}^1 \to X$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$

2) If we want a real Einstein manifold, X must have an anti-holomorphic involution

M, the Einstein manifold, is

$$M = \{ \mathbb{CP}^1\text{'s in } X \text{ deforming the section } \sigma \}$$

n X.

$\mathbb{CP}^1$

ection

$\mathcal{O}(1)\oplus\mathcal{O}(1)$

M, the Einstein manifold, is

$$M = \left\{ \mathbb{CP}^1\text{'s in } X \text{ deforming the section } \sigma \right\}$$

This is a 4 parameter space

Normal bundle is $\mathcal{O}(1)\oplus\mathcal{O}(1)$

$\exists$ normal vectors n_1, n_2 on a patch of $\mathbb{CP}^1$ $(z \neq \infty)$

which vanish as $z \to \infty$

$\mathcal{O}(1) \oplus \mathcal{O}(1)$ rank 2

change patches, there's a 2×2 matrix

$\mathcal{O}(1) \oplus \mathcal{O}(1)$ means the matrix is

$$\begin{pmatrix} \tilde{z} & 0 \\ 0 & \tilde{z} \end{pmatrix}$$

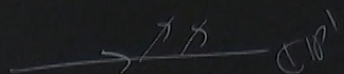
$\sigma \}$

atch of $\mathbb{CP}^1$

$$X \supseteq \mathbb{CP}^1 = \operatorname{Im} \sigma$$

On $\mathbb{CP}^1$, have

$$T\mathbb{CP}^1 \subseteq TX$$



N = quotient $TX/T\mathbb{CP}^1$

On $z \neq \infty$

n_1, n_2 a basis of normal vectors

$z \neq 0$

$\tilde{n}_1, \tilde{n}_2$ a basis

$$n_i = \tilde{z}\, \tilde{n}_i = \frac{1}{z}\tilde{n}_i$$

Normal vectors which have no poles

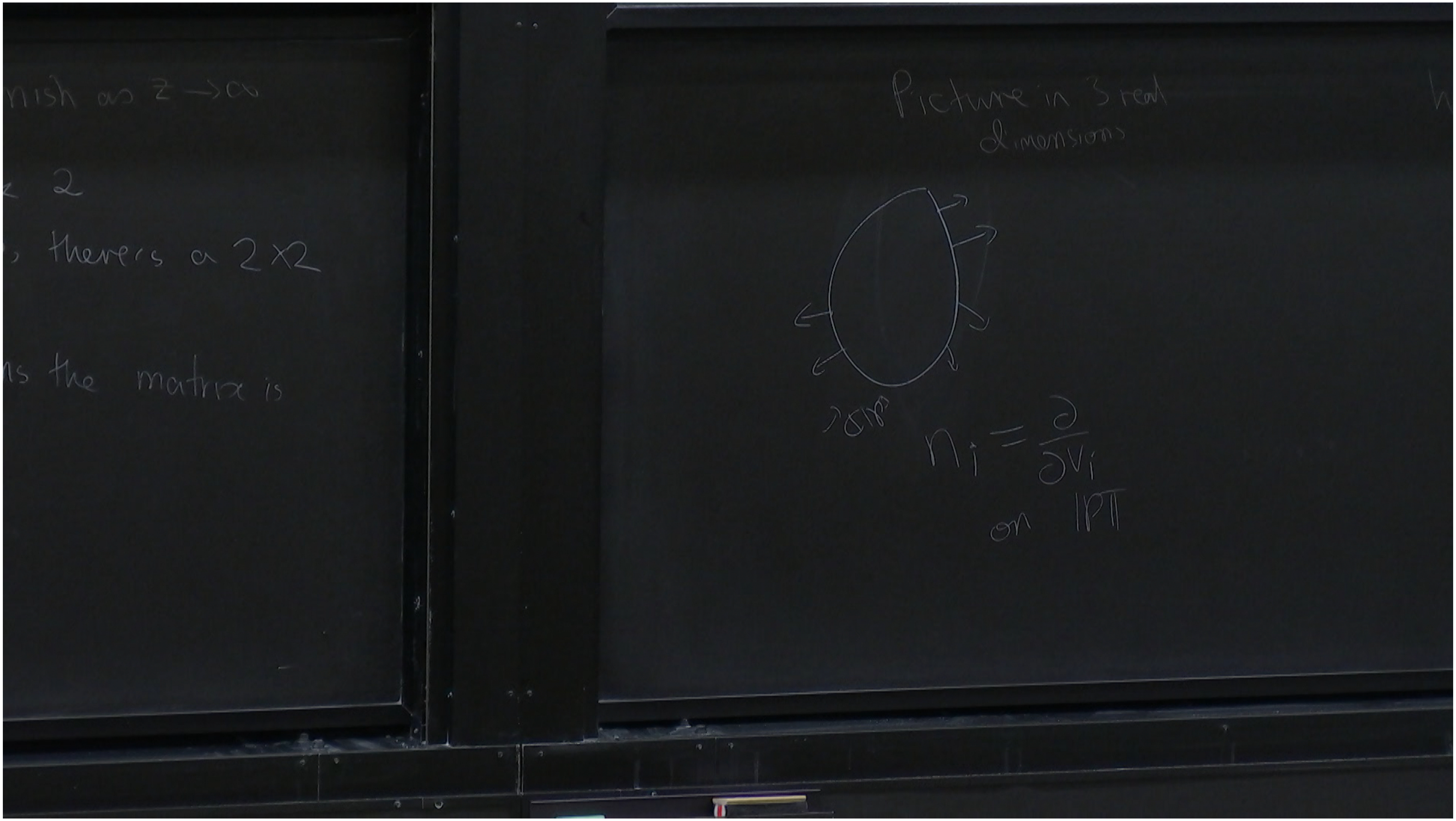
4 of them

n_1, n_2

$z n_1, n_2$

ctors

$\Rightarrow$ we can move our $\mathbb{CP}^1$ in a 4 parameter family of holomorphic $\mathbb{CP}^1$'s


nish as z → ∞
2
there's a 2×2
ns the matrix is
Picture in 3 real
dimensions
∂Ω
n_i = ∂/∂v_i
on ℝP

Where does the metric come from?

If $\mathbb{CP}^1 \subseteq X$

$\Longleftrightarrow$ it's a point $p \in M$

$T_p M$ = holomorphic sections of the normal bundle

$n_1, n_2, z n_1, z n_2$

Want to build $\langle \, , \rangle$ inner product

$$\begin{tikzcd} X \arrow[r, shift left] & \mathbb{CP}^1 \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

Normal bundle = vectors along fibre of $X \to \mathbb{CP}^1$

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Want to build $\langle\,,\rangle$ inner product.

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Normal bundle = vectors along fibre of $X \longrightarrow \mathbb{CP}^1$

Data on X means there's
a bundle map $\Lambda^2 N_\sigma \cong \mathcal{O}(2)$
(symplectic form on fibres)

This allows us to define
ε^{ij}, an anti-sym pairing
on n_1, n_2

(On flat space had $\varepsilon_{\dot\alpha\dot\beta}$ on $\frac{\partial}{\partial v_{\dot\alpha}}$)

Inner pro

means there's

$\Lambda^2 N_\sigma \cong \mathcal{O}(2)$ dimensional is ma

n on fibres)

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anti-sym pairing $\mathcal{O}(2)$

ad $\quad \varepsilon_{\dot\alpha\dot\beta}$ on $\frac{\partial}{\partial v_{\dot\alpha}}$)

Inner product on 4d space with basis

$n_1, n_2, z n_1, z n_2$

$$\langle n_i, z n_j \rangle = \omega_{ij}$$

$$\langle z n_i, n_j \rangle = -\omega_{ij}$$

More clever formula

$N_i(z)$ section of normal bundle

$$\langle N_i(z), N_j(z) \rangle = \oint N_i(z) \partial N_j(z) \omega^{ij} \frac{dz}{z}$$

It is really not obvious that this solves Einstein equations.

$$H^0(\mathbb{CP}^1, \mathcal{O}(1)) = \{1, z\}$$

SL_2

this is fundamental rep. of SL_2

has natural antisymmetric inner product.

ent

ons.

N normal bundle

is $V \otimes \mathcal{O}(1)$

V symplectic v. space of dim^n 2

$$H^0(\mathbb{CP}^1, N)$$

$$= \underbrace{V \otimes H^0(\mathbb{CP}^1, \mathcal{O}(1))}_{\text{symmetric inner product}}$$

Role of W_∞

The terminology is confusing + not consistent.

W_∞ = Lie algebra of Holomorphic Hamiltonian symmetries of $\mathbb{C}^2$

i.e. symmetries like

$\{h, -\}$

h is hol. function on $\mathbb{C}^2$

We can build an X as above by gluing:

$X = U \cup V$

$U = \mathbb{C}^3$ coords

$v_{\dot\alpha}, z$

$V = \mathbb{C}^3$ coordinates

$\tilde{v}_{\dot\alpha}, \tilde{z}$

Given

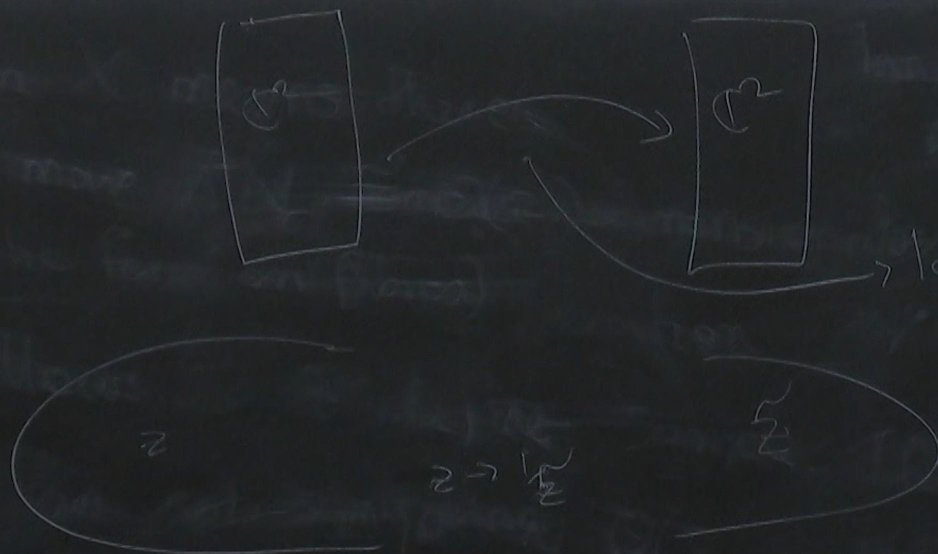
$$h(z) : \mathbb{C}^2 \times \mathbb{C}^\times \longrightarrow \mathbb{C}$$

holomorphic,

we can identify the coordinates by

$$\tilde{z} = 1/z$$

$$\tilde{v}_{\dot\alpha} = \frac{1}{z} e^{\{h,-\}} v_{\dot\alpha}$$



identified by a holomorphic transformation preserving $\{\ \}$ on fibres

$z) : \mathbb{C}^2 \times \mathbb{C}^\times \longrightarrow \mathbb{C}$

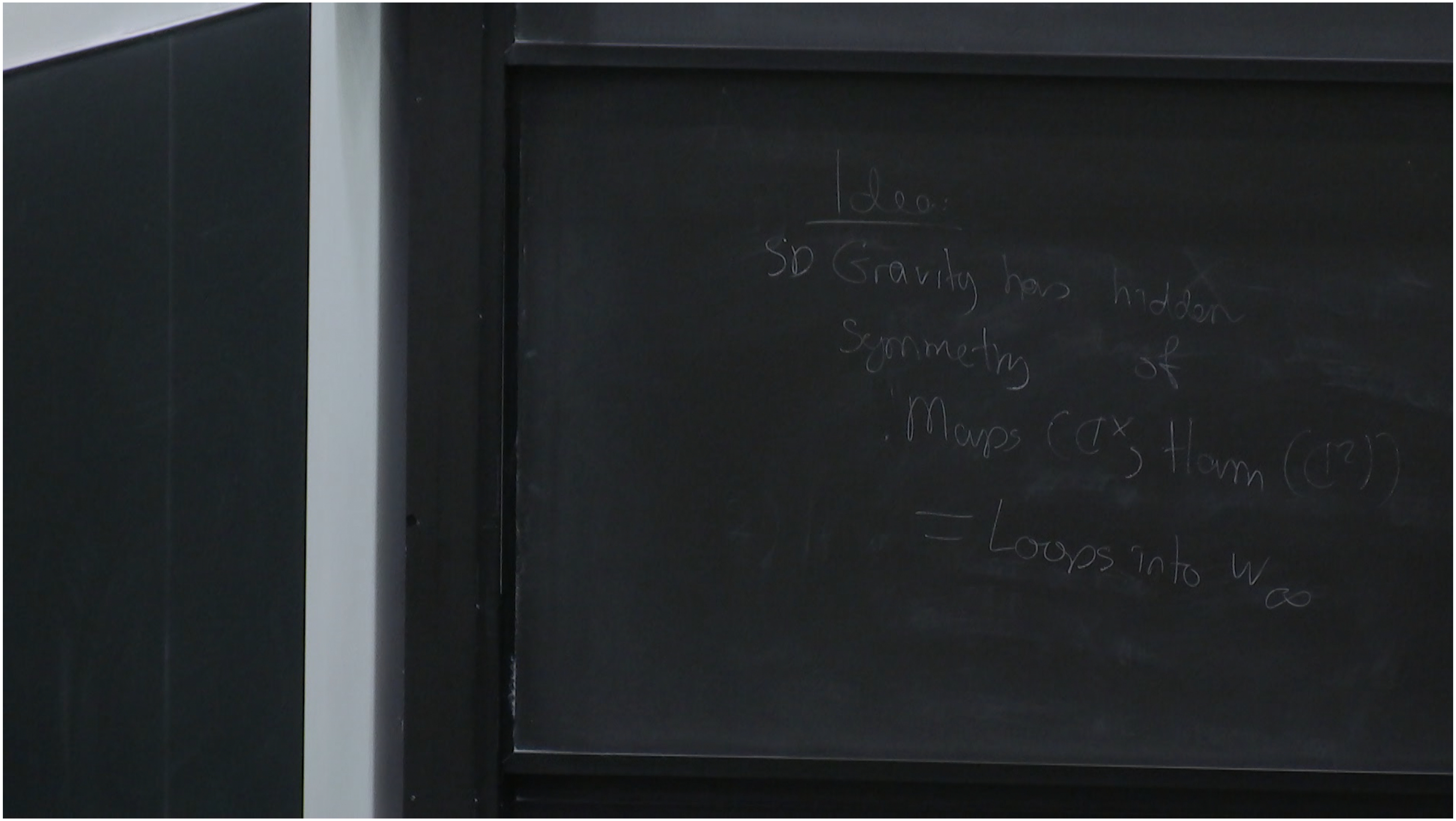
lomorphic,

n identify the coordinates by

$= 1/z$

$= \frac{1}{z} e^{\{h,-\}} v_{\dot\alpha}$

$\{h,-\} = \varepsilon_{\dot\alpha\dot\beta} \frac{\partial h}{\partial v_{\dot\alpha}} \frac{\partial}{\partial v_{\dot\beta}}$


Idea:
5D Gravity has hidden symmetry of
Maps ($\mathbb{C}^\times$, Hom ($\mathbb{C}^2$))
= Loops into W_∞

$$\left.\begin{array}{l} \dfrac{\partial}{\partial v_\alpha} \\ z\dfrac{\partial}{\partial v_\alpha} \end{array}\right\}$$ translations

$v_\alpha \dfrac{\partial}{\partial v_\beta}$ rotations

$\dfrac{1}{z}\dfrac{\partial}{\partial v_\alpha}$ supertranslations

....

Also

$$v_\alpha v_\beta \frac{\partial}{\partial v_\gamma}$$

tions

ations

ertranslations

Also,

$$\left\{ v_{\alpha}\, v_{;\beta}\, v_{;\gamma}, \ldots \right\}$$

$$v\, v \frac{\partial}{\partial v}$$

not a rotation!

These are part of the BMS asymptotic symmetry group

$v_{;\beta} v_{;\gamma}, - \}$

$\frac{\partial}{\partial v}$

a rotation!

e are part

f the BMS

mptotic symmetry group

$h : \mathbb{S}^2 \times \mathbb{C}^\times \to \mathbb{C}$

Gauge trans. on

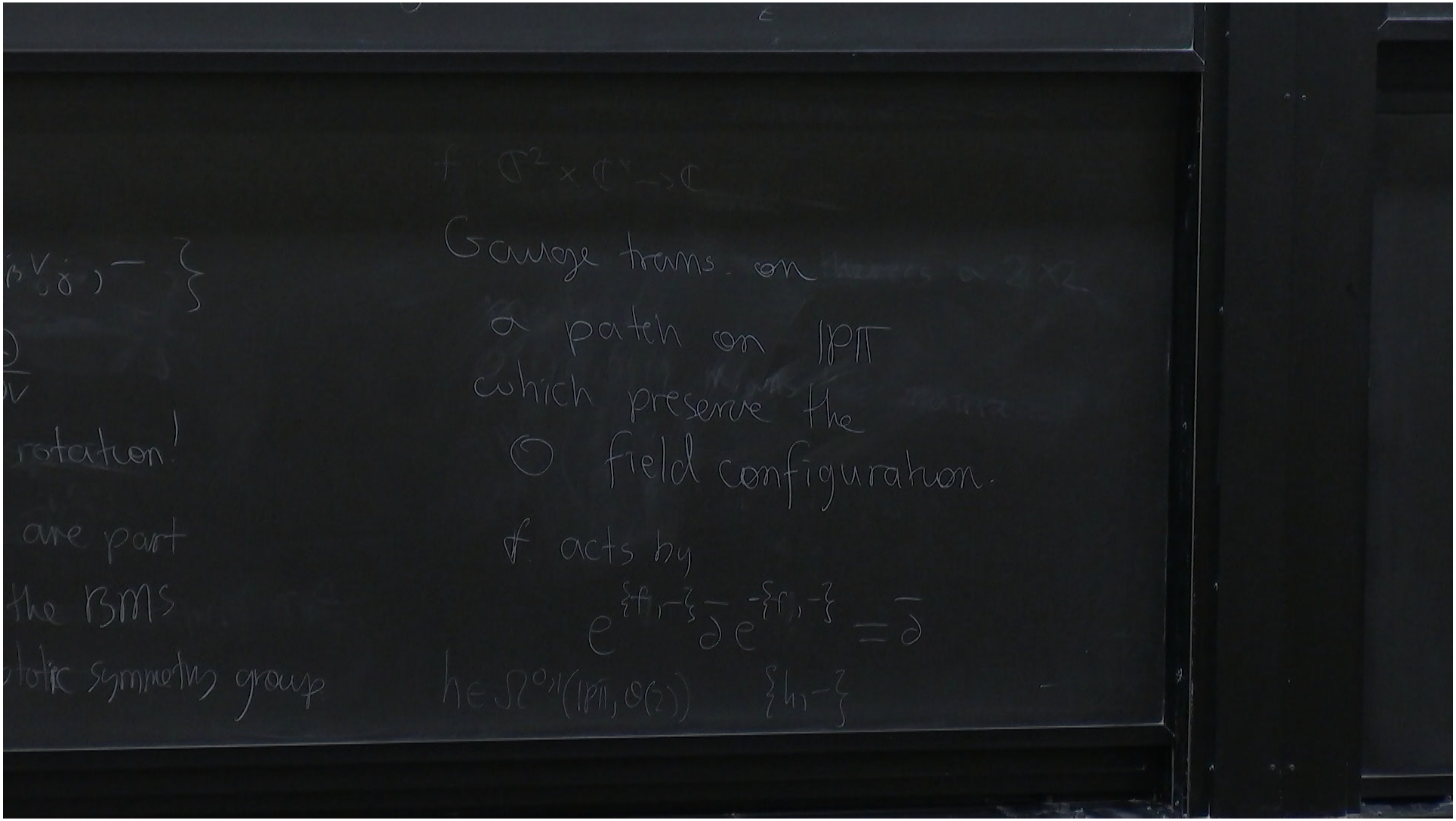
a patch on $\mathbb{PT}$

which preserve the

⊙ field configuration.

h acts by

$$e^{\{h,-\}}\bar{\partial}\, e^{-\{h,-\}} = \bar{\partial}$$


rotation!
are part
the BMS
totic symmetry group
Gauge trans. on
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f acts by
$e^{\{f,-\}}\bar\partial e^{-\{f,-\}} = \bar\partial$
$h \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(2))$ $\{h,-\}$