Title: Mathematical Physics Lecture (230419)

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Collection: Mathematical Physics - Elective (2022/2023)

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Formula:

Gauge invariant defect on $\mathbb{PT}$

$$\sum \int_{z_1 \dots z_n} \operatorname{tr} \frac{B(z_1) A(z_2) \dots B(z_m) \dots A(z_n)}{z_{12} \dots z_{n1}} z_{1m}^4$$

Want to show this reproduces scattering in 4d

SDYM in the presence of the operator $\mathrm{tr}\, B^2$

Step 1

Lift states on $\mathbb{R}^4$ to twistor space

$p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$

State becomes

$\int_{z=z_0} e^{\nu_{\dot\alpha} \tilde\lambda_{\dot\beta} \varepsilon^{\dot\alpha\dot\beta}}$ where $\lambda = (1, z_0)$

Note:

If $\lambda^{(i)} = (1, z_i)$

then $\langle i\, j \rangle$

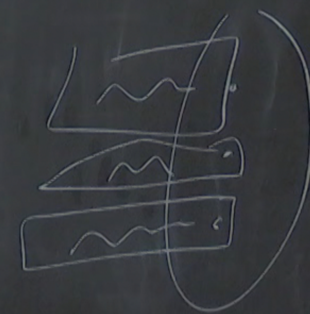
$= \varepsilon^{\alpha\beta} \lambda^{(i)}_{\alpha} \lambda^{(j)}_{\beta}$

$= z_j - z_i$

Step 2:

On $\mathbb{PT}$, the states are localized on codimension 2 loci

$\delta_{z = z_i}$



The states don't interact or scatter except off the holomorphic Wilson line

Step 3

On $\mathbb{PT}$, the scattering is computed by inserting our wave functions into the hol. Wilson line.

$$\int \frac{B(z_1)A(z_2)\cdots A(z_n)\, z_{1n}^4}{z_{12}\cdots z_{n1}}$$

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Step 3

On $\mathbb{PT}$, the scattering is computed by inserting our wave functions into the hol. Wilson line.

$$\int \frac{B(w_1)\, A(w_2) \cdots A(w_n)\, w_{1n}^4}{w_{12} \cdots w_{n1}}$$

has n external fields,
we take them to be
the n massless particles
we are scattering
1st $B(\omega_1) = \delta_{\omega_1 = z_1} e^{\tilde{\lambda}^{(1)}_v}$

$A(\omega_2) = \delta_{\omega_2 = z_2} e^{\tilde{\lambda}^{(2)} \cdot v}$

Then, the amplitude
involves replacing
$\omega_i - \omega_j$ by $\langle ij \rangle$
we get

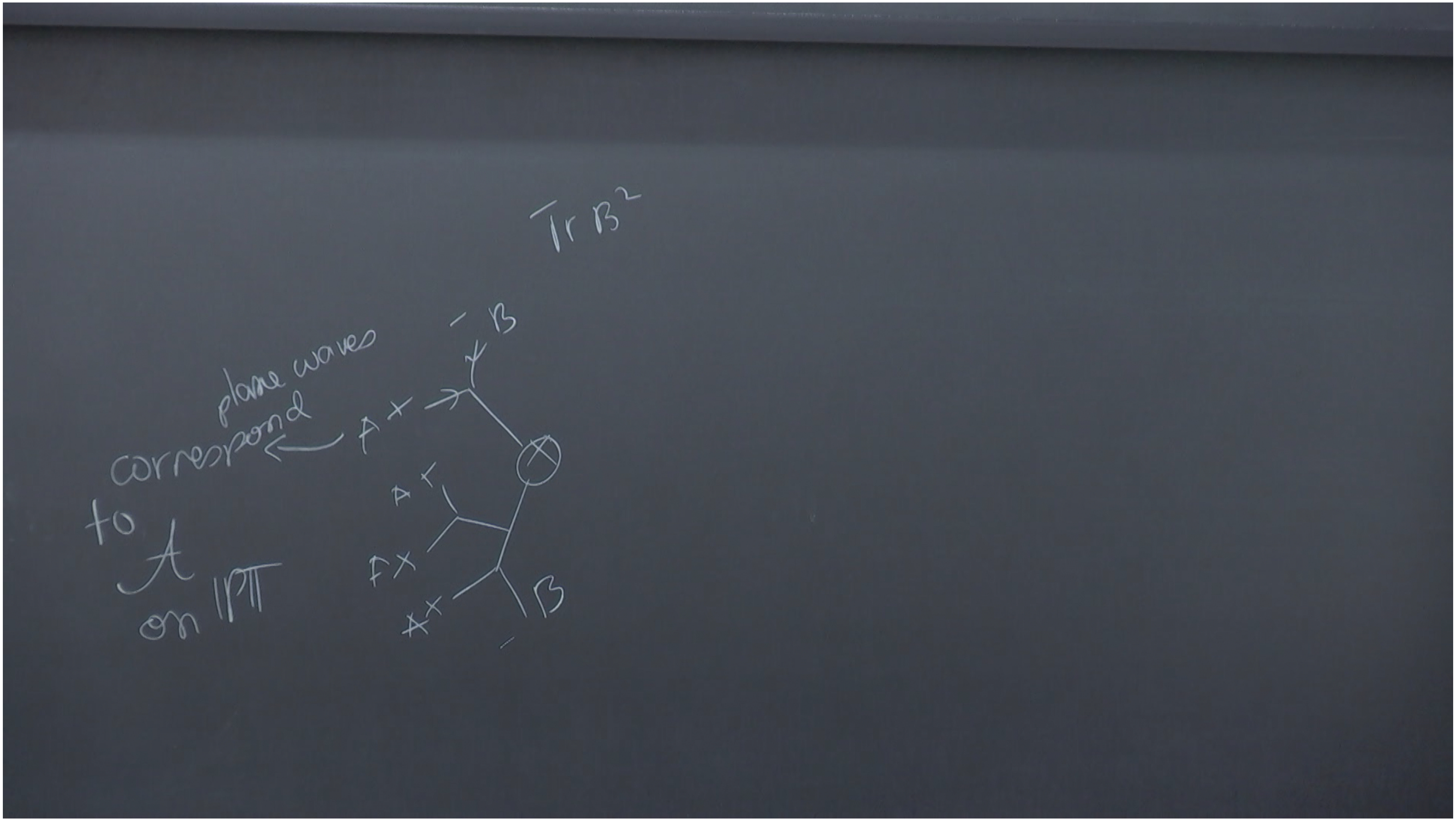
$$\frac{\langle lm \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

The states don't interact or scatter <u>except</u> off the holomorphic Wilson line

<u>Step 3</u>

On $\mathbb{PT}$, the scattering is computed by inserting our wave functions into the hol. Wilson line.

$$\int_{w_1 \cdots w_n} \frac{B(w_1)\, A(w_2) \cdots ABA \cdots A(w_n)\, \omega_{1n}^4}{\omega_{12} \cdots \omega_{n1}}$$


Tr B²
plane waves
correspond
to 𝒜
on ℙT
A
A
A
A
B
B

$$\delta_{w=z} = \bar{\partial} \frac{1}{w-z}$$

On $\mathbb{CP}^1$ we integrate over

$$e^{\tilde{\lambda}^{(i)}_\alpha v_\beta \varepsilon^{\alpha\beta}} = 1$$

If our hol. Wilson line had

$$\cdots \frac{\partial}{\partial v_{\dot\alpha}} A \frac{\partial}{\partial v_\beta} A \, \varepsilon_{\dot\alpha\beta} \cdots$$ in it, we would find

$[ij]$

Exercise:

Compute the scattering in SDYM at tree level off the operator

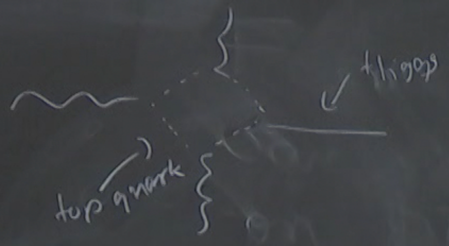
$$\mathrm{tr}\, B^3 = \mathrm{tr}\left(B_{\alpha_1\beta_1} B_{\alpha_2\beta_2} B_{\alpha_3\beta_3} \right) \varepsilon^{\beta_1\alpha_2}\varepsilon^{\beta_2\alpha_3}\varepsilon^{\beta_3\alpha_1}$$

Can compute scattering of 3 -ve helicity

$n-3$ +ve helicity

The twistor lift of this starts as

$$\int_{z_1,z_2,z_3 \in \mathbb{CP}^1} B(z_1)B(z_2)B(z_3)\, z_{12} z_{13} z_{23}\, dz_1\, dz_2\, dz_3$$



$\varphi_{\text{Higgs}} \operatorname{tr} B^3$

Gravity

(Bittleston's recent arxiv paper)

$$h \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(2))$$

$$g \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-6))$$

then $\int_{\mathbb{PT}} g \bar{\partial} h$

and solns of EOM/gauge are

h : Linearized Einstein equation
with $C_- = 0$ (C = Weyl tensor)

g : " with $C_+ = 0$.

Question

Can we make this interacting so that we see non-linear Einstein equations?

The interacting Lagrangian is the following

$$\int \overset{-6}{g}\,\bar\partial h + \frac{1}{2}\, g\, \overset{1}{\frac{\partial}{\partial v_\alpha} h}\; \overset{1}{\frac{\partial}{\partial v_\beta} h}\; \varepsilon_{\alpha\beta}$$

$$= \int g\,\bar\partial h + \frac{1}{2} g \{h, h\}$$

$\{\ ,\ \}$ is Poisson bracket on V plane.

Note:

$h \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(2))$

$\frac{\partial h}{\partial v_\alpha} \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(1))$

weights always sum to -4

__Theorem (Penrose)__ Non-linear Graviton

Those $h \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(2))$

satisfying $\bar{\partial} h + \frac{1}{2}\{h,h\} = 0$,

up to gauge transformations

$=$ Sol'ns to Einstein equation with $C_- = 0$

Gauge transformations

$\chi \in \Omega^{0,0}(\mathbb{PT}, \mathcal{O}(2))$

$$\{\chi, -\} = \varepsilon_{\alpha\beta} \frac{\partial \chi}{\partial v_\alpha} \frac{\partial}{\partial v_\beta}$$

this is a v. field

with no weight

Gauge transf.

= coordinate transformations by vector field

$\{\chi, -\}$

for $\chi \in \Omega^{0,0}(\mathbb{PT}, \mathcal{O}(2))$)

$h \in \Omega^{0,1}(\mathbb{P}\Pi, \mathcal{O}(2))$

$\{h, -\} = \varepsilon_{\alpha\beta} \frac{\partial h}{\partial v_\alpha} \frac{\partial}{\partial v_\beta}$

$\underbrace{\in \Omega^{0,1}(\mathbb{P}\Pi, T\mathbb{P}\Pi)}$

This is the space of Beltrami differentials

If X is a complex manifold

$V \in \Omega^{0,1}(X, TX)$

we can deform X by deforming

$\bar\partial \longrightarrow \bar\partial + V$

$V = V_j^i \, d\bar z_i \frac{\partial}{\partial z_j}$

$\bar\partial = d\bar z_i \frac{\partial}{\partial \bar z_i}$

This is a deformation if and only if

$$(\bar{\partial}+V)^2 = 0$$

New hol. fns satisfy

$$(\bar{\partial}+V)F = 0$$

The eq^n

$$\bar\partial h + \frac{1}{2}\{h,h\} = 0$$

$$\Longleftrightarrow$$

$$(\bar\partial + \{h,-\})^2 = 0$$

$\Longleftrightarrow$ $\bar\partial + \{h,-\}$

defines a new complex

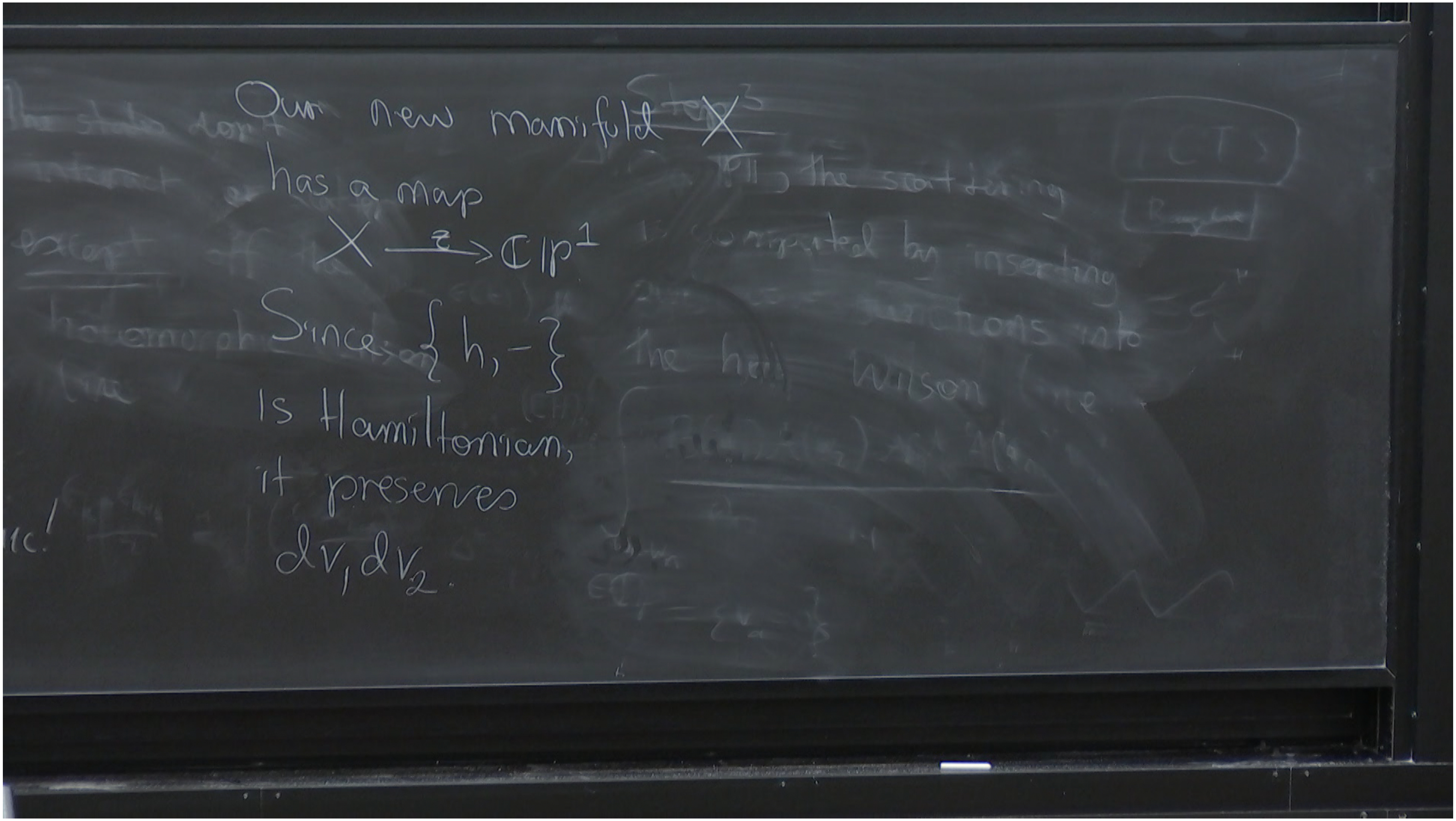
manifold

A function is holomorphic if

$$\bar\partial F + \{h,F\} = 0$$

Note $\{h, z\} = 0$

So, z is still holomorphic!


Our new manifold X
has a map
X —z→ ℂℙ¹
Since {h, −}
is Hamiltonian,
it preserves
dv₁, dv₂.

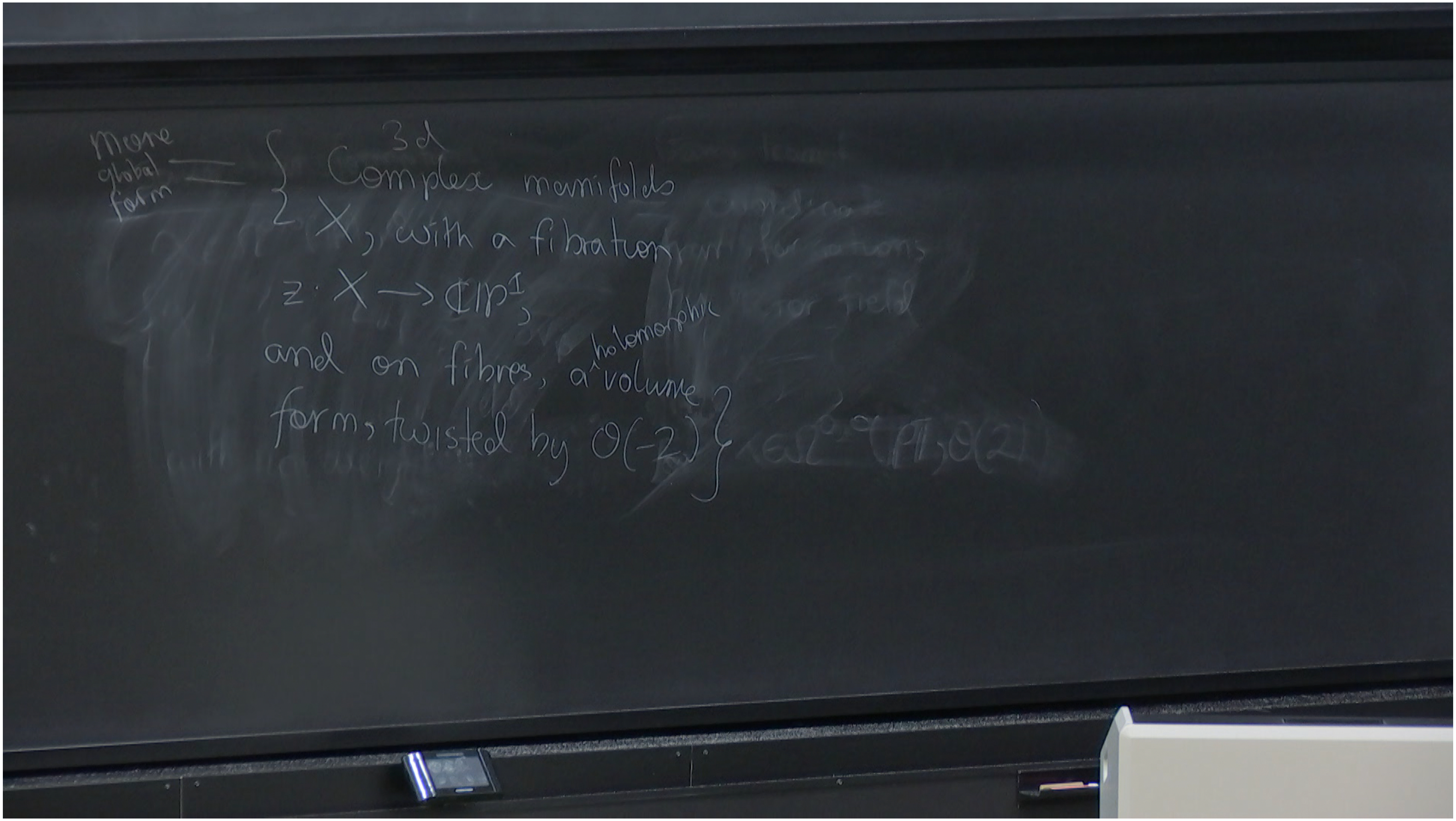
Theorem (Penrose) Non-linear Graviton

Those $h \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(2))$

satisfying $\bar{\partial} h + \frac{1}{2}\{h, h\} = 0$,

up to gauge transformations

$=$ Sol'ns to Einstein equation with $C_{-} = 0$


More global form
3d
Complex manifolds
X, with a fibration
z: X → ℂP¹,
and on fibres, a holomorphic volume form, twisted by O(-2)

Our new manifold X^3
has a map
$X \xrightarrow{z} \mathbb{C}\mathbb{P}^1$
Since $\{h, -\}$
is Hamiltonian,
it preserves
$dv_1 dv_2$.

$(\bar\partial + v)F = 0$
$\Rightarrow (\bar\partial + v)^2 F = 0$
If are n hol. sol'ns,
independent,
$\Rightarrow (\bar\partial + v)^2 = 0$ on anything.