Title: Mathematical Physics Lecture (230414)

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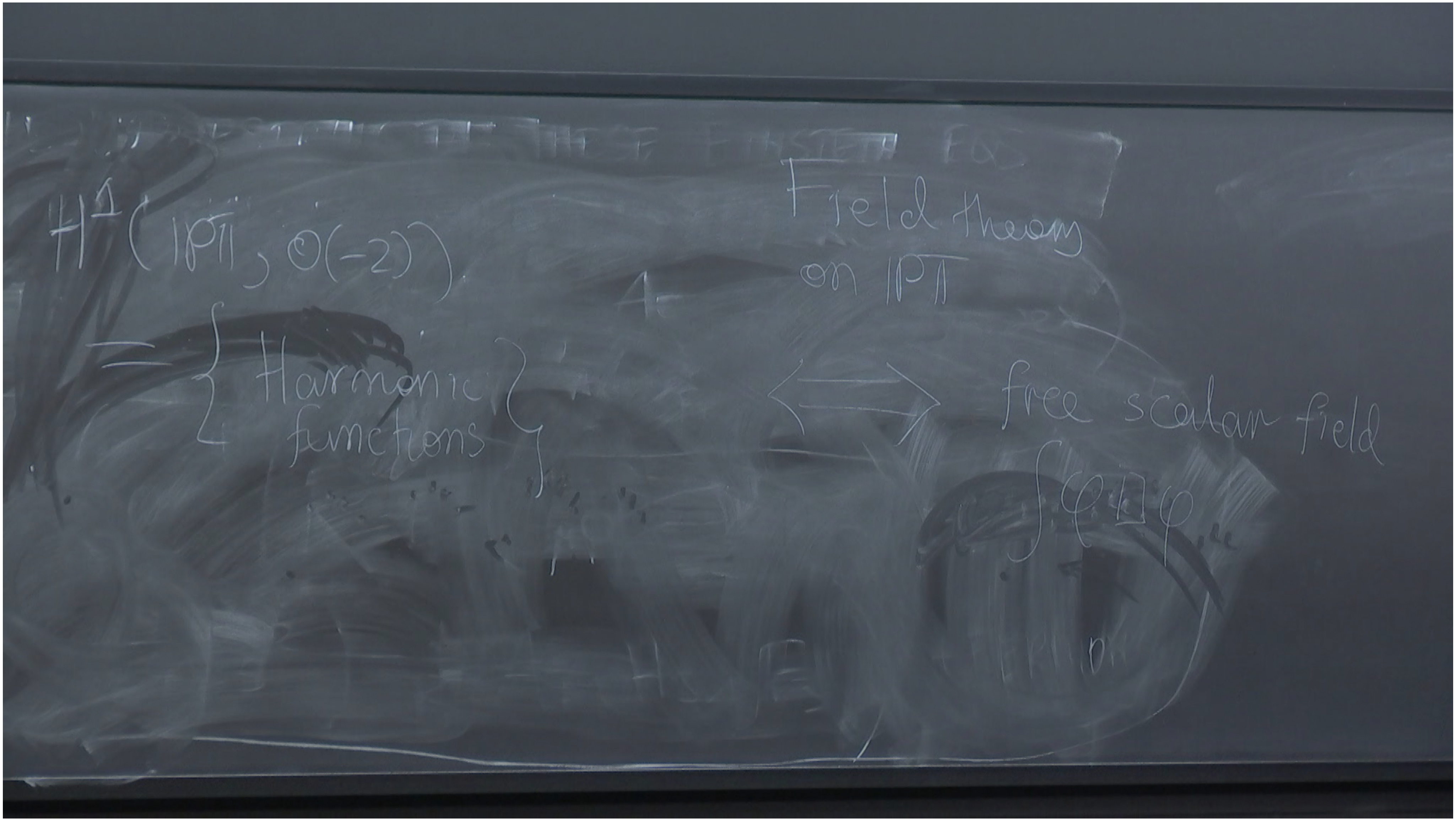

$H^1(\mathbb{P}^1, \mathcal{O}(-2))$
$= \{$ Harmonic functions $\}$
Field theory on $\mathbb{P}^1$
$\Longleftrightarrow$ free scalar field
$\int \varphi \Box \varphi$

$H^1(\mathbb{PT}, \mathcal{O}(-2))$

comes from Dolbeault cohomology representative is

$$A \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-2))$$

In other words,

$$A = (d\bar{z}\, f + d\bar{v}_i\, g^i)\, dz$$

We would like the eq$^n$ $\bar\partial A = 0$ to be the EL equation.

$$S(\mathcal{A}) = \int_{\mathbb{PT}} \mathcal{A} \wedge \bar{\partial} \mathcal{A}$$

$$\mathcal{A} \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-2))$$

$$\bar{\partial} \mathcal{A} \in \Omega^{0,2}(\mathbb{PT}, \mathcal{O}(-2))$$

$$\mathcal{A} \bar{\partial} \mathcal{A} \in \Omega^{0,3}(\mathbb{PT}, \mathcal{O}(-4))$$

$dz\, dv_1\, dv_2$ is a section of $\mathcal{O}(-4)$

$dz$ contributes $z^{-2}$

$dv_i$ " $z^{-1}$

when we change patches.

$A\bar{\partial}A$ locally

$F \quad dz\, d^2v \quad d\bar{z}\, d^2\bar{v}$

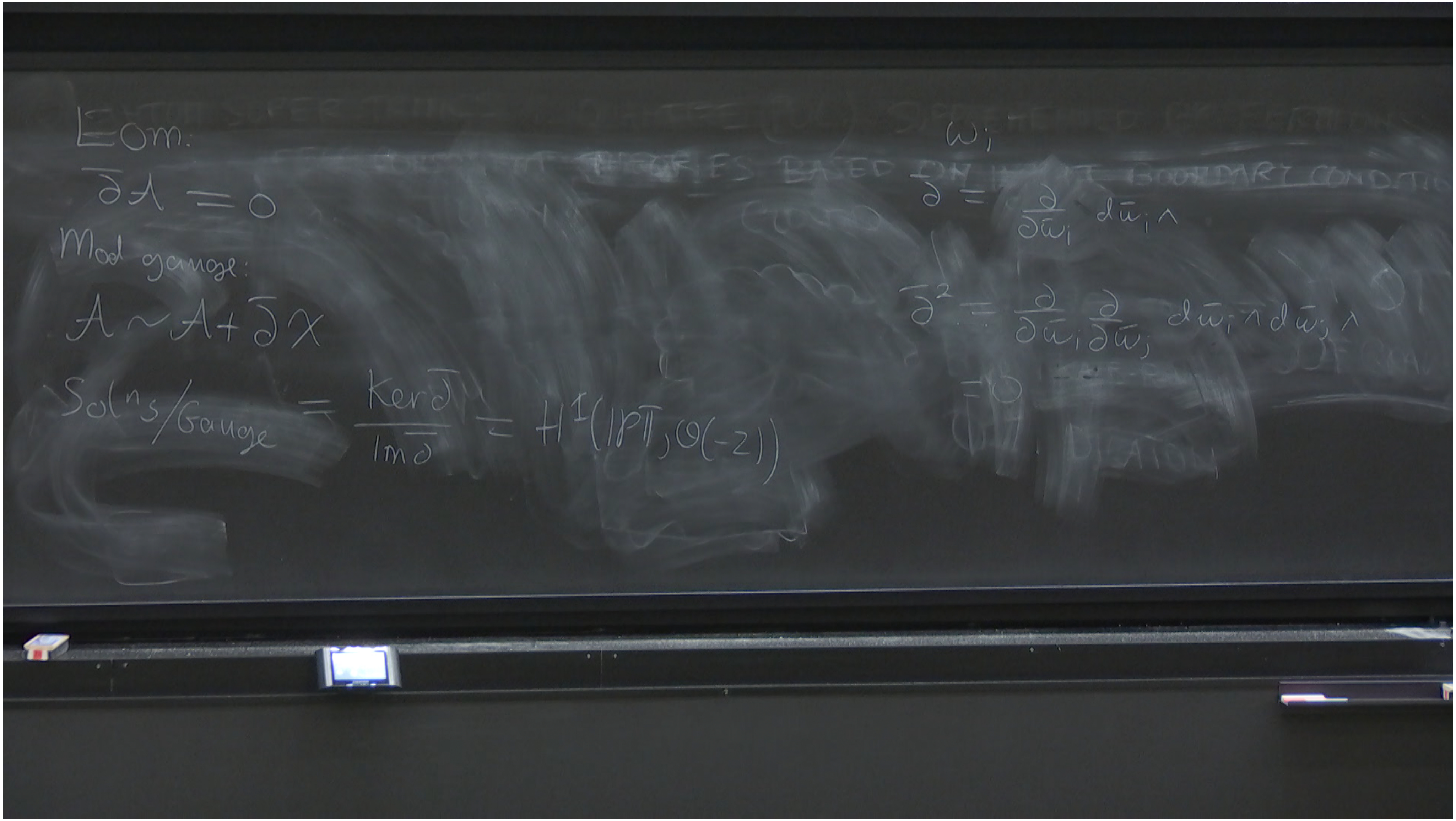
So we can integrate

$\int A\bar{\partial}A$

is invariant under

$A \longrightarrow A + \bar{\partial}\chi$

$\chi \in \Omega^{0,0}(\mathbb{PT}, \mathcal{O}(-2))$


EOM:
$\bar{\partial} A = 0$
Mod gauge:
$A \sim A + \bar{\partial} \chi$
$\text{Sol}^{\text{n}}\text{s}/\text{Gauge} = \frac{\text{Ker}\,\bar{\partial}}{\text{Im}\,\bar{\partial}} = H^1(\mathbb{PT}, \mathcal{O}(-2))$
$\omega_i$
$\bar{\partial} = \frac{\partial}{\partial \bar{\omega}_i} d\bar{\omega}_i \wedge$
$\bar{\partial}^2 = \frac{\partial}{\partial \bar{\omega}_i}\frac{\partial}{\partial \bar{\omega}_j} d\bar{\omega}_i \wedge d\bar{\omega}_j \wedge$
$= 0$

## Lemma

Sol'ns to the EOM
modulo gauge

$$= H^1(\mathbb{PT}, \mathcal{O}(-2))$$

$=$ Harmonic functions
on $\mathbb{R}^4$.

EOM:

$$\bar{\partial} \mathcal{A} = 0$$

Mod gauge:

$$\mathcal{A} \sim \mathcal{A} + \bar{\partial} \chi$$

$$\text{Sol}^{\text{ns}}/\text{Gauge} = \frac{\text{Ker}\,\bar{\partial}}{\text{Im}\,\bar{\partial}} = H^1(\mathbb{PT}, \mathcal{O}(-2))$$

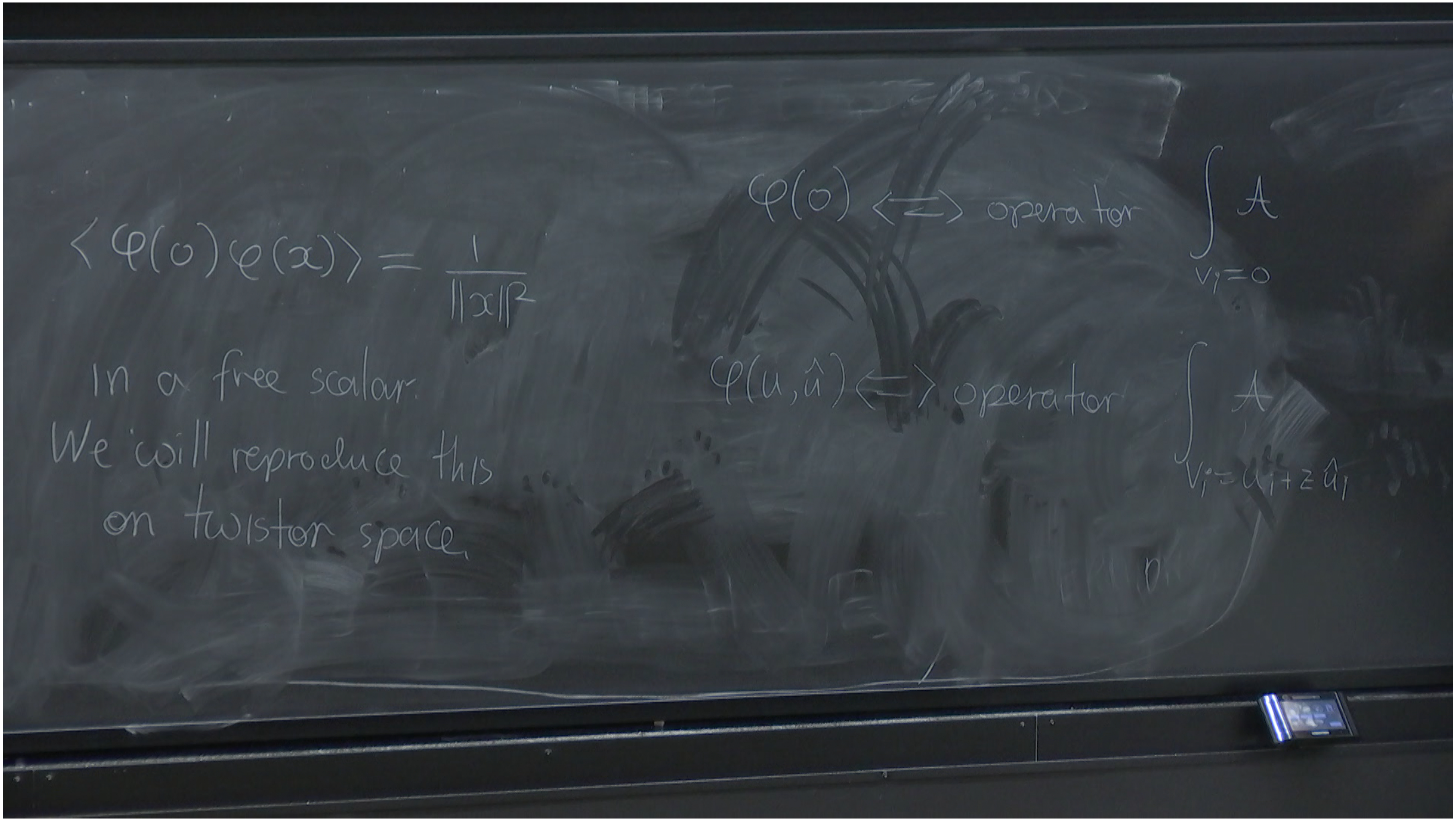
$\omega_i$

$$\bar{\partial} = \frac{\partial}{\partial \bar{w}_i} \, d\bar{w}_i \wedge$$

$$\bar{\partial}^2 = \frac{\partial}{\partial \bar{w}_i} \frac{\partial}{\partial \bar{w}_j} \, d\bar{w}_i \wedge d\bar{w}_j \wedge$$

$$= 0$$

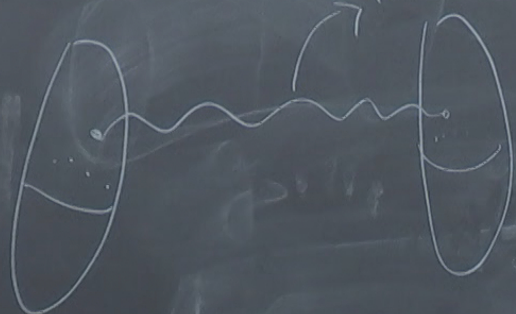
$T, \mathcal{O}(-2))$


$\langle \varphi(0)\varphi(x)\rangle = \frac{1}{\|x\|^2}$
In a free scalar.
We will reproduce this on twistor space.
$\varphi(0) \Longleftrightarrow$ operator $\int_{v_1=0} A$
$\varphi(u,\hat{u}) \Longleftrightarrow$ operator $\int_{v_1=u_1+z\hat{u}_1} A$

$\langle \varphi(0)\varphi(x)\rangle$

$\Longleftrightarrow$ 2 point fn of non-local operators

Twistor space propagator.



$$\langle \varphi(0)\varphi(x)\rangle = \frac{1}{\|x\|^2}$$

In a free scalar.

We will reproduce this on twistor space.

$$\varphi(0) \Longleftrightarrow \text{operator } \int_{v_1=0} A$$

$$\varphi(u,\hat{u}) \Longleftrightarrow \text{operator } \int_{v_1 = u_1 + z\hat{u}_1} A$$

$$\mathcal{A} = F(v_1, z)\, dz \ \delta_{|z|=1}\, d\bar{z}$$

$$\oint_{|z|=1} F\, dz = \int_{\mathbb{CP}^1} \mathcal{A}$$

Dolbeault: $\int$ over a $\mathbb{CP}^1$.

**Trick:**

Don't compute Propagator directly.

Instead, look at field sourced by the defect:

$$= \int_{\text{defect}} P$$

Lagrangian + defect is

$$-\frac{1}{2}\int_{\mathbb{PT}} \mathcal{A}\bar{\partial}\mathcal{A} + \int_{v_1=0} \mathcal{A}$$

Field eqns with source:

$$\bar{\partial}\mathcal{A} = \delta_{v_1=0,\, v_2=0}$$

Sol'n to this equation

$$\mathcal{A} = \frac{1}{(4\pi)^2}\frac{1}{v_1}\bar{\partial}\frac{1}{v_2}\, dz$$

Using the fact

$$\boxed{\bar{\partial}\frac{1}{z} = 2\pi i\,\delta_{z=0}}$$

Lagrangian + defect is

$$-\frac{1}{2}\int_{\mathbb{P}\mathbb{T}} \mathcal{A}\,\bar{\partial}\mathcal{A} + \int_{v_1=0} \mathcal{A}$$

Field eqns with source:

$$\bar{\partial}\mathcal{A} = \delta_{v_1=0,\, v_2=0}$$

Sol$^n$ to this equation

$$\mathcal{A} = \frac{1}{(4\pi)^2}\frac{1}{v_1}\,\bar{\partial}\,\frac{1}{v_2}\,dz$$

Using the fact

$$\boxed{\bar{\partial}\,\frac{1}{z} = 2\pi i\,\delta_{z=0}}$$

$$A = \frac{1}{V_1}\,\bar\partial\,\frac{1}{V_2}\,dz$$

$$= \frac{1}{V_1}\,\delta_{V_2=0}\,dz$$

$$\bar\partial A = \delta_{V_1=0}\,\delta_{V_2=0}\,dz$$

$$\bar\partial\,\frac{1}{V_1} = \delta_{V_1=0}$$

2 point fn

$$A = \frac{1}{V_1}\,\bar\partial\,\frac{1}{V_2}\,dz$$

sourced by defect at $V_i = 0$

2 pt function is

$$\int_{V_i = a_i + z\hat{a}_i} A$$

2 point fn

$A = \frac{1}{v_1} \bar\partial \frac{1}{v_2} dz$

sourced by defect at $v_i = 0$

2 pt function is

$$\int_{v_i = u_i + z\hat{u}_i} A$$

$v_1 = u_1 + \bar{u}_2 z$

$v_2 = u_2 - \bar{u}_1 z$

We want

$$\int_z \frac{1}{u_1 + \bar{u}_2 z} \bar\partial \left( \frac{1}{u_2 - \bar{u}_1 z} \right) dz$$

$$\bar\partial \frac{1}{u_2 - \bar{u}_1 z} = \frac{-1}{\bar{u}_1} \delta_{z = u_2/\bar{u}_1}$$

$V_1 = u_1 + \bar{u}_2 z$

$V_2 = u_2 - \bar{u}_1 z$

We want

$$\int_Z \frac{1}{u_1 + \bar{u}_2 z} \, \bar{\partial}\left(\frac{1}{u_2 - \bar{u}_1 z}\right) dz$$

$$\bar{\partial} \frac{1}{u_2 - \bar{u}_1 z} = \frac{-1}{\bar{u}_1} \, \delta_{z = u_2/\bar{u}_1}$$

Answer is

$$-\frac{1}{\bar{u}_1} \, \frac{1}{u_1 + \bar{u}_2 u_2/\bar{u}_1}$$

$$= -\frac{1}{\|u\|^2}$$

(really $\frac{1}{4\pi^2} \frac{1}{\|u\|^2}$)

2 point fn

$\langle \varphi(0)\varphi(x)\rangle$

$= G(x)$

$G$ is Green's function

By definition

$$\Delta G(x) = \delta_{x=0}$$

$G(x)$ is the field sourced by $\varphi(0)$:

$$-\frac{1}{2}\int \varphi \Delta \varphi + \varphi(0)$$

Solve EOM, get $G(x)$

What is

$$\left\langle \left(\int_{\mathbb{R}} \varphi(x)\right) \varphi(y) \right\rangle$$

$$= \int_{x\in\mathbb{R}} G(y-x)\,dx$$

Equivalently:

Solve EoM in presence of

$$\int_{\mathbb{R}} \varphi(x)\,dx$$

$$\Delta_y \int G(y-x)\,dx$$

$$= \int_x \delta_{y=x}\,dx = \delta_{y\in\mathbb{R}}$$

Lagrangian + defect is

$$-\frac{1}{2}\int_{\mathbb{PT}} \mathcal{A}\,\bar\partial\mathcal{A} + \int_{\mathbb{PT}} \mathcal{A}\,\delta_{v_1=v_2=0}$$

Field eqns with source:

$$\bar\partial \mathcal{A} = \delta_{v_1=0,\, v_2=0}$$

Soln to this equation

$$\mathcal{A} = \frac{1}{(4\pi^2)}\frac{1}{v_1}\,\bar\partial\,\frac{1}{v_2}\,dz$$

Using the fact

$$\boxed{\bar\partial\,\frac{1}{z} = 2\pi i\,\delta_{z=0}}$$

Spin 1 : +ve helicity, $H^1(\mathbb{PT}, \mathcal{O}(0))$ $\quad A \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O})$

-ve " $\quad H^1(\mathbb{PT}, \mathcal{O}(-4))$ $\quad B \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-4))$

Build a theory on twistor space
for Maxwell theory

$$A \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O})$$

$$A_{\bar z}\, d\bar z + A_{\bar v_1}\, d\bar v_1$$

$$B \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-4))$$

$$\left(B_{\bar z}\, d\bar z + B_{\bar v_1}\, d\bar v_1\right) dz\, dv_1\, dv_2$$

Lagrangian is

$$\int B \bar\partial A$$

$$A \sim A + \bar\partial \chi \qquad B \sim B + \bar\partial \eta$$

$$\chi \in \Omega^{0,0}(\mathbb{PT}) \qquad \eta \in \Omega^{0,0}(\mathbb{PT}, \mathcal{O}(-4))$$

The Lagrangian gives a twistor version of Maxwell theory.

We can make this interacting in a very simple way

$\mathfrak{g}$ a Lie algebra, basis $t_a$.

As before,

Solns to EOM/Gauge

$= H^1(\mathbb{P}T, \mathcal{O}) \oplus H^1(\mathbb{P}T, \mathcal{O}(-4))$

$=$ Solns of Maxwells eqns.

The Lag

Version

We can

in a ver

$g$ a Lie

$A \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{g})$

$B \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{g} \otimes \mathcal{O}(-4))$

Lagrangian is

$$\int \mathrm{tr}\left(B\, F^{0,2}(A)\right)$$

$A = t_a A^a_{\bar z}\, d\bar z + \dots$

$B = \left(t_a B^a_{\bar z}\, d\bar z + \dots\right) dz\, dv_1\, dv_2$

$$F^{0,2}(A) = \bar\partial A + \frac{1}{2}[A, A]$$

Gauge trans. for $A$ have to be modified:

$$\delta A = \bar\partial \chi + [\chi, A]$$

**Theorem** (Ward)

Sol$^{n}$s to EOM on twistor space with $B=0$
(i.e. $\mathcal{A}$, $F^{0,2}(\mathcal{A})=0$) / Gauge

$=$ Sol$^{n}$s to the Self-Dual Yang-Mills eq$^{n}$ on $\mathbb{R}^4$

$A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{g}$
with $F_-(A) = 0$ This implies YM equation