

Title: Mathematical Physics Lecture (230412)

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OBJECTS

In General

$$H^1(\mathbb{P}^1, \mathcal{O}(2sh - 2))$$

= Solns of free field eqns

for a field of spin $s = 0, \frac{1}{2}, 1, \dots$

helicity $h = \pm 1$

Recall,

$$\text{Spin}(4, \mathbb{C}) = \text{SL}_2^+ \times \text{SL}_2^-$$

There are 2 spin reps.

S_+ , S_- fundamental
reps of SL_2^+ and SL_2^-

$\dim_{\mathbb{C}} S_{\pm}$

$$\dim_{\mathbb{C}} S_{\pm} = 2.$$

$$SL_2^+ \times SL_2^-$$

spin reps.

fundamental
and SL_2^-

$$\psi_{\alpha}^- \in S_{-} \quad \alpha = 1, 2$$

$$\alpha = 1, 2$$

$$\psi_{\alpha}^+ \in S_{+} \quad \alpha = 1, 2$$

$$\alpha = 1, 2$$

are a basis.

$$\dim_{\mathbb{C}} S_{\pm} = 2.$$

-ve helicity $\psi_{\alpha}^{-} \in S_{-}$ $\alpha = 1, 2$

pos helicity $\psi_{\dot{\alpha}}^{+} \in S_{+}$ $\dot{\alpha} = 1, 2$

are a basis.

Spinor field is in

$$S_{+} \oplus S_{-}$$

(4 components)

$$\mathbb{Z}/2 \subseteq \text{Spin}(4)$$

In $SL_2^{+} \times SL_2^{-}$, $\mathbb{Z}/2$ is

$$(-\text{Id}, -\text{Id}) \quad SO(4) =$$

What is helicity for photons?

Solns of Maxwell eqⁿ

$$= \left\{ F \in \Omega^2(\mathbb{R}^4) \text{ satisfying} \right. \\ \left. \begin{array}{l} dF = 0 \\ d\star F = 0 \end{array} \right\}$$

SIDER FOR SPIN

$$\star : \Omega^2(\mathbb{R}^4) \rightarrow \Omega^2(\mathbb{R}^4)$$

The two helicity fields are

$$F_+ = \frac{1}{2}(1+\star)F$$

-ve helicity are

$$F_- = \frac{1}{2}(1-\star)F$$

In components:

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon_{\alpha\beta} F_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}} F_{\alpha\beta}$$

The Maxwell eqⁿ is

$$\frac{\partial F_{\dot{\alpha}\dot{\beta}}}{\partial x_{\alpha\dot{\alpha}}} = 0$$

$$\frac{\partial F_{\alpha\beta}}{\partial x_{\alpha\dot{\alpha}}} = 0$$

Dirac
 $\frac{\partial \psi}{\partial x_{\alpha\dot{\alpha}}}$
 $\frac{\partial \psi}{\partial x_{\alpha\dot{\alpha}}}$
" "
" "
 $\Gamma_{\mu}^{\alpha\dot{\alpha}}$
 $\frac{\partial}{\partial x_{\mu}}$

Dirac eqn

$$\frac{\partial \psi_\alpha}{\partial x_{\alpha\dot{\alpha}}} = 0$$

$$\frac{\partial \psi_{\dot{\alpha}}}{\partial x_{\alpha\dot{\alpha}}} = 0$$

"

$$\pi^{\alpha\dot{\alpha}} \frac{\partial}{\partial x_\mu}$$

$$\langle \psi_{\alpha\dot{\alpha}}, \psi_{\beta\dot{\beta}} \rangle$$

$$= \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}}$$

$$x_{1\dot{\alpha}} = u_{\dot{\alpha}}$$

$$x_{2\dot{\alpha}} = \hat{u}_{\dot{\alpha}}$$

R_{abcd} has 20 components

$$Ric_{ab} = 0$$

Are 10 remaining components.

Weyl tensor

Field of Spin 2

Flat metric $g_{\mu\nu}$ perturb to $g + \delta g$ to first order, so the Einstein eqⁿ holds

What is the Diff invariant info (Really) diffeos that \downarrow Id + something sm

Answer:

Curvature of $g + \delta g$

What is the
Diff invariant information?
(Really, diffeos that are
Id + something small)

Answer:

Curvature of $g + \delta g$

Weyl tensor

in a 5+5 dim

of Spin(4)

with spinor notation

$$= C_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^- + C_{\dot{\alpha}_1 \dot{\alpha}_2 \dot{\alpha}_3 \dot{\alpha}_4}^+$$

C^+ , C^- are symmetric in spinor indices.

Branch + Einstein says

$$\frac{\partial}{\partial x_{\dot{\alpha}\dot{\alpha}}} C_{\alpha\beta\gamma\delta}^- = 0$$

$$\frac{\partial}{\partial x_{\dot{\alpha}\dot{\alpha}}} C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}^+ = 0$$

Let us consider

$$S=1$$

when $\mathcal{O}(-1), \mathcal{O}(-3) \rightsquigarrow$ spinors

$$H^1(\mathbb{P}^1, \mathcal{O}(-1)) \simeq \{ \psi_{\dot{\alpha}}, \text{ satisfying Dirac eqn} \}$$

$$\oint z^n F(u_2 + z\hat{u}_2) dz$$

doesn't make sense,
as we have \sqrt{dz} not dz .

(Doesn't give us 0 on
a quantity extending
across ∞)

But:

$$\sqrt{dz} \frac{\partial}{\partial V_\alpha}$$

extends across ∞

\sqrt{dz} pole

$\frac{\partial}{\partial V}$ zero

$$\oint \sqrt{dz} z^n \sqrt{dz} \left(\frac{\partial}{\partial v_i} F \right) (u_1 + z\hat{u}_1, u_2 + z\hat{u}_2) = \gamma \alpha$$

picks up
z

$O(-3)$

we have an expression like

$$F(v_i) (dz)^{3/2}$$

We need to remove a
 dz by multiplying by
something with no poles.

$\frac{1}{(dz)^{1/2}}$ has a zero at $z=0$

2 possibilities

$$\frac{1}{(dz)^{1/2}} \quad \frac{z}{(dz)^{1/2}}$$

$\frac{1}{(dz)^{1/2}}$ has a zero at $z = \infty$

2 possibilities

$\frac{1}{(dz)^{1/2}}$ $\frac{z}{(dz)^{1/2}}$

\rightarrow SL_2 acts by conformal transformations on z -plane, these tensors are a doublet

$$\Psi_1 = \oint F(u+za_i) dz$$

$$\Psi_2 = \oint z F(u+za_i) dz$$

both expressions vanish on tensors that extend to $z = \infty$