Title: Mathematical Physics Lecture (230412)

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Collection: Mathematical Physics - Elective (2022/2023)

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## In General

$$H^1(\mathbb{PT}, \mathcal{O}(2sh-2))$$

$= \text{Sol}^{n}\text{s}$ of free field $\text{eq}^{n}\text{s}$ for a field of spin $s = 0, \tfrac{1}{2}, 1, \ldots$

helicity $h = \pm 1$

Recall,

$$\mathrm{Spin}(4,\mathbb{C}) = SL_2^+ \times SL_2^-$$

There <u>are</u> 2 spin reps.

$S_+, S_-$ fundamental reps of $SL_2^+$ and $SL_2^-$

$\dim_{\mathbb{C}} S_\pm$ -

$SL_2^+ \times SL_2^-$

spin reps.

fundamental

and $SL_2^-$

$\dim_{\mathbb{C}} S_\pm = 2.$

$\psi^-_\alpha \in S_- \qquad \alpha = 1, 2$

$\psi^+_{\dot\alpha} \in S_+ \qquad \dot\alpha = 1, 2$

are a basis.

$\dim_{\mathbb{C}} S_{\pm} = 2.$

$-$ve helicity $\psi^{-}_{\alpha} \in S_{-}$ $\quad \alpha = 1,2$

$+$ve helicity $\psi^{+}_{\dot\alpha} \in S_{+}$ $\quad \dot\alpha = 1,2$

are a basis.

Spinor field is in $S_{+} \oplus S_{-}$ (4 components)

$\mathbb{Z}/2 \subseteq \mathrm{Spin}(4)$

In $SL_2^{+} \times SL_2^{-}$, $\mathbb{Z}/2$ is $(-\mathrm{Id}, -\mathrm{Id})$ $\quad SO(4) =$

## What is helicity for photons?

Sol$^{n}$s of Maxwell eq$^{n}$

$$= \left\{ F \in \Omega^2(\mathbb{R}^4) \text{ satisfying } \left\{ \begin{array}{l} dF = 0 \\ d{\star}F = 0 \end{array} \right. \right\}$$

$$\star : \Omega^2(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4).$$

The +ve helicity fields are

$$F_+ = \tfrac{1}{2}(1+\star)F$$

-ve helicity are

$$F_- = \tfrac{1}{2}(1-\star)F$$

In components:

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \varepsilon_{\alpha\beta} F^{+}_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}} F^{-}_{\alpha\beta}$$

The Maxwell eq$^n$ is

$$\frac{\partial F^{+}_{\dot{\alpha}\dot{\beta}}}{\partial x_{\alpha\dot{\alpha}}} = 0 \qquad \frac{\partial F^{-}_{\alpha\beta}}{\partial x_{\alpha\dot{\alpha}}} = 0$$

Dira

$$\frac{\partial}{\partial x_{\alpha\dot{\alpha}}}$$

$$\frac{\partial}{\partial x_{\alpha\dot{\alpha}}}$$

$$\Gamma^{\alpha\dot{\alpha}}_{\mu} \frac{\partial}{\partial x_{\mu}}$$

Dirac eqn

$$\frac{\partial \psi_\alpha}{\partial x_{\alpha\dot\alpha}} = 0$$

$$\frac{\partial \tilde\psi_{\dot\alpha}}{\partial x_{\alpha\dot\alpha}} = 0$$

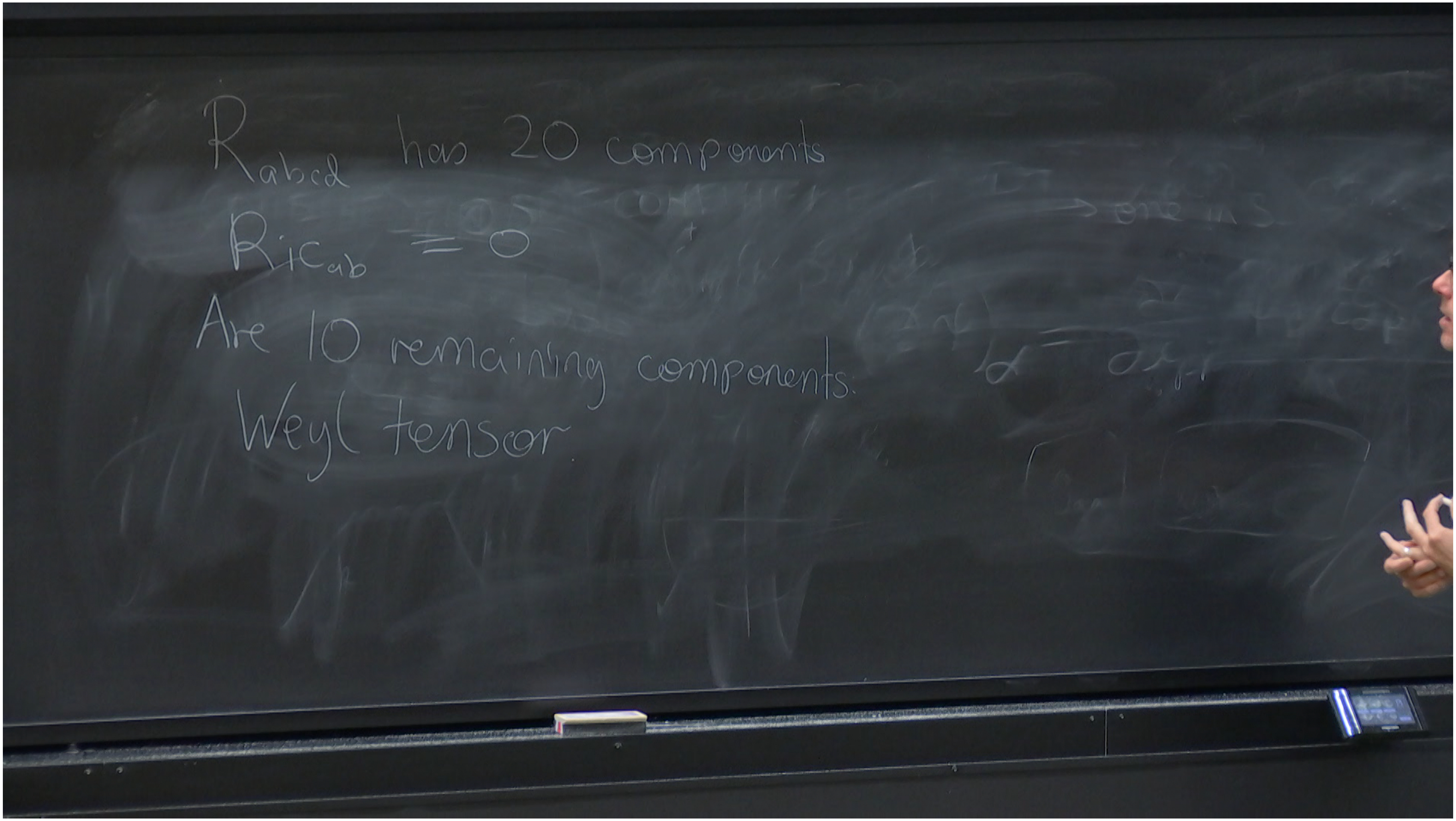
"

$$\Gamma_\mu^{\alpha\dot\alpha} \frac{\partial}{\partial x_\mu}$$

$$\langle x_{\alpha\dot\alpha}, x_{\beta\dot\beta} \rangle = \varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}$$

$$x_{1\dot\alpha} = u_{\dot\alpha}$$

$$x_{2\dot\alpha} = \hat{u}_{\dot\alpha}$$


$R_{abcd}$ has 20 components
$Ric_{ab} = 0$
Are 10 remaining components:
Weyl tensor

## Field of Spin 2

Flat metric

$g_{\mu\nu}$ perturb to

$g + \delta g$ to fist order,

so the Einstein eq$^n$ holds

What is the

Diff invariant info

(Really diffeos that

$Id$ + something sm

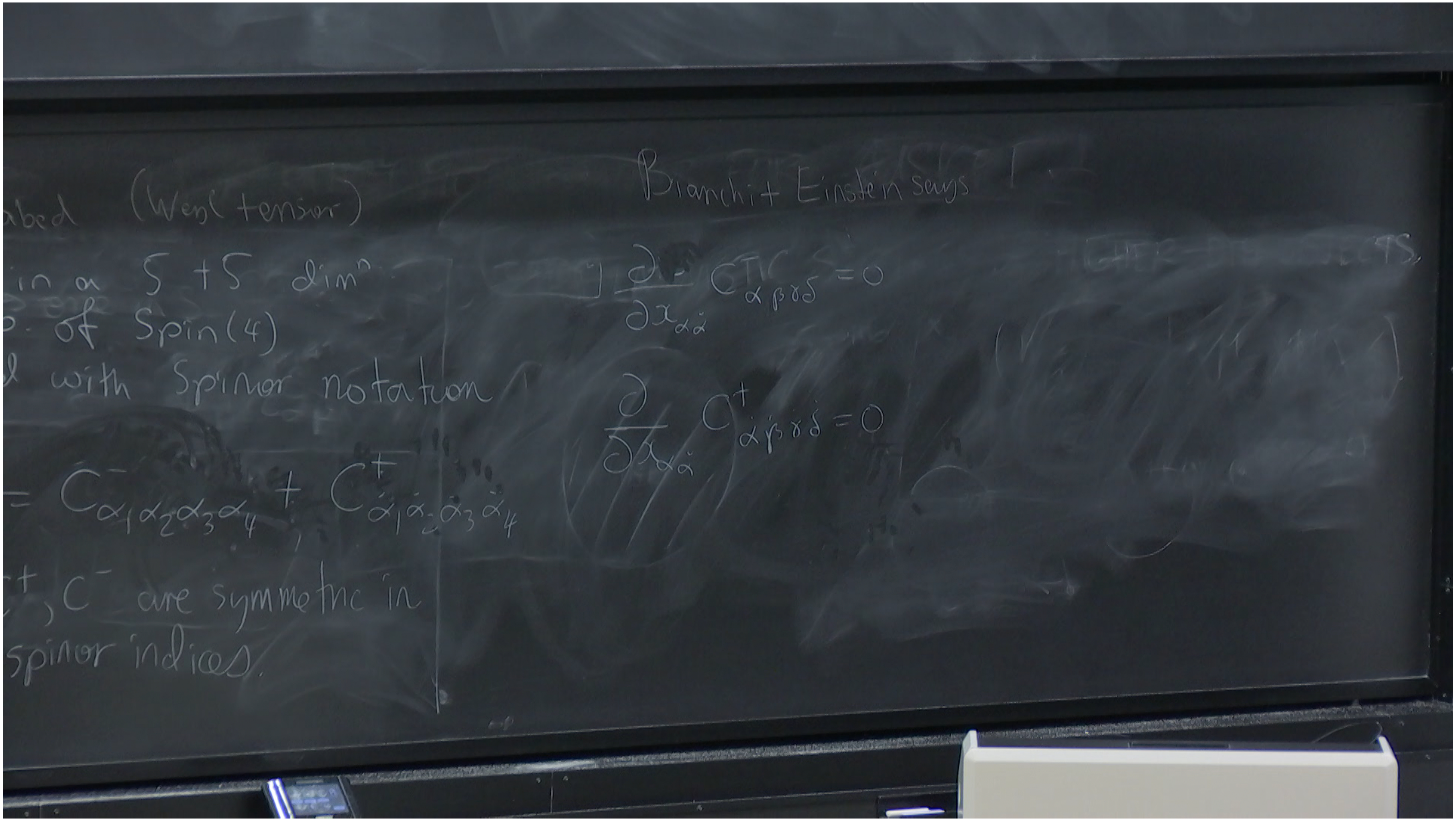
Answer:

Curvature of $g+\delta$

What is the
Diff invariant information?
(Really, diffeos that are
Id + something small)

Answer:

Curvature of $g + \delta g$


abcd (Weyl tensor)
in a 5 + 5 dim$^n$
of Spin(4)
with Spinor notation
$= C^-_{\alpha_1\alpha_2\alpha_3\alpha_4} + C^+_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}$
$C^+, C^-$ are symmetric in
spinor indices.
Bianchi + Einstein says
$\frac{\partial}{\partial x_{\alpha\dot\alpha}} C^-_{\alpha\beta\gamma\delta} = 0$
$\frac{\partial}{\partial x_{\alpha\dot\alpha}} C^+_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = 0$

Let's consider

$S = 1$

when $\mathcal{O}(-1), \mathcal{O}(-3) \rightsquigarrow$ spinors

$$H^1(\mathbb{PT}, \mathcal{O}(-1)) \simeq \{ \psi_{\dot\alpha}, \text{ satisfying Dirac eq}^n \}$$

$$\oint z^n F(u_\alpha + z\hat{u}_\alpha)\, dz$$

doesn't make sense,
as we have $\sqrt{dz}$ not $dz$.
(Doesn't give us 0 on a quantity extending across $\infty$)

But:

$$\sqrt{dz}\,\frac{\partial}{\partial V_\alpha}$$

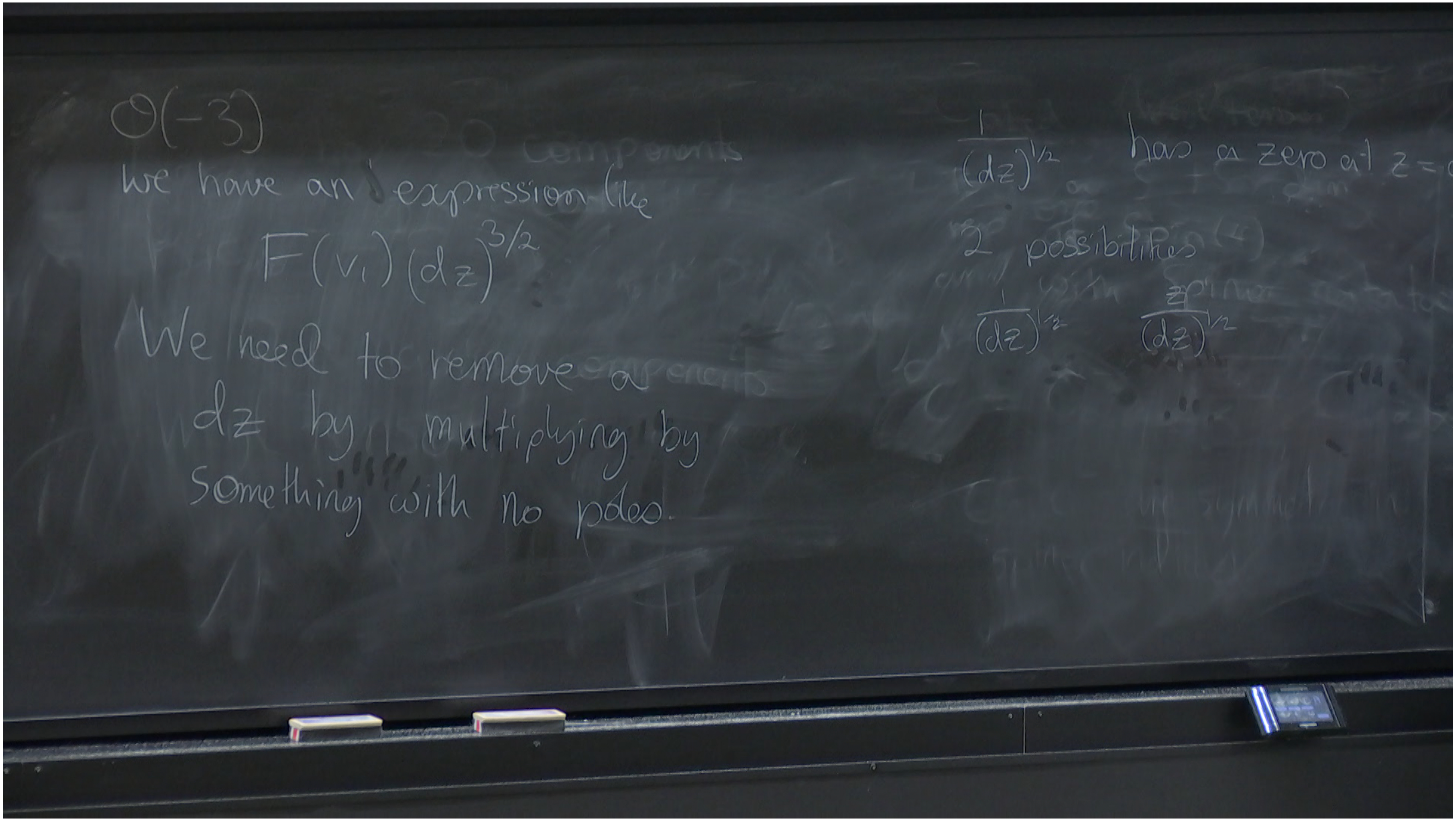
extends across $\infty$

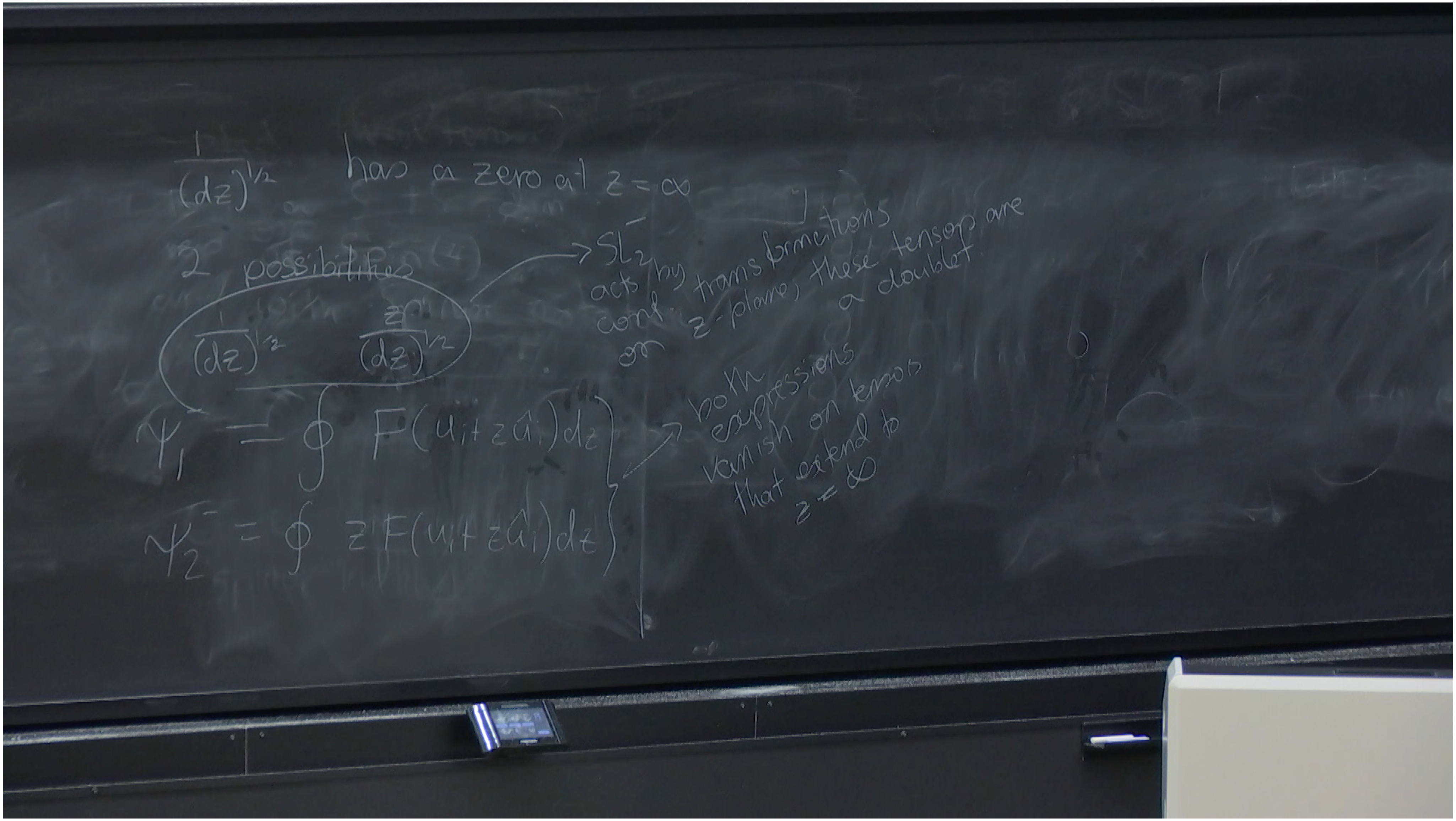
$\sqrt{dz}$ pole

$\frac{\partial}{\partial V}$ zero

$$\oint \sqrt{d\lambda z}\, z^n \sqrt{d\lambda z} \left(\frac{\partial}{\partial v_{\dot\alpha}} F\right)(u_1 + z\hat{u}_1, u_2 + z\hat{u}_2) = \tilde{\psi}^{\dot\alpha}$$

picks up $z$


O(-3)
we have an expression like
F(v1)(dz)3/2
We need to remove a
dz by multiplying by
something with no poles.
1/(dz)1/2 has a zero at z=
2 possibilities
1/(dz)1/2
z/(dz)1/2


1/(dz)^1/2 has a zero at z = ∞
2 possibilities
1/(dz)^1/2 z/(dz)^1/2
SL2 acts by conf. transformations on z-plane, these tensors are a doublet.
Ψ1 = ∮ F(ui + z ûi) dz
Ψ2 = ∮ z F(ui + z ûi) dz
both expressions vanish on tensors that extend to z = ∞