Title: Mathematical Physics Lecture (230405)

Speakers: Kevin Costello

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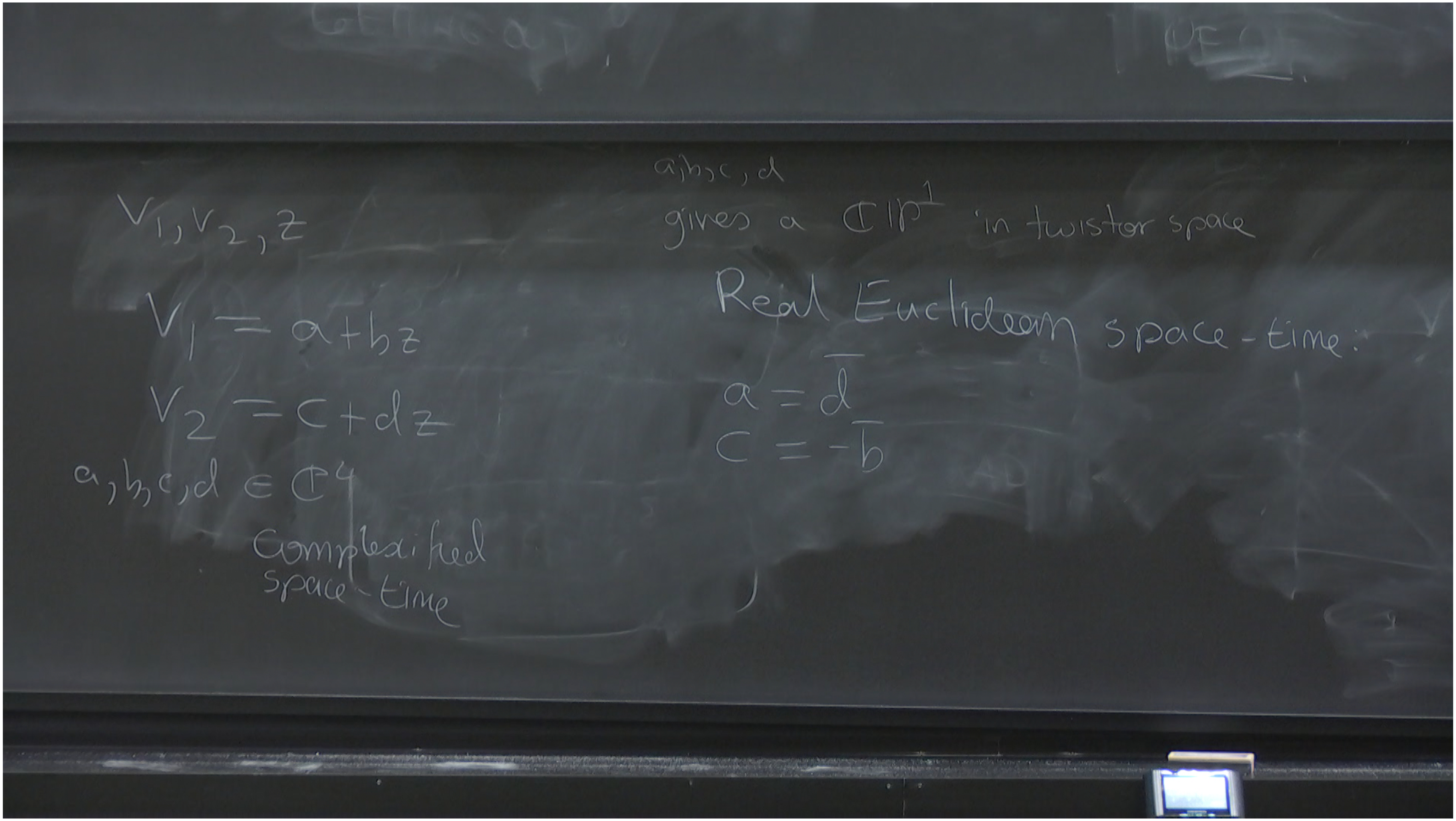

v_1, v_2, z
$v_1 = a + bz$
$v_2 = c + dz$
$a, b, c, d \in \mathbb{C}^4$
Complexified space-time
a, b, c, d
gives a $\mathbb{CP}^1$ in twistor space
Real Euclidean space-time:
$a = \bar{d}$
$c = -\bar{b}$

Proposition

There is a diffeo of real manifolds

$$\mathbb{R}^4 \times \mathbb{CP}^1 \longrightarrow \mathbb{PT}$$

$$(a, b, z) \longmapsto (a + bz, -\bar{b} + \bar{a}z, z)$$

Content:
Every (v_1, v_2, z) is on a unique twistor line for a point in Euclidean $\mathbb{R}^4$

(Twistor line $= \mathbb{CP}^1$ associated to $(a,b,c,d) \in \mathbb{C}^4$)

Uniqueness: If (v_1, v_2, z) was on 2 Euclidean twistor lines, they would be null separated.

Existence

Given v_1, v_2, z

Can we find a, b solving the eqns

$$v_1 = a + bz$$

$$\bar{v}_2 = a\bar{z} - b$$

This happen is if

$$\det\begin{pmatrix} 1 & z \\ \bar{z} & -1 \end{pmatrix} \neq 0$$

$$\det(\ \ ) = -1 - z\bar{z} < 0$$

If we fix $z \in \mathbb{CP}^1$

then v_1, v_2 are coords on $\mathbb{C}^2$

this map gives an isomorphism

$$\mathbb{R}^4 \xrightarrow{\simeq} \mathbb{C}^2$$

by expressing v_1, v_2 in terms of a, b

Each z gives a complex structure on $\mathbb{R}^4$

holomorphic functions are

$$V_1 = a + bz$$

$$V_2 = -\bar{b} + \bar{a}z$$

$z=0$ $(a, \bar{b})$ holomorphic

$z=\infty$

$b, \bar{a}$ are holomorphic

$$\{\text{Complex structures on } \mathbb{R}^4\}$$

$$\cong SO(4)/U(2)$$

$$\cong S^2$$

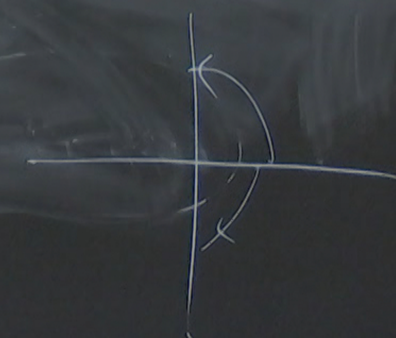
Twistor space

$$= \{ x \in \mathbb{R}^4 , \text{ and a complex structure on } T_x \mathbb{R}^4 \}$$

Complex str. on V, a real v. space

is a map $J \in SO(V)$

$$J^2 = -1$$



So the set of complex structures is

$SO(4)/U(2)$

Minkowski space in 2d:

Euclidean sig, $J(dx_1 + i dx_2) = i(dx_1 + i dx_2)$

dz

If we Wick rotate, $x_2 \rightsquigarrow it$,

$dz, d\bar{z} \Rightarrow dx_+, dx_-$ null and real.

Penrose Transform

Dolbeault / Čech cohomology

of twistor space

$= \text{sol}^{\text{ns}}$ of field eq^{ns} on $\mathbb{R}^4$

Dolbeault Cohomology

If X is a complex manifold,

$$\Omega^{0,q}(X) = \left\{ \text{tensors which look locally like } \sum f_{i_1 \ldots i_q} \, d\bar{z}_{i_1} \wedge \ldots \wedge d\bar{z}_{i_q} \right\}$$

$\Omega^{0,q}(X) \subseteq \Omega^{q}(X)$ space of q-forms

(with $\mathbb{C}$ coefficients)

where in a local patch, we only have

$d\bar{z}_i$.

There's a map

$$\bar{\partial} : \Omega^{0,q}(X) \longrightarrow \Omega^{0,q+1}(X)$$

given by

$$\bar{\partial}\left(\sum f_I \, d\bar{z}_I\right)$$

$$= \sum \frac{\partial f_I}{\partial \bar{z}_i} \, d\bar{z}_i \wedge d\bar{z}_I$$

Basically same as de Rham operator.

The composition

$$\Omega^{0,q}(X) \xrightarrow{\bar\partial} \Omega^{0,q+1}(X) \xrightarrow{\bar\partial} \Omega^{0,q+2}(X)$$

is zero.

The Dolbeault cohomology

$$H^{q}_{\bar\partial}(X) = \frac{\operatorname{Ker} \bar\partial : \Omega^{0,q} \to \Omega^{0,q+1}}{\operatorname{Im} \bar\partial : \Omega^{0,q-1} \to \Omega^{0,q}}$$

Examples

f is holomorphic

On $\mathbb{C}^n$, $f \in \mathcal{H}^0_{\bar\partial}(\mathbb{C}^n)$

is a function $f: \mathbb{C}^n \to \mathbb{C}$,
such that

$$\sum d\bar{z}_i \frac{\partial f}{\partial \bar{z}_i} = 0$$

This means $\dfrac{\partial f}{\partial \bar{z}_i} = 0$, so

We can also make sense of

e.g. $\Omega^{0,q}(X, TX)$

are things locally like

$$f^{I}_{i}\, d\bar{z}_{I}\, \partial_{z_i}$$

or $\Omega^{0,q}(X, T^{*}X)$

locally looks like

$f^{I,i}\, d\bar{z}_I \wedge dz_i$

$\bar{\partial} : \Omega^{0,q}(X, TX) \longrightarrow \Omega^{0,q+1}(X, TX)$

is defined as before.

We can define

$H^q_{\bar{\partial}}(X, TX)$

exactly as before.

Theorem

1) On $\mathbb{C}^n$, $H^q(X, \text{tensor bundle}) = 0$ for $q>0$

$H^0(X, \text{'' \quad ''}) =$ holomorphic tensors.

2) If $X = U \cup V$, U, V are both copies of $\mathbb{C}^n$

$\mathbb{CP}^1 = U \cup V$

$U = \{z \neq \infty\}$

$V = \{z \neq 0\}$

$H^1_{\bar\partial}(\mathbb{CP}^1, T^*\mathbb{CP}^1)$?

$U \cap V = \mathbb{C} \setminus 0$

Hol. tensors here are

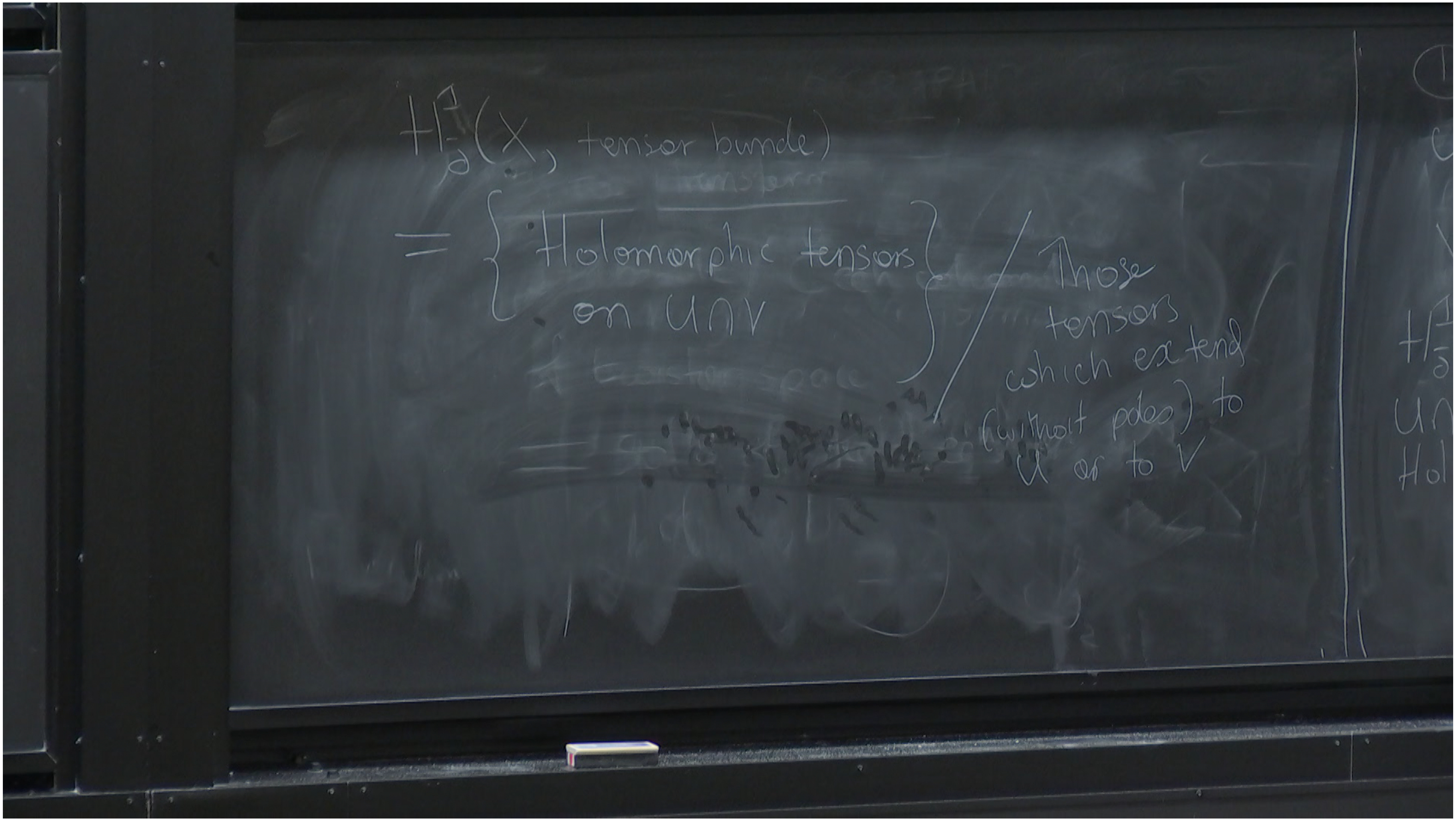
$z^n dz \quad n \in \mathbb{Z}$

This extends across 0 if $n \geq 0$

Extends across ∞ if $n \leq -2$

$H^1_{\bar\partial}(\mathbb{CP}^1, T^*\mathbb{CP}^1) = \mathbb{C}$

basis $z^{-1} dz$


$H^q_{\bar\partial}(X$, tensor bundle)
$= \{$ Holomorphic tensors on $U \cap V \}$ / Those tensors which extend (without poles) to U or to V