Title: Mathematical Physics Lecture (230403)

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Collection: Mathematical Physics - Elective (2022/2023)

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TWISTOR THEORY

Complex Geometry

$\longleftrightarrow$ PDEs such as

Einstein equation

YM equation

...

Part 1 Complex Manifolds

If $U \subseteq \mathbb{C}^n$ is open

$f: U \longrightarrow \mathbb{C}$

f is holomorphic if it satisfies the CR equation $\frac{\partial f}{\partial \bar{z}_i} = 0$ for $i = 1, \ldots, n$

If $x_i = \mathrm{Re}\, z_i$

$y_i = \mathrm{Im}\, z_i$

then the CR eqn is

$$\frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j} = 0$$

$$\frac{\partial \mathrm{Re} f}{\partial x_i} - \frac{\partial \mathrm{Im} f}{\partial y_i} = 0$$

$$\frac{\partial \mathrm{Im} f}{\partial x_i} + \frac{\partial \mathrm{Re} f}{\partial y_i} = 0$$

Def^n

A complex manifold of (complex) dimension n, is a manifold of real dim^n $2n$, with local coords

$z_1, \ldots, z_n : U \longrightarrow \mathbb{C}$

($U \subseteq M$ open)

Such that if U, V are coordinate patches,

z_i coords on U

w_i " " V

then $w_i(z_1, \dots, z_n)$

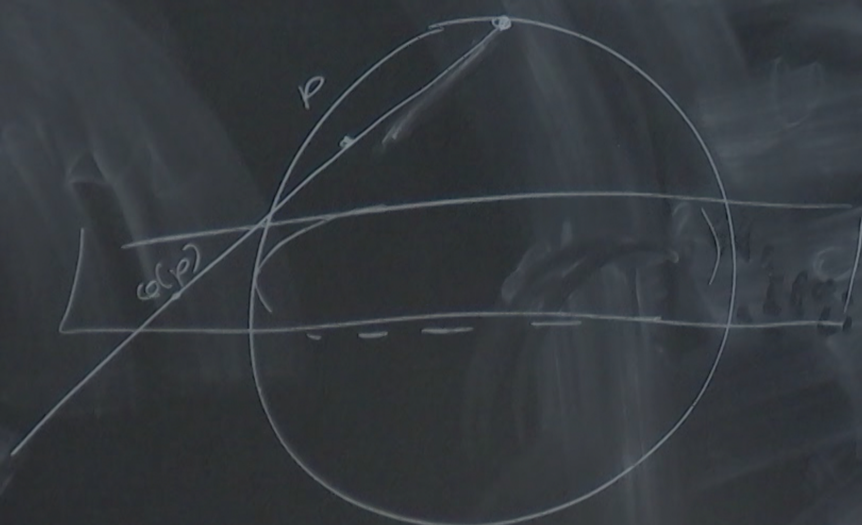
is a hol. function of the z_i

Examples

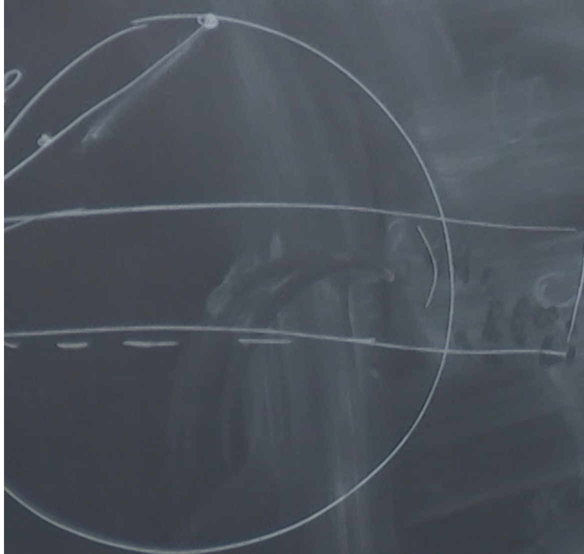
$\mathbb{C}P^1 = S^2 =$ Riemann sphere

$S^2 \setminus N = \mathbb{R}^2 = \mathbb{C}$

by stereographic projection.



$\mathbb{CP}^1 = S^2 =$ Riemann sphere



$S^2 \setminus N = \mathbb{R}^2 = \mathbb{C}$

by stereographic projection.

$p \longrightarrow \varphi(p) =$ point on plane on the line through p and N

$S^2 \setminus N = \mathbb{C}$

with coordinate z

In the same way,

$S^2 \setminus S = \mathbb{C}$ with coord w.

south pole

$U = S^2 \setminus N$

$V = S^2 \setminus S$

$U \cap V$ we have coords w and z.

You can check $w = 1/z$

This is holomorphic, so

$S^2 = \mathbb{CP}^1$ is a complex manifold

$\mathbb{C}_z \setminus 0 \simeq \mathbb{C}_w \setminus 0$

$w = 1/z$

$$\mathbb{C}P^n = \left\{ (z_0 : \dots : z_n) \right\} \Big/ (z_0 : \dots : z_n) \sim (\lambda z_0 \dots \lambda z_n)$$

not all z_i are 0

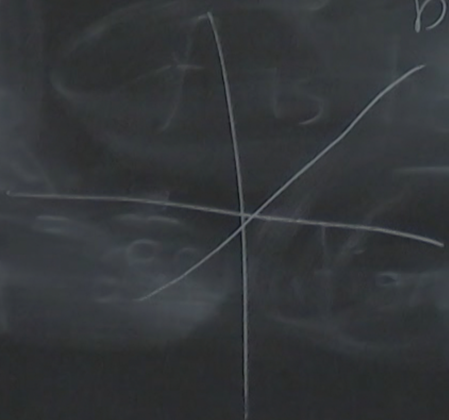
for $\lambda \in \mathbb{C} \setminus 0$

$$= S^{2n-1} / S^1$$

$\mathbb{CP}^1 = \{$ lines through 0 in $\mathbb{C}^2 \}$

$\cup|$

U = patch where it can be represented by $(z : 1)$



V patch where rep. by $(1 : \omega)$

$(z : 1) \sim (1 : \frac{1}{z})$

Coord transformation is $\omega = \frac{1}{z}$

gh 0 in $\mathbb{C}^2$ }

epresented

rep. by $(1 : \omega)$

$$S^3 = \{ (z,w) \mid z\bar{z} + w\bar{w} = 1 \}$$

$$\mathbb{CP}^1 = \{ (z,w) \in S^3 \} / (z,w) \sim (\lambda z, \lambda w)$$

$$\lambda \bar{\lambda} = 1$$

manifold

patches

ds V_1, V_2, z

$\tilde{V}_1, \tilde{V}_2, \tilde{z}$

The intersection

$U \cap V$ is the locus $z \neq 0$

$\tilde{z} \neq 0$

Coords are identified by

$\tilde{z} = 1/z$

$\tilde{V}_i = V_i / z$

$z \neq 0$

$\bar{z} \neq 0$

ed by

Tensor fields on complex manifolds M

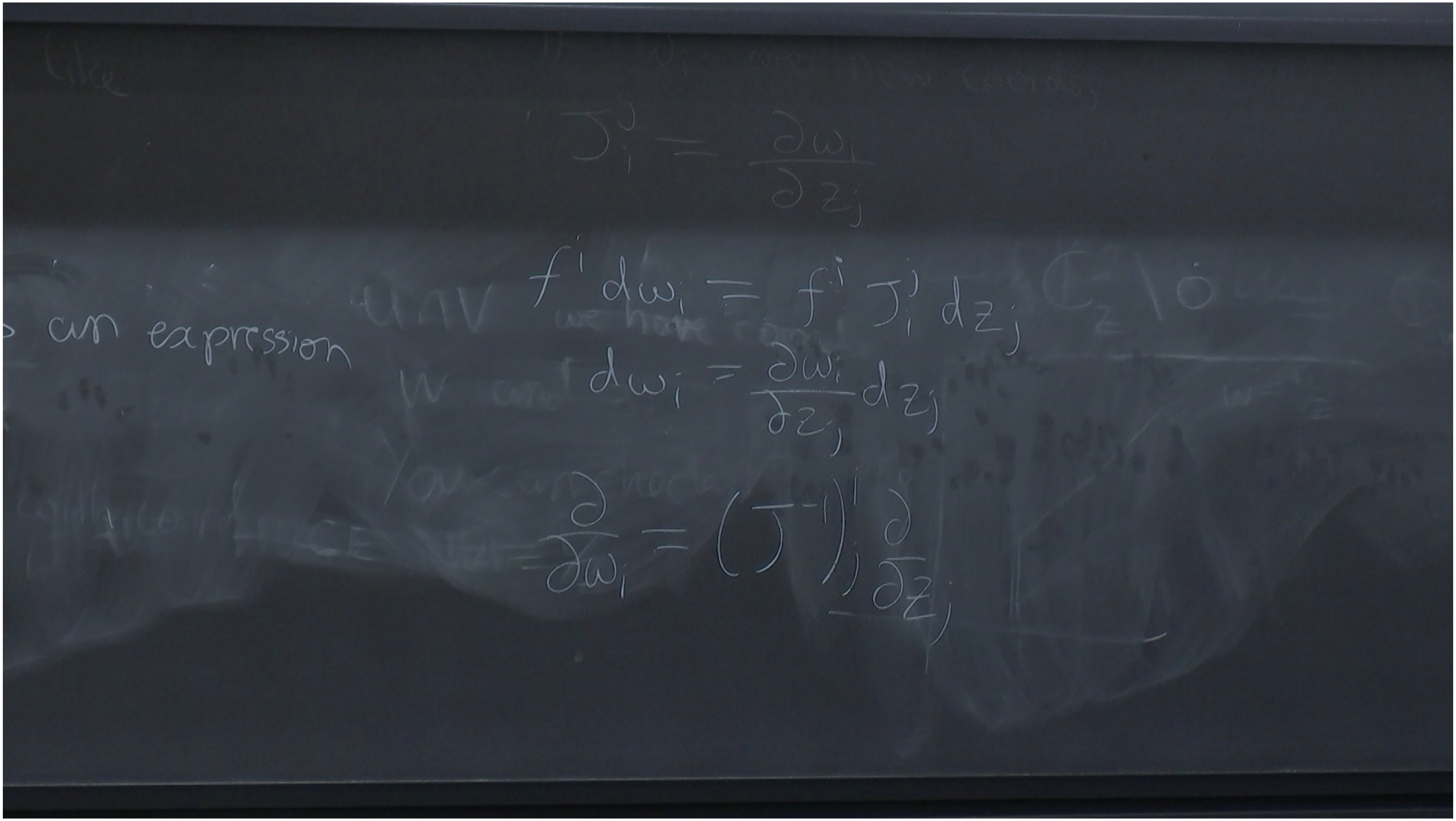
If z_i are local coords. on a complex manifold

Analog of a 1 form is a (1,0) form:

is an expression like

$f^i \, dz_i$

A (1,0) vector is an expression like $f_i \dfrac{\partial}{\partial z_i}$


like
$J_i^j = \frac{\partial \omega_i}{\partial z_j}$
an expression
$f^i d\omega_i = f^i J_i^j dz_j$
$d\omega_i = \frac{\partial \omega_i}{\partial z_j} dz_j$
$\frac{\partial}{\partial \omega_i} = (J^{-1})_j^i \frac{\partial}{\partial z_j}$

Example

$\mathbb{CP}^1$

z, w coords on patches

$w = \frac{1}{z}$

$dw = -z^{-2}\, dz \qquad \frac{\partial}{\partial z} = -w^2 \frac{\partial}{\partial w}$

$dz = -w^{-2}\, dw$

$\frac{\partial}{\partial w} = -z^2 \frac{\partial}{\partial z}$

Example

$\mathbb{CP}^1$

z, ω coords on patches

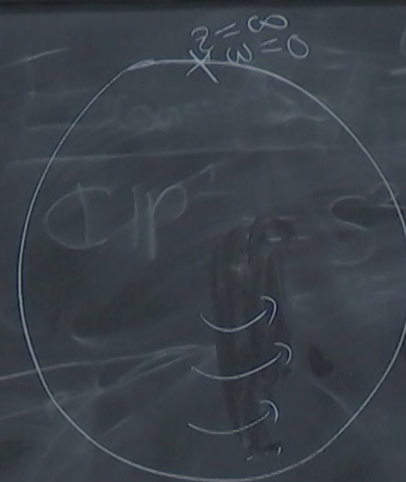
$$\omega = \frac{1}{z}$$

$$d\omega = -z^{-2}\,dz \qquad \frac{\partial}{\partial z} = -\omega^2 \frac{\partial}{\partial \omega}$$

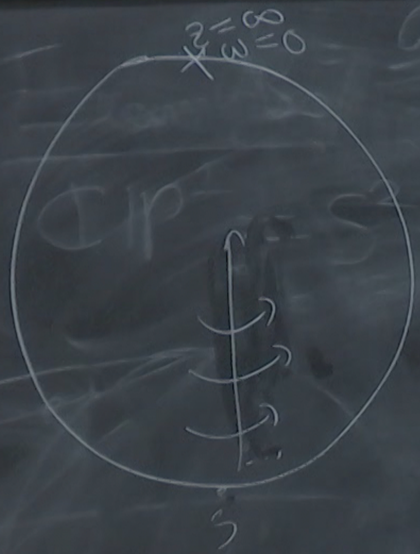
$$dz = -\omega^{-2}\,d\omega$$

$$\frac{\partial}{\partial \omega} = -z^2 \frac{\partial}{\partial z} \qquad z\partial_z = -\omega\partial_\omega$$

$\mathbb{CP}^2$



$$\frac{\partial}{\partial z}$$



$\frac{\partial}{\partial z}$

$\mathrm{Re}\, z\,\partial_z = r\partial_r$

$\mathrm{Im}\, z\,\partial_z = \partial_\theta$

A tensor field like

$f^i dz_i$ is holomorphic

if in each patch, f^i are

holomorphic.

Qn Are there holomorphic $(1,0)$ forms on $\mathbb{CP}^1$?

On z patch, it would be

$$f(z)\,dz$$

ω patch: $g(\omega)\,d\omega$

These must agree

$$f(z) = -z^2 g(1/z)$$

$z \to \infty$,

$f(z) \to 0$ (like z^{-2})

as $g(1/z)$ is regular at $z = \infty$

Qn Are there holomorphic $(1,0)$ forms on $\mathbb{CP}^1$?

On z patch, it would be

$$f(z)\,dz$$

w patch: $g(w)\,dw$

Maximum Principle

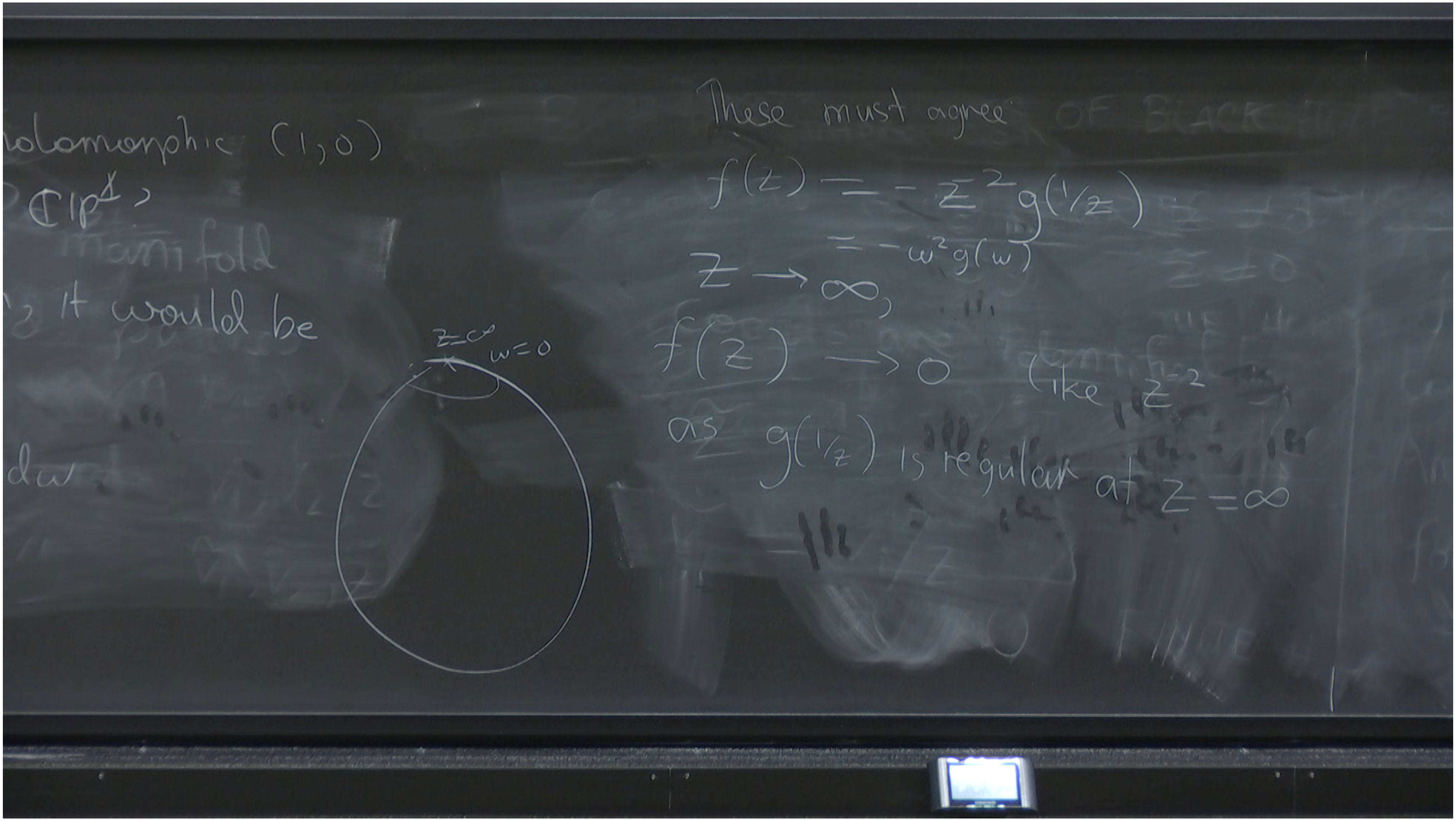
f hol. is bounded on $\mathbb{C}$

then it is constant.

f is bounded

and $\to 0$ at $z = \infty$

so $f \equiv 0$


olomorphic (1,0)
$\mathbb{CP}^1$,
manifold
, it would be
dw
$z=\infty$
$w=0$
These must agree
$f(z) = -z^2 g(1/z)$
$= -w^2 g(w)$
$z \to \infty$,
$f(z) \to 0$ (like z^{-2}
as $g(1/z)$ is regular at $z=\infty$

Example.

What about vectors?

$$\{ f(z)\,\partial_z \,,\; g(w)\,\partial_w \}$$

define a hol. vector field on $\mathbb{CP}^1$.

$$f(z) = -z^2 g(1/z) = -\frac{1}{w^2} g(w)$$

$f(z)$

$f(z)$ can have a pole of order 2 at ∞

$f(z)\partial_z$ must be

$$a\partial_z + bz\partial_z + cz^2\partial_z$$

tors?

or field on

$(1/z)$

Example: $g(w)\partial_w = -w^2 f(1/w)\partial_w$

What about vectors?

$$\{f(z)\partial_z\,,\ g(w)\partial_w\}$$

define a hol vector field on $\mathbb{C}\mathbb{P}^1$.

$$f(z) = -z^2 g(1/z)$$
$$= -\frac{1}{w^2} g(w)$$

$f(z)$ can ha

of order 2 o

$f(z)\partial_z$ must

$a\partial_z + b z$

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Fact

The holomorphic vector fields on $\mathbb{PT}$ = The conformal vector fields on $\mathbb{R}^4$, with complex coefficients.

4 translations

6 rotations

1 dilation

4 special conformal transformations

Rules	Pole/zero at $z=\infty$
∂_z	zero order 2 at $z=\infty$
v_i	pole order 1
∂_{v_i}	zero of order 1
z	pole of order 1

$F\partial_z + G_i\partial_{v_i}$ which have no poles at $z=\infty$