Title: Quantum Gravity Lecture (230424)

Speakers: Aldo Riello

Collection: Quantum Gravity (2022/2023)

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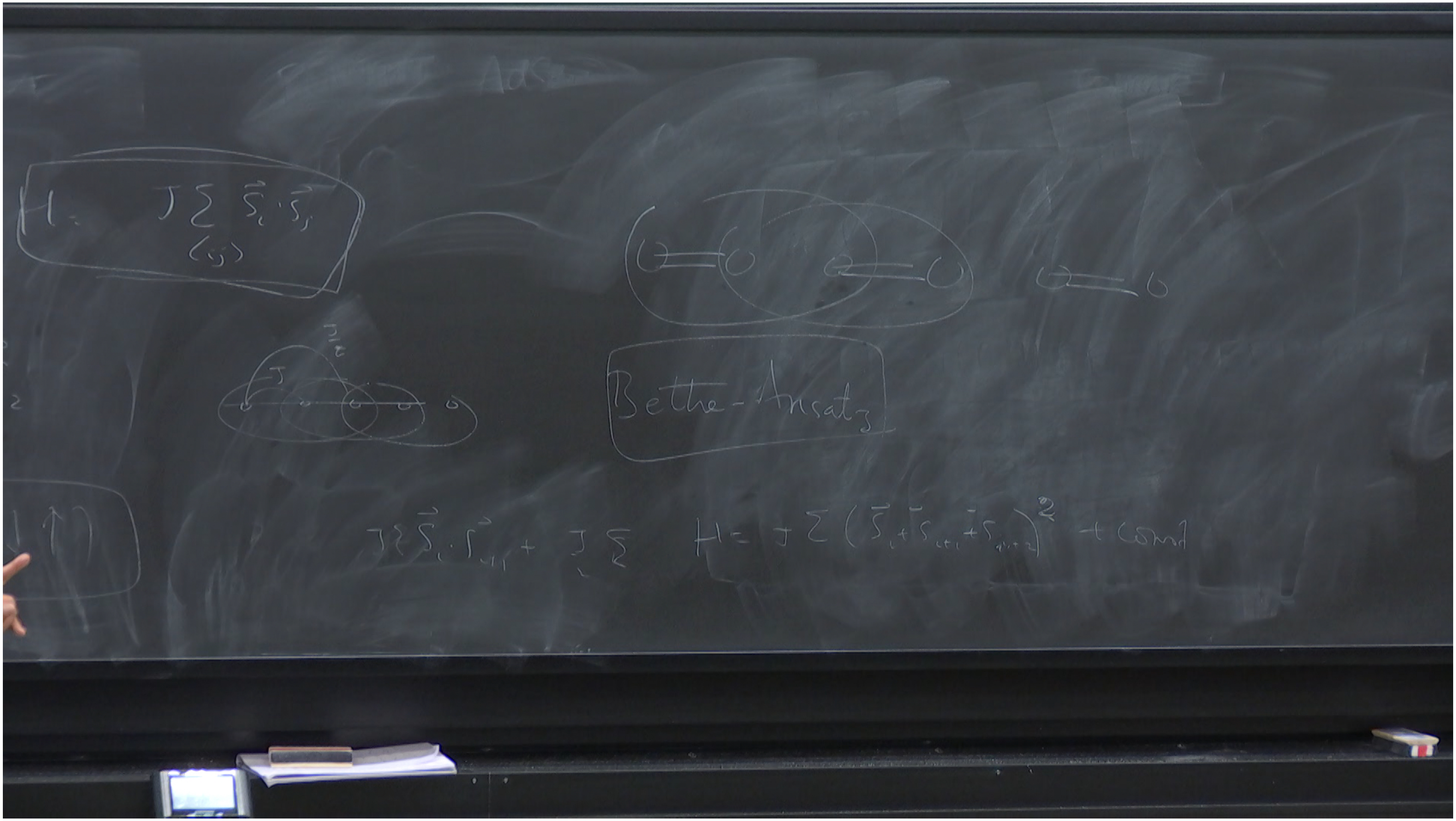
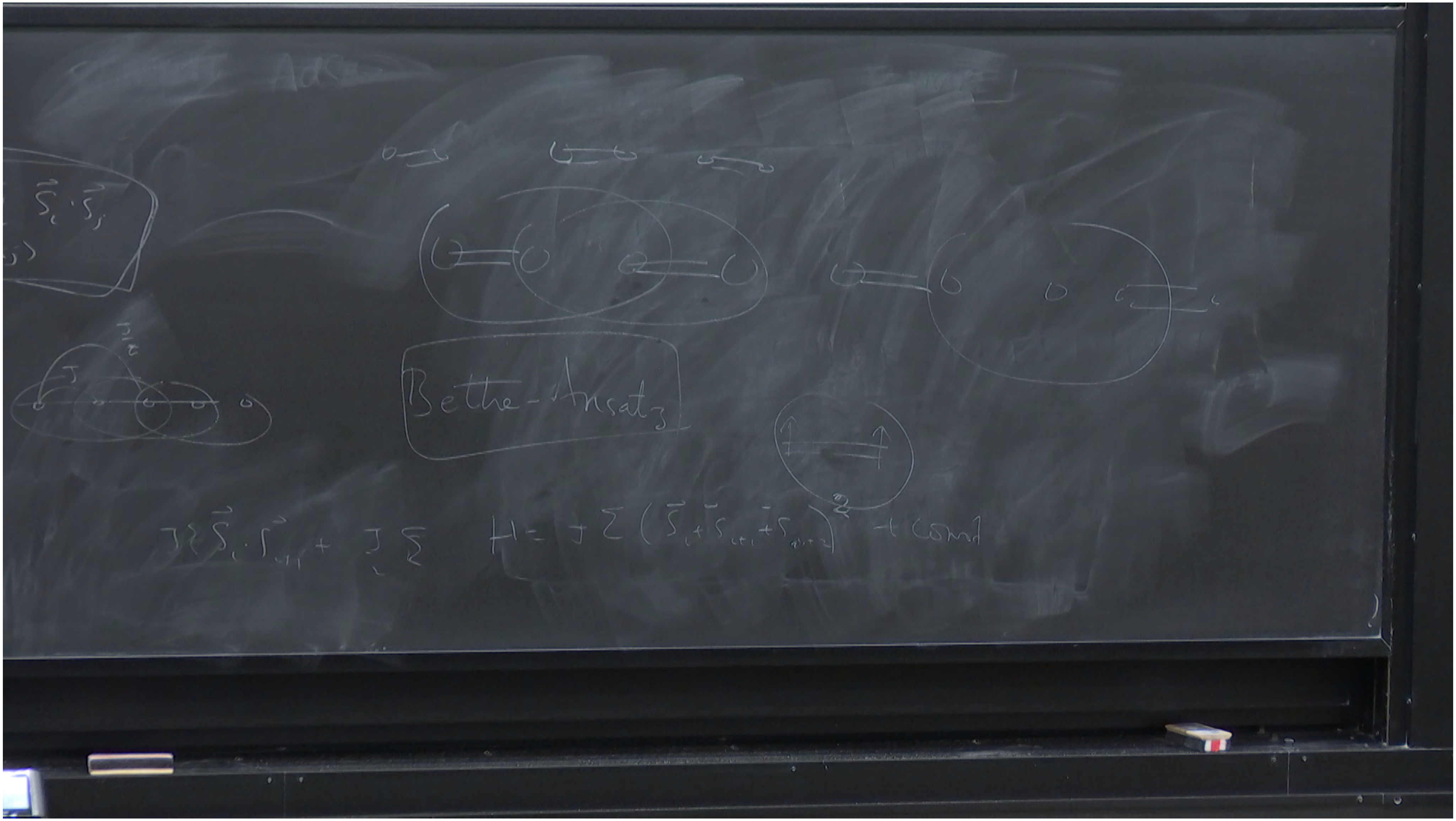


$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$
Bethe-Ansatz
$J \sum \vec{S}_i \cdot \vec{S}_{i+1} + J \sum_i$
$H = J \sum (\vec{S}_i + \vec{S}_{i+1} + \vec{S}_{i+2})^2 + \mathrm{const}$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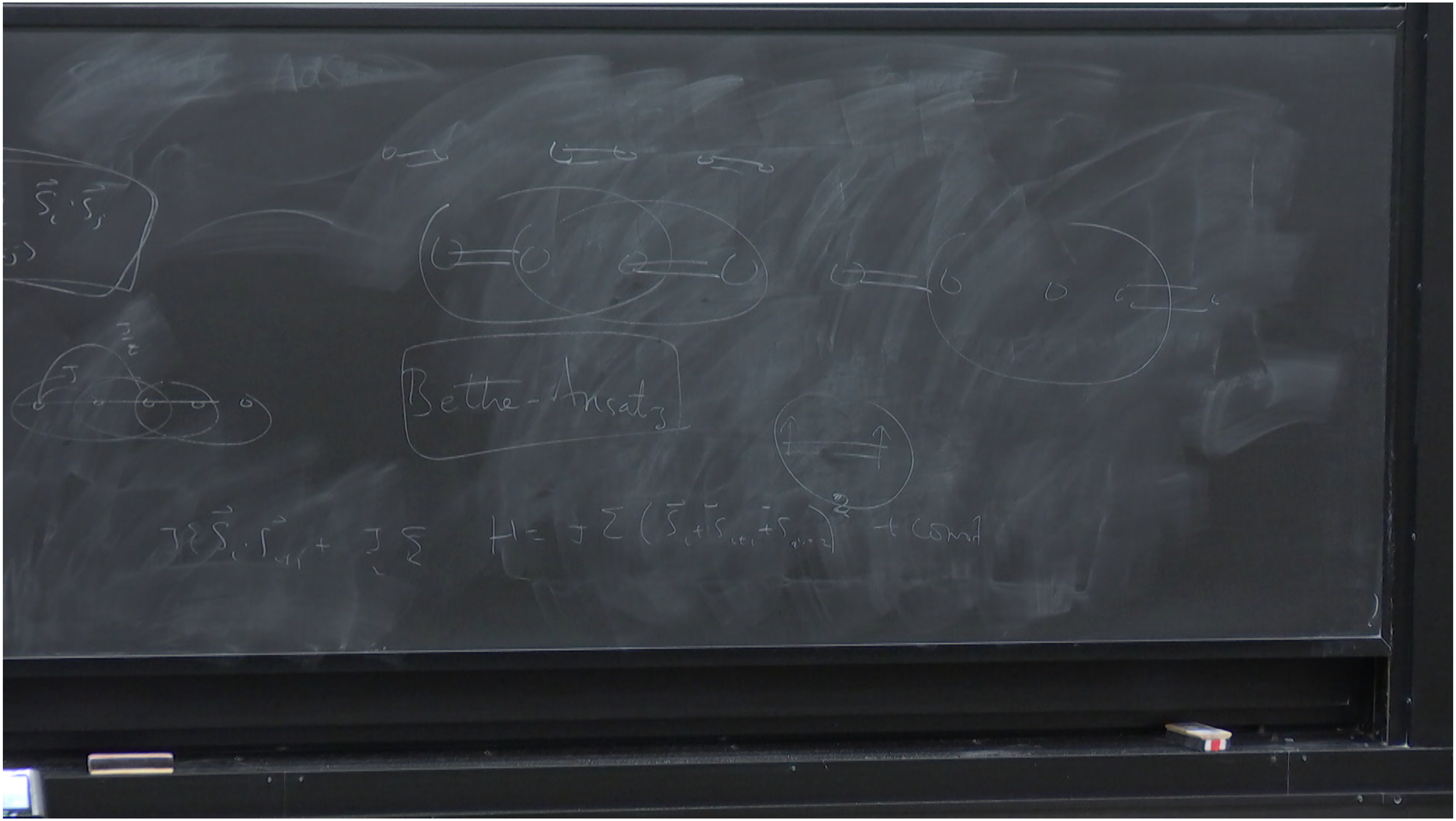
Bethe-Ansatz

RECAP [Lee-Wald 1991]

Noether 1: (g,ρ) Lagrangian sym $\left(\mathcal{L}_{\rho(\xi)} \underline{L} = d\underline{R}(\xi)\right)$

$$d\underline{J}(\xi) = -\underline{E}_I \, \delta_\xi \varphi^I \approx 0$$

Generator:

$$\mathbb{I}_{\rho(\xi)} \underline{\Omega} = -d\underline{J}(\xi) + \underline{e}(\xi) + d\underline{s}(\xi)$$

$$\underline{e}(\xi) \approx 0$$

$\int_\Sigma$

$$\mathring{\mathbb{I}}_{\rho(\xi)} \Omega_\Sigma = -\mathbb{d} Q_\Sigma(\xi) + \int_\Sigma \underline{e}(\xi) + \int_{\partial\Sigma} s(\xi)$$

① $\partial\Sigma = \phi$, Σ Cauchy: $\mathring{\mathbb{I}}_{\rho(\xi)} \Omega_\Sigma \approx -\mathbb{d} Q_\Sigma(\xi)$

② Possible complications at $\partial\Sigma$.

$\Longrightarrow$ (

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$$\mathring{I}_{\rho(\xi)} \Omega_\Sigma = -\mathbb{d} Q_\Sigma(\xi) + \int_\Sigma \underline{e}(\xi) + \int_{\partial\Sigma} s(\xi)$$

① $\partial\Sigma = \emptyset$, Σ Cauchy: $\mathring{I}_{\rho(\xi)} \Omega_\Sigma \approx -\mathbb{d} Q_\Sigma(\xi)$

② Possible complications of $\partial\Sigma$.

$\Rightarrow \left(\mathcal{F}, \iota_{EL}^{A} \Omega_\Sigma\right)$ always carry a Ham. generator for a Lagrangian sym.
$\partial\Sigma = \phi$

Noether 2 : (G, ρ) local Lagrangian sym
$\xi = \xi(x)$

& we write $\delta_\xi \varphi^I = \mathcal{D}^I_\alpha \xi^\alpha$

then : $\mathcal{D}^{\dagger I}{}_\alpha E_I = 0$ [offshell]

Ex: (N2)
Maxwell $\nabla^a \nabla^b F_{ab} = 0$
YM $(D^a D^b F_{ab})^\alpha = 0$ $\quad D := \nabla + [A, \cdot]$
GR $\nabla^a G_{ab} = 0$

(ξ)

a Ham. generator
ian sym.

agrangian sym

$\xi(x)$

[offshell]

0

$= 0$ $\quad D := \nabla + [A, \cdot]$

Consequence
less eom than fields to evolve
$\mapsto$ dynamics is underdetermined
$\rightarrow$ some data can be made evolve to diff conf.

$\Longrightarrow$ naive phase space too big!
Let's see this

a Ham. generator
ian sym.

agrangian sym

$\xi(x)$

[offshell]

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$= 0 \quad D := \nabla + [A, \cdot]$

Consequence

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$\mapsto$ dynamics is underdetermined

$\rightsquigarrow$ same data can be made evolve to diff conf.

$\Rightarrow$ naive phase space too big!

Let's see this

Lemma (ξ, ρ) local sym & $\partial\Sigma = \emptyset$

$$\Rightarrow Q_\Sigma(\xi) \approx 0$$

Pf: N1 + locality of ξ

$$0 \overset{\uparrow}{\underset{(N1)}{\approx}} \int_M dJ(\xi) = Q_{\Sigma_f}(\xi) - Q_{\Sigma_i}(\xi) \quad \forall \xi \in G$$

Choose ξ arbitrary in a neighborhood of Σ_{in}

& vanishing —— " —— Σ_{fin}

$\to Q_{\Sigma_{in}}(\xi) \approx 0$ $\boxtimes$

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Corollary $(\mathcal{F}, \iota^*_{EL}\Omega_\Sigma)$ is degenerate along the symmetries

i.e. $\rho(\xi) \in \ker(\iota^*_{EL}\Omega_\Sigma)$

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Corollary $(\mathcal{F}, \iota^*_{EL}\Omega_\Sigma)$ is degenerate along the symmetries

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To find a symplectic space need to quotient out the local symmetry.

i.e. identify sym-related config.

$$\delta_\xi \varphi^I = \mathcal{D}^I_\alpha \xi^\alpha \equiv A^I_\alpha \xi^\alpha + B^{I a}_\alpha \nabla_a \xi^\alpha$$

$$\underline{B}^I_\alpha = * \, g_{ab} B^{I a}_\alpha \, dx^b$$

Corollary: local sym, then

$$\underline{J}(\xi) = \boxed{E_I \underline{B}^I_\alpha} \xi^\alpha + dj(\xi)$$

$$\equiv \underline{C}_\alpha \approx 0$$

$$C_{\Sigma\alpha} = \iota^*_\Sigma \underline{C}_\alpha \Longleftarrow \boxed{\text{constraint}}$$

top form on $\Sigma \Leftrightarrow$ scalar on Σ: $C_{\Sigma\alpha} = E_I B^{I b}_\alpha n_b$

$$\delta_\xi \varphi^I = D^I_\alpha \xi^\alpha \equiv A^I_\alpha \xi^\alpha + B^{Ia}_\alpha \nabla_a \xi^\alpha$$

$$\underline{B}^I_\alpha = * g_{ab} B^{Ia}_\alpha dx^b$$

Corollary: $\underline{\text{local}}$ sym, then

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Corollary: local sym, then

$$\underline{J}(\xi) = \boxed{E_I \underline{B}^I_\alpha} \xi^\alpha + d\underline{j}(\xi) \quad , \quad \underline{j}(\xi) \in \Omega^{top-2}(M)$$

$$\equiv \underline{C}_\alpha \approx 0$$

$$\underline{C}_{\Sigma\alpha} = \iota^*_\Sigma \underline{C}_\alpha \Longleftarrow \boxed{\text{constraint}}$$

top form on $\Sigma \Leftrightarrow$ scalar on Σ: $C_{\Sigma\alpha} = E_I B^{Ib}_\alpha n_b$

Pf sketch

N1 $\underset{\overset{\shortparallel}{\nabla_a J^a(\xi)}}{dJ(\xi)} = E\,D\,\xi = \underbrace{D^{\dagger}E}_{\underset{N2}{=0}}\,\xi + \nabla_a(E\,B^a\xi)$

$d(\underline{J} - E\,\underline{B}\,\xi) = 0$ $\square$

Ex:

YM: $J^a(\xi) = \mathrm{tr}\left(F^{ab} D_b \xi\right)$

$= -\mathrm{tr}\left(\left(\underbrace{D_b F^{ab}}_{C_a}\right)\xi\right)$

$+ \nabla_b\, \mathrm{tr}\left(\underbrace{F^{ab}\,\xi}_{j^{ab}(\xi)}\right)$

$Q_\Sigma(\xi) = -\int_\Sigma \mathrm{tr}\left(\left(\underbrace{n_a D_b F^{ab}}_{\text{Gauss constraint}}\right)\xi\right)$

$+ \int_{\partial\Sigma} \frac{1}{2}\,\mathrm{tr}\left(F^{ab}\,\xi\right)\underline{\varepsilon}_{ab}$

Pf sketch

N1 $\underset{\text{N1}}{dJ(\xi)} = E\,D\xi = \underbrace{D^\dagger E}_{=0 \text{ N2}}\,\xi + \nabla_a(E\,B^a\xi)$

$\nabla_a J^a(\xi)$

$d(\underline{J} - E\underline{B}\,\xi) = 0$ ☐

$D_b\xi + [A_b, \xi]$

Ex:

YM: $J^a(\xi) = \mathrm{tr}\left(F^{ab} D_b \xi\right)$

$= -\mathrm{tr}\left(\underbrace{(D_b F^{ab})}_{C_a}\xi\right)$

$+ \nabla_b \underbrace{\mathrm{tr}\left(F^{ab}\xi\right)}_{j^{ab}(\xi)}$

$Q_\Sigma(\xi) = -\int_\Sigma \mathrm{tr}\left(\underbrace{(n_a D_b F^{ab})}_{\text{Gauss constraint}}\xi\right)$

$+ \int_{\partial\Sigma} \frac{1}{2}\,\mathrm{tr}\left(F^{ab}\xi\right)\epsilon_{ab}$ electric flux through $\partial\Sigma$

Pf sketch

N1 $\underset{\nabla_e J^e(\xi)}{dJ(\xi)} = E\,D\xi = \underbrace{D^\dagger E}_{=0 \;\; N2}\,\xi + \nabla_e(E B^e \xi)$

$d(J - E B \xi) = 0$ □

$D_b\xi + [A_b, \xi]$

Ex:

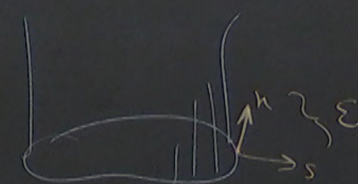
YM: $J^a(\xi) = tr(F^{ab} D_b \xi)$

$= -tr(\underbrace{(D_b F^{ab})}_{C_a}\xi) + \nabla_b\, \underbrace{tr(F^{ab}\xi)}_{j^{ab}(\xi)}$

$Q_\Sigma(\xi) = -\int_\Sigma tr(\underbrace{(n_a D_b F^{ab})}_{\text{Gauss constraint}}\xi)$

$+ \int_{\partial\Sigma} \frac{1}{2} tr(F^{ab}\xi)\,\epsilon_{ab}$

electric flux through $\partial\Sigma$



$Q = -\int_\Sigma tr(D{*F}\,\xi) + \int_{\partial\Sigma} tr({*F}\,\xi)$

Conventions

$$\begin{cases} \mathbb{d} g^{ab} := g^{aa'} g^{bb'} \mathbb{d} g_{a'b'} = -\mathbb{d}(g^{ab}) \\ \mathbb{d} g := g^{ab} \mathbb{d} g_{ab} \end{cases}$$

(exercise)

Symmetries : diffeos

$$G = \mathfrak{X}^1(M) \qquad [\xi, \eta] = \mathcal{L}_\xi \eta = -\mathcal{L}_\eta \xi$$

action $\rho(\xi) = \int \delta_\xi g_{ab} \dfrac{\delta}{\delta g_{ab}}$, $\delta_\xi g_{ab} = \mathcal{L}_\xi g_{ab} = 2 \nabla_{(a} \xi_{b)}$

RMk 1 antihomomorphism

$$[\rho(\xi), \rho(\eta)] = -\rho([\xi, \eta])$$

RMk 2 these diffeos act on $\mathcal{F} = \{ g_{ab}(x) \}$, not on M nor on x^a.

GENERAL RELATIVITY

$$S = \frac{1}{8\pi G}\int \underline{\mathcal{L}}$$

$$\underline{\mathcal{L}} = \mathcal{L}\,\underline{\epsilon} \qquad \underline{\epsilon}: \text{vol. form} = \sqrt{g}\, d^4x$$

$$\mathcal{L}: \text{scalar} = \frac{1}{2}(R - \Lambda$$

<u>Rmk</u>: For a well def action principle with fixed induced metric at $\partial\Sigma$, we need to add the York-Gibbons-Hawking term

$$S_\partial = \frac{1}{2}\int_{\partial\Sigma}\sqrt{h}\,K \qquad K = h^{ij}K_{ij}$$

(We will not add it)