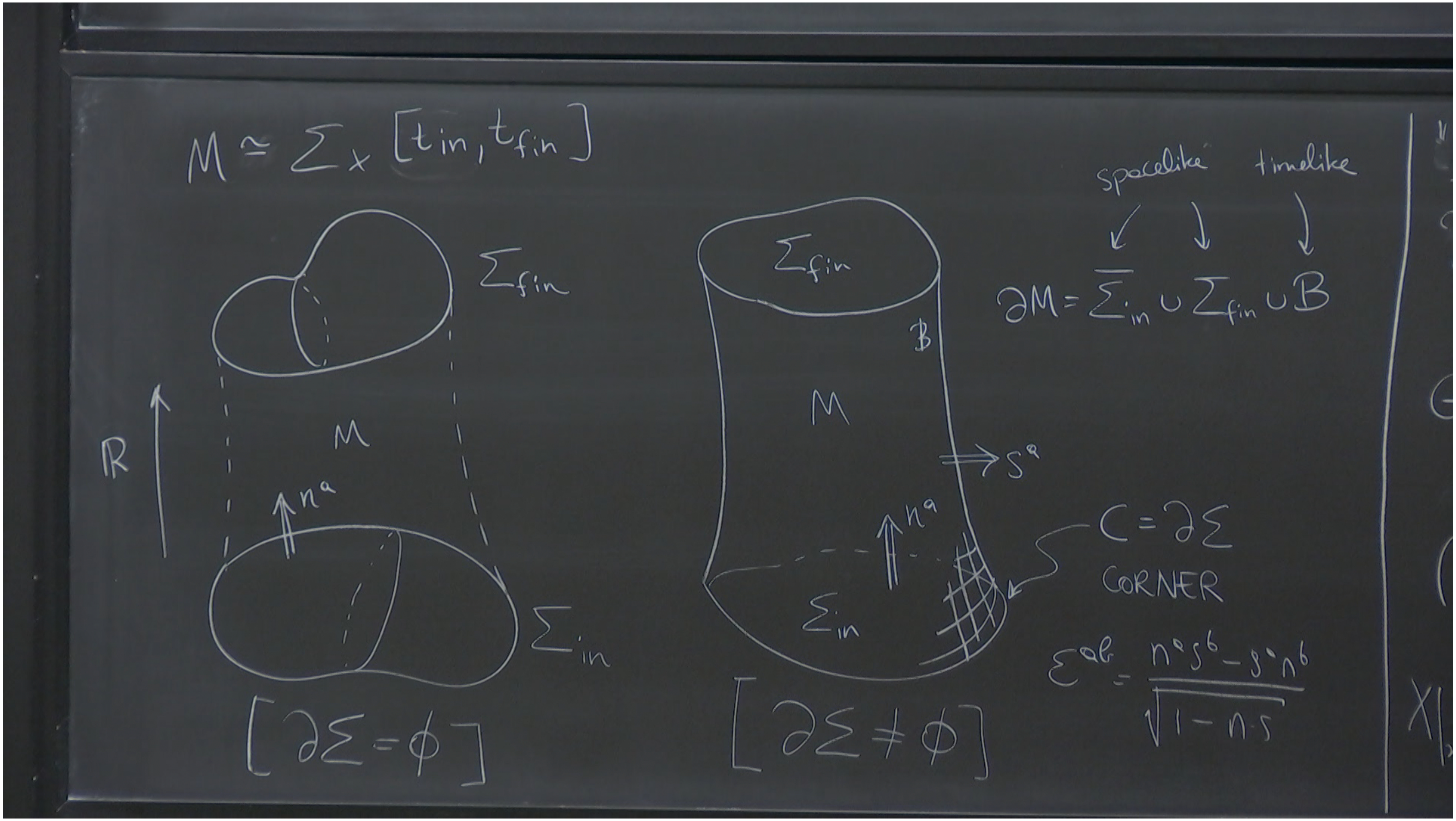
Title: Quantum Gravity Lecture (230417)

Speakers: Aldo Riello

Collection: Quantum Gravity (2022/2023)

Date: April 17, 2023 - 2:00 PM

URL: https://pirsa.org/23040028


$M \simeq \Sigma \times [t_{in}, t_{fin}]$
Σ_{fin}
$\mathbb{R}$
M
n^a
Σ_{in}
$[\partial\Sigma = \phi]$
Σ_{fin}
$\mathcal{B}$
M
s^a
n^a
Σ_{in}
$[\partial\Sigma \neq \phi]$
spacelike
timelike
$\partial M = \overline{\Sigma}_{in} \cup \Sigma_{fin} \cup B$
$C = \partial\Sigma$
CORNER
$\varepsilon^{ab} = \frac{n^a s^b - s^a n^b}{\sqrt{1 - n\cdot s}}$

...elike

$\downarrow$

B

"Field space"

$\mathbb{F}$ = sp. of field config. φ

[sections of some fibre bundle] e.g. $C^\infty(M,\mathbb{R})$ SCALAR FIELD

Geometry on both M & $\mathbb{F}$:

M	$\mathbb{F}$
(d, i_X, L_X)	$(\mathbb{d}, \mathbb{i}_{\mathbb{X}}, \mathbb{L}_{\mathbb{X}})$
$X\big\vert_x = \sum_i X^i(x)\frac{\partial}{\partial x^i}$	$\mathbb{X}\big\vert_\varphi = \int_M (\delta_{\mathbb{X}}\varphi)(x)\frac{\delta}{\delta\varphi(x)}$

function on M

$f: M \longrightarrow \mathbb{R}$

$X(f) = \sum X^i \frac{\partial f}{\partial x^i}$

$\mathbb{X}(S)$

$(M, \mathbb{R})$
IELD

function on M

$$f : M \longrightarrow \mathbb{R}$$

$$X(f)\big|_x = \sum X^i_{(x)} \frac{\partial f}{\partial x^i}$$

$$\downarrow$$

$$\underbrace{i_X df} = L_X f$$

$$i_X \sum_i \frac{\partial f}{\partial x^i} dx^i = \sum \frac{\partial f}{\partial x^i} X^i$$

$$\equiv X^i$$

(local) functional on $\mathcal{F}$

$$S[\varphi] = \int_M \mathcal{L}(x, \varphi(x), \partial\varphi(x))$$

$$S : \mathcal{F} \longrightarrow \mathbb{R}$$

$$\mathbb{X}(S) = \int_M \frac{\delta S}{\delta\varphi(x)} \left(\delta_{\mathbb{X}} \varphi(x)\right)$$

$$= \int_M \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_e \frac{\partial \mathcal{L}}{\partial(\partial_e \varphi(x))} \right) \delta_{\mathbb{X}} \varphi(x) + \oint_{\partial M} n^e \frac{\partial \mathcal{L}}{\partial(\partial_e \varphi(x))} \delta_{\mathbb{X}} \varphi(x)$$

$$i_{\mathbb{X}} \mathbb{d}\varphi(x) \equiv \delta_{\mathbb{X}} \varphi(x)$$

$$\mathbb{d}S = \int_M \frac{\delta S}{\delta\varphi(x)} (\mathbb{d}\varphi(x))$$

(x)

$\mathbb{L}_{\mathbb{X}} = \mathbb{d}\, \mathbb{i}_{\mathbb{X}} + \mathbb{i}_{\mathbb{X}}\, \mathbb{d}$

when acting on

$\Omega^k(\mathcal{F})$

(Cartan's formula)

<u>Detail</u>: $\partial_a \mathbb{d}\varphi \equiv \mathbb{d}\,\partial_a \varphi$

$$dx^a \partial_a \mathbb{d}\varphi = d\,\mathbb{d}\varphi \equiv \mathbb{d}\, d\varphi \in \Omega^{1,1}(M \times \mathcal{F})$$

$$\rightsquigarrow \Omega^{k,\ell}(M \times \mathcal{F}) \sim \underbrace{dx \wedge \cdots \wedge dx}_{k} \wedge \underbrace{\mathbb{d}\varphi \wedge \cdots \wedge \partial\mathbb{d}\varphi}_{\ell}$$

R is a

$\Omega^{\text{top}-k}$

dim M

$\Omega^{\text{top}-1}($

$\Omega^{\text{top}}(\Sigma$

R is a n_R-dim mfd

$$\Omega^{top-k}(R) \equiv \Omega^{n_R-k}(R)$$

$\dim M = n+1$

$$\Omega^{top-1}(M) = \Omega^{n}(M)$$

$$\Omega^{top}(\Sigma) = \Omega^{n}(\Sigma)$$

$$\Omega^{top,1}(M,\mathcal{F}) = \Omega^{top,1}_{source}(M\times\mathcal{F}) \oplus \Omega^{top,1}_{bdry}(M\times\mathcal{F})$$

[Takens thm]

$$d\,\Omega^{top-1,1}(M\times\mathcal{F})$$

there is <u>No</u> derivative acting on $d\varphi$

COVARIANT PHASE SPACE

action $S = \int_M \underline{\mathcal{L}}(\varphi, \partial\varphi)$

$\underline{\mathcal{L}} = \Omega^{top,0}(M \times \mathcal{F})$

Lagrangian density

$\underline{\mathcal{L}} = \mathcal{L}\, \epsilon_M$

$\uparrow$ vol. form

$\Omega^{top,}$

Ψ

$d\underline{\mathcal{L}} =$

$\underline{\Theta} \in$

$$\underline{\Omega}^{top,1}(M \times \mathcal{F})$$

$$\overset{\Psi}{d\underline{\mathcal{L}}} \overset{\text{Takens}}{=} \underbrace{\underline{E}_I \, d\varphi^I}_{\text{source form}} + \underbrace{d\,\underline{\Theta}}_{\text{bdry form}}$$

$\underline{\Theta} \in \Omega^{top-1,1}(M \times \mathcal{F})$ covariant sympl. potential current

Example

Example (scalar field)

$$\mathcal{L} = \sum_I \frac{1}{2} \nabla_a \varphi^I \nabla^a \varphi^I$$

$$E_I = -\Box\varphi$$

$$E_I \,\mathbb{d}\varphi^I = -(\Box\varphi)\,\mathbb{d}\varphi$$

$$\Theta^a = \frac{\partial \mathcal{L}}{\partial(\partial_a\varphi)}\,\mathbb{d}\varphi = (\nabla^a\varphi)\,\mathbb{d}\varphi$$

$$\underline{L} = \frac{1}{2}\, d\varphi^I \wedge * d\varphi^I$$

$$\underline{E} = (d * d\varphi)\,\mathbb{d}\varphi$$

$$\underline{\Theta} = (* d\varphi)\,\mathbb{d}\varphi$$

$$\mathbb{d}\mathcal{L} = -(\Box\varphi)\,\mathbb{d}\varphi + \nabla_a\big((\nabla^a\varphi)\,\mathbb{d}\varphi\big)$$

$$\mathbb{d}\underline{L} = -\underbrace{(d*d\varphi)\,\mathbb{d}\varphi}_{\text{Source}} + \underbrace{d\big((*d\varphi)\,\mathbb{d}\varphi\big)}_{\text{bdry.}}$$

$$\Omega^{top,1}(M \times \mathcal{F})$$

ψ Tokens

$$d\underline{L} = \underbrace{\underline{E}_I \, d\varphi^I}_{\underline{E} = \text{source form}} + \underbrace{d\,\underline{\Theta}}_{\text{bdry form}}$$

$$\underline{\Theta} \in \Omega^{top-1,1}(M \times \mathcal{F}) \quad \text{covariant sympl potential current}$$

Example (scalar field)

$$\mathcal{L} = \sum_I \frac{1}{2} \nabla_a \varphi^I \nabla^a \varphi^I$$

$$E_I = -\Box\varphi$$

$$E_I \, d\varphi^I = -(\Box\varphi)\, d\varphi$$

$$\Theta^a = \frac{\partial \mathcal{L}}{\partial(\partial_a \varphi)} d\varphi = (\nabla^a \varphi)$$

$$d\underline{L} = -(\Box\varphi)\, d\varphi + \nabla_a (\nabla^a$$

$$d\underline{L} = -\underbrace{(d{*}d\varphi)\, d\varphi}_{\text{Source}} + \underbrace{d((*}_{\text{bo}}$$

DEF: Space of "phys. config" or the "shell" as the space of sols to the EOM:

embedding ι_{EL} : $\bar{\mathcal{F}} := \{\varphi \in \mathcal{F} : \underline{E}|_{\varphi} = 0\} \hookrightarrow \mathcal{F}$

Rmk: even if $\mathcal{F}$ is a space of sections of some f.b. over M this is not the case for $\bar{\mathcal{F}}$.

DEF Cov. sympl (2-)form current $\quad \underline{\Omega} := \mathbb{d}\Theta \in \Omega^{top-1,2}(M \times \mathcal{F})$

Ex: $\underline{\Omega} = \mathbb{d}(\ast d\varphi) \wedge \mathbb{d}\varphi = (\ast d\, \mathbb{d}\varphi) \wedge \mathbb{d}\varphi$

$\ldots\big|_\varphi = \int_M (\partial_X \varphi)(x) \frac{\delta}{\delta\varphi^I(x)}$ $\qquad \equiv X^i$ $\qquad \mathbb{d}S = \int_M \frac{\delta S}{\delta\varphi(x)} (\mathbb{d}\varphi(x))$

<u>Example</u> (scalar field)

$$\mathcal{L} = \sum_I \frac{1}{2} \nabla_a \varphi^I \nabla^a \varphi^I$$

$$E_I = -\Box\varphi$$

$$E_I \mathbb{d}\varphi^I = -(\Box\varphi)\,\mathbb{d}\varphi$$

$$\Theta^a = \frac{\partial \mathcal{L}}{\partial(\partial_a\varphi)} \mathbb{d}\varphi = (\nabla^a\varphi)\,\mathbb{d}\varphi$$

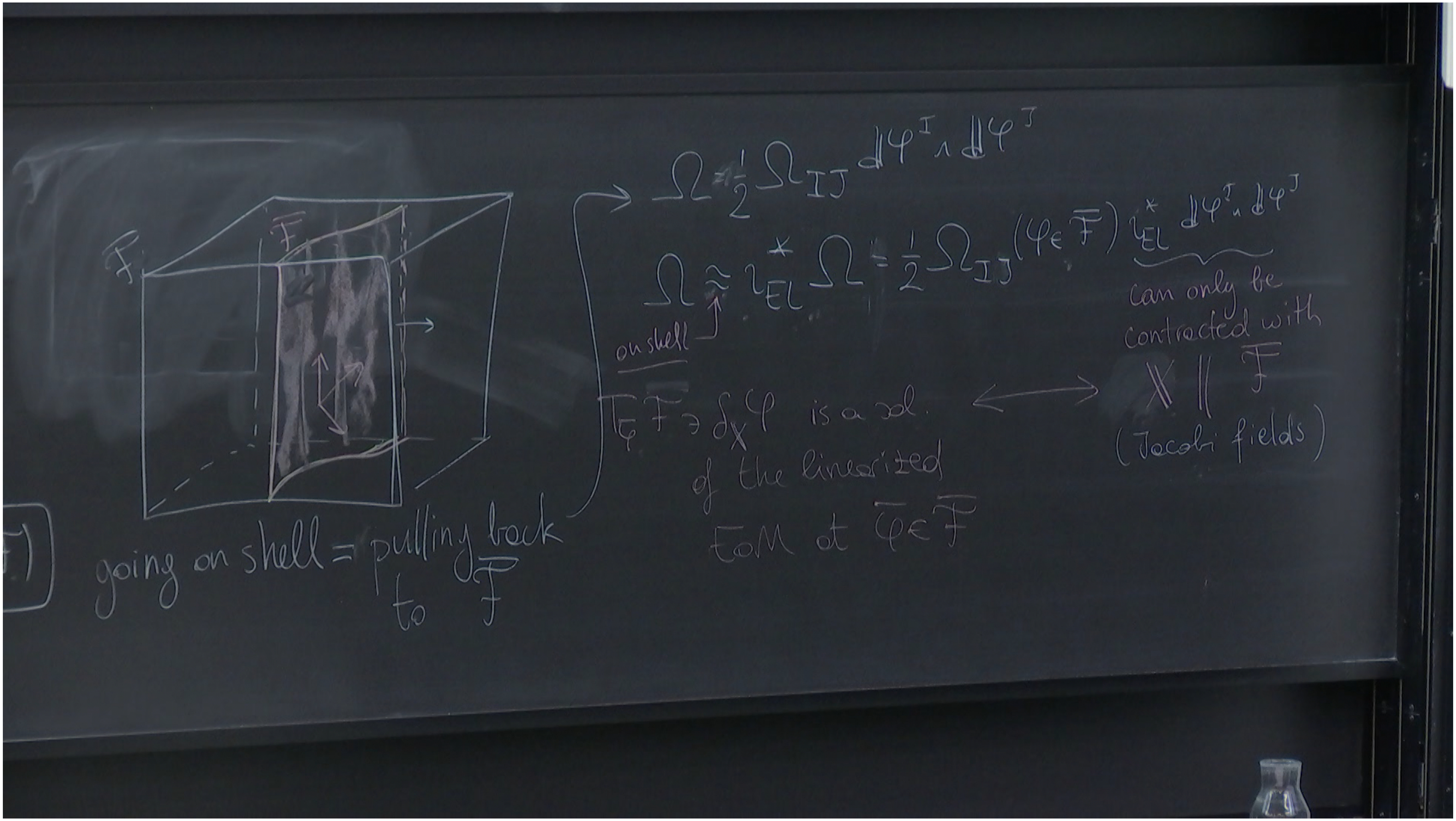
$$\underline{\mathcal{L}} = \frac{1}{2} d\varphi^I \wedge * d\varphi^I$$

$$\underline{E} = (d * d\varphi)\,\mathbb{d}\varphi$$

$$\underline{\Theta} = (*d\varphi)\,\mathbb{d}\varphi + d(\ \cancel{\ }\ \mathbb{d}\varphi)$$

$$\mathbb{d}\mathcal{L} = -(\Box\varphi)\,\mathbb{d}\varphi + \nabla_a\big((\nabla^a\varphi)\,\mathbb{d}\varphi\big)$$

$$\mathbb{d}\underline{\mathcal{L}} = -\underbrace{(d*d\varphi)\,\mathbb{d}\varphi}_{\text{Source}} + \underbrace{d\big((*d\varphi)\,\mathbb{d}\varphi\big)}_{\text{bdry.}}$$


$\Omega = \frac{1}{2}\Omega_{IJ}\, d\varphi^I \wedge d\varphi^J$
$\underline{\Omega} \approx \iota_{EL}^* \Omega = \frac{1}{2}\Omega_{IJ}(\varphi \in \mathcal{F})\, \underbrace{\iota_{EL}^* d\varphi^I \wedge d\varphi^J}$
on shell
can only be contracted with
$\mathbb{X} \parallel \mathcal{F}$
(Jacobi fields)
$T_{\bar\varphi}\mathcal{F} \ni \delta_X \varphi$ is a sol. of the linearized EoM at $\bar\varphi \in \mathcal{F}$
$\mathcal{F}$
going on shell = pulling back to $\mathcal{F}$

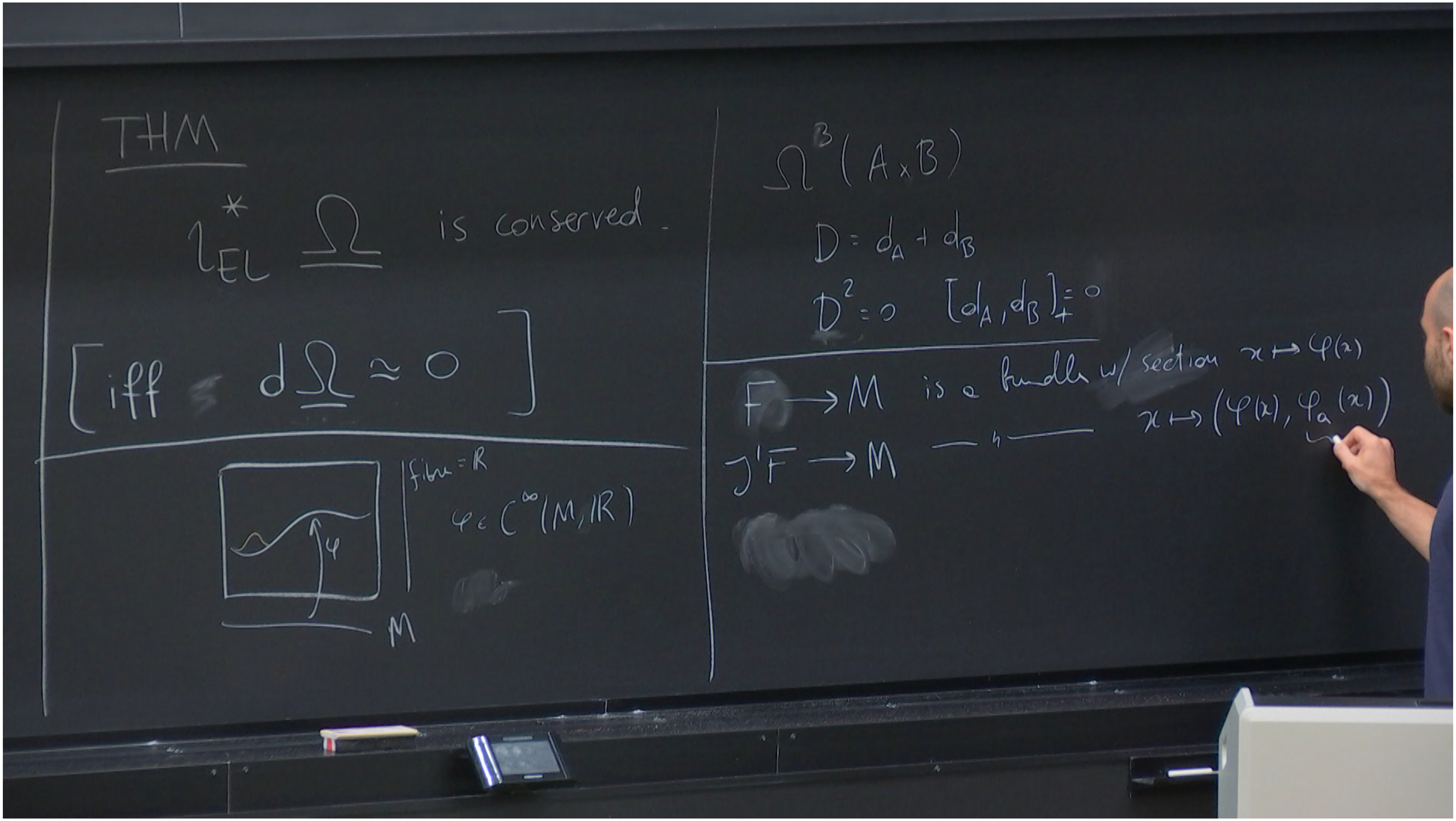
<u>Pf</u> (pink)

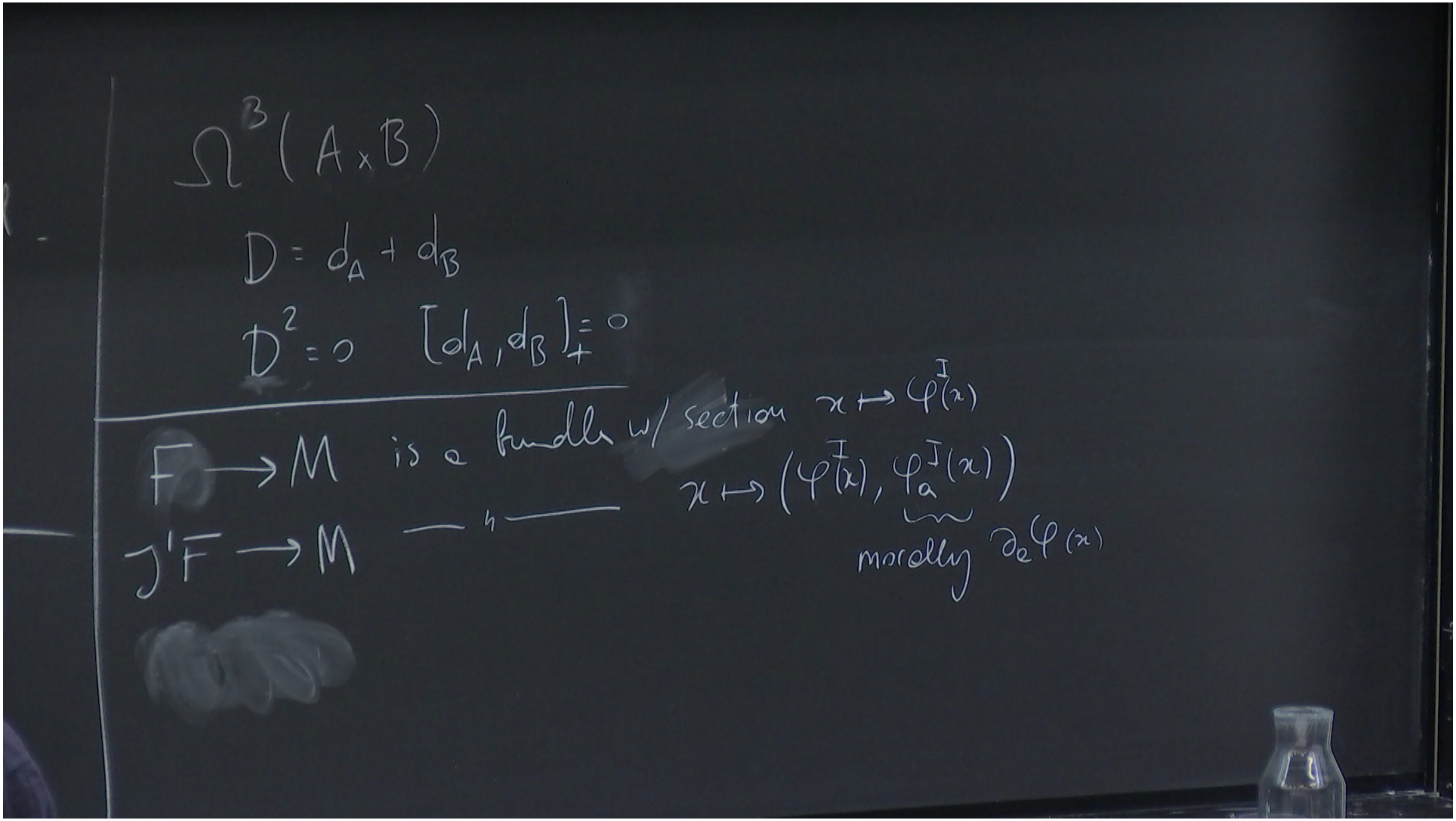
$$\mathcal{F} \supset \bar{\mathcal{F}} = \{ E_I = 0 \} \ni \bar{\varphi}$$

$\mathbb{X}|_{\bar{\varphi}}$ is tangent to $\bar{\mathcal{F}}$ iff $\mathbb{X}(E_I)|_{\bar{\varphi}} = 0$

In an ex: $\quad E = \Box \varphi + \lambda \varphi^3$

$$\mathbb{X}(E)|_{\bar{\varphi}} = \Box \delta_{\mathbb{X}} \varphi + 3\lambda \bar{\varphi}^2 \delta_{\mathbb{X}} \varphi$$


THM
ι*EL Ω is conserved.
[iff dΩ ≃ 0]
fibre = ℝ
φ ∈ C∞(M, ℝ)
φ
M
ΩB(A × B)
D = dA + dB
D² = 0 [dA, dB]± = 0
F → M is a bundle w/ section x ↦ φ(x)
J¹F → M — " — x ↦ (φ(x), φa(x))


$\Omega^B(A \times B)$
$D = d_A + d_B$
$D^2 = 0 \quad [d_A, d_B]_+ = 0$
$F \longrightarrow M$ is a bundle w/ section $x \mapsto \varphi^I(x)$
$J^1F \longrightarrow M$ — " — $x \mapsto (\varphi^I(x), \varphi^I_a(x))$
morally $\partial_a \varphi(x)$