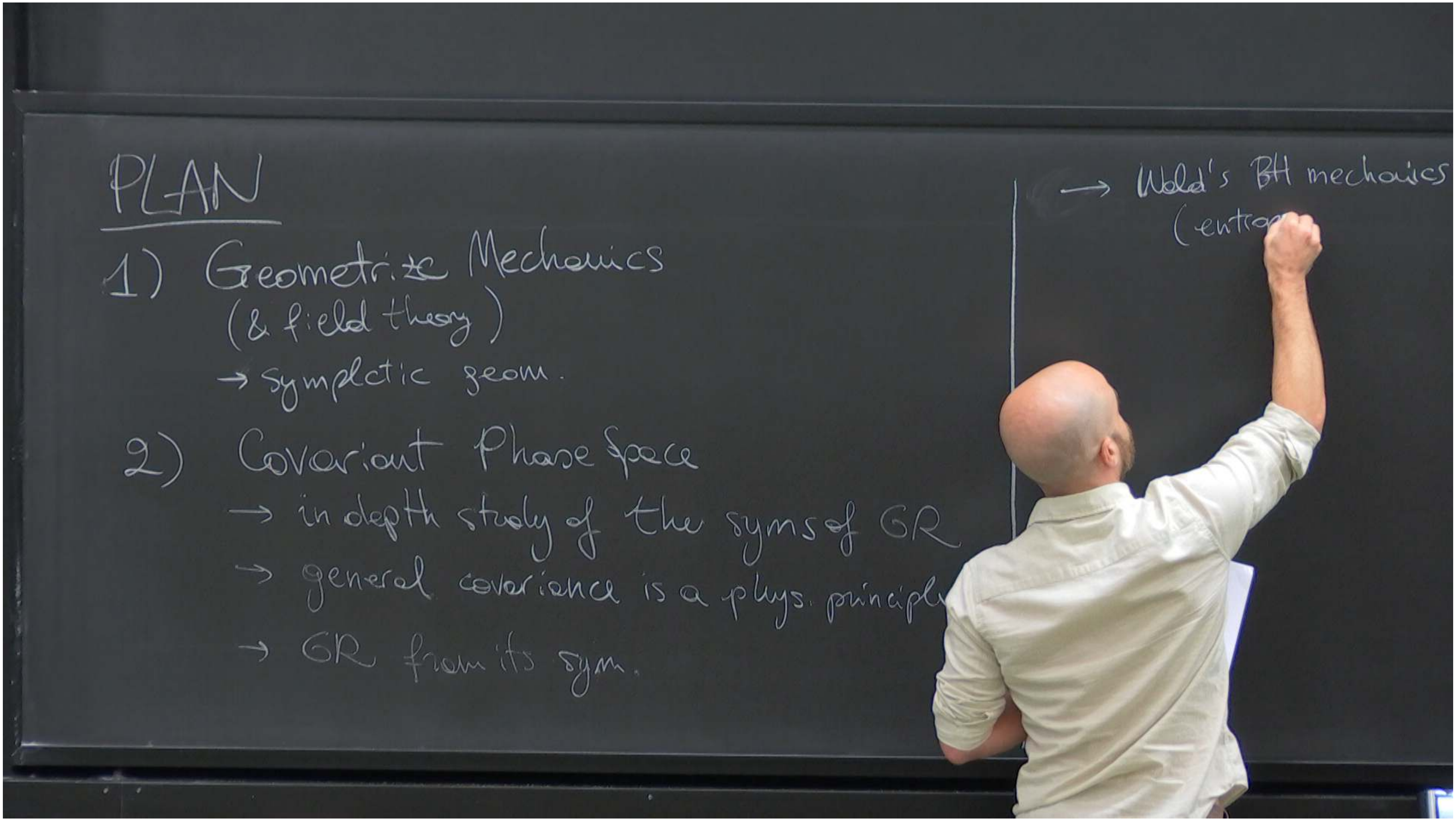
Title: Quantum Gravity Lecture (230403)

Speakers: Aldo Riello

Collection: Quantum Gravity (2022/2023)

Date: April 03, 2023 - 2:00 PM

URL: https://pirsa.org/23040026


PLAN
1) Geometric Mechanics
(& field theory)
→ symplectic geom.
2) Covariant Phase Space
→ in depth study of the syms of GR
→ general covariance is a phys. princip
→ GR from its sym.
→ Wald's BH mechanics

GR

inciple

→ Wald's BH mechanics
(entropy as a N. ch.)

3) CPS to Canonical Ph Sp.
↳ GR constr algebra
→ WdW eq.
→ Ashtekar's variables for GR

...ics
(N. ch.)

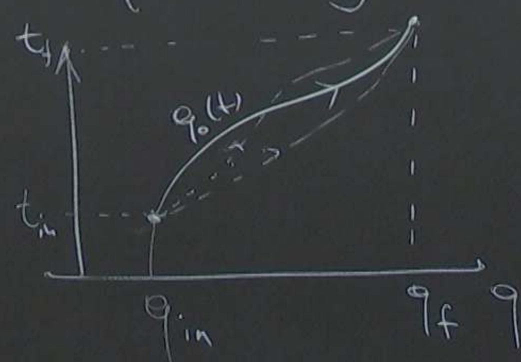
...nical Ph Sp.

...tr algebra

...'s variables for

ACTION PRINCIPLE

$$S[q(t)] = \int dt \; \mathcal{L}(q(t), \dot{q}(t))$$



Extremality = phys path

$$0 \overset{!}{=} \delta S\big|_{q_0} = \int_{t_{in}}^{t_f} \overbrace{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)}^{EL} \delta q \; +$$

$$+ \Big[\underbrace{\frac{\partial L}{\partial \dot{q}}}_{\equiv\, p} \underbrace{\delta q}_{\equiv 0} \Big]_{t_{in}}^{t_{fin}}$$

Hamiltonian

$p = \frac{\partial \mathcal{L}}{\partial \dot q}$ canonical momentum

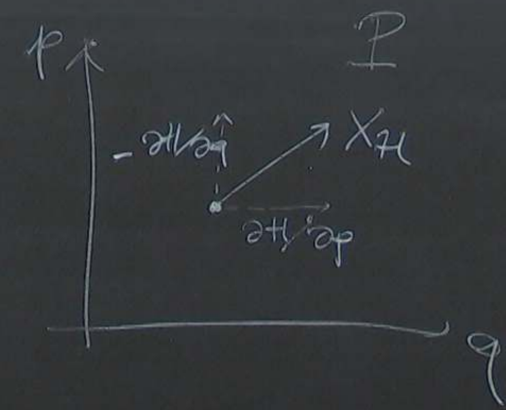
$\hookrightarrow \dot q = \dot q(p,q)$

Hamilt.: $\mathcal{H}(p,q) = p\dot q - \mathcal{L}(q,\dot q)$

Eom $\begin{cases} \dot p = -\frac{\partial H}{\partial q} \\ \dot q = \frac{\partial H}{\partial p} \end{cases} \rightarrow \frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = X_H \in \mathfrak{X}^1(P)$

"Velocity in Ph Sp"

HAM V.F.

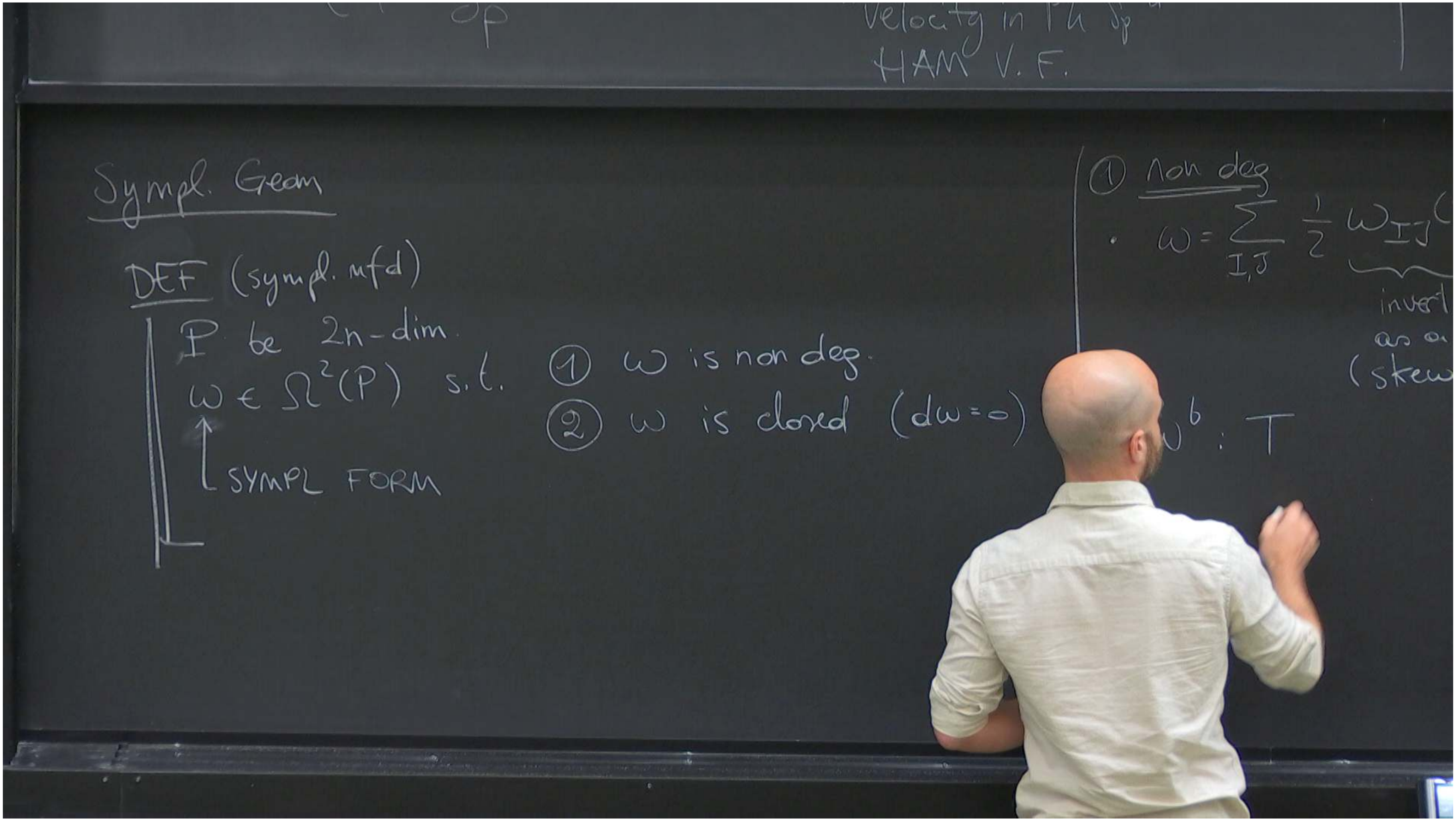


Poisson Brackets

$$\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \longrightarrow C^\infty(P)$$
$$(f, g) \longmapsto \{f,g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i}$$

$$P \ni (p, q)$$

$$X_H = \{H, \cdot\} = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$


Velocity in Ph Sp
HAM V. F.
Sympl. Geom
DEF (sympl. mfd)
P be 2n-dim.
ω ∈ Ω²(P) s.t.
① ω is non deg.
② ω is closed (dω=0)
SYMPL FORM
① Non deg
ω = Σ_{I,J} ½ ω_{IJ}
invert
(skew

Velocity in Tu Sp

HAM V. F.

① ω is non deg.

② ω is closed ($d\omega = 0$)

① Non deg

- $\omega = \sum_{I,J} \frac{1}{2} \underbrace{\omega_{IJ}}_{\text{invertible at all } z \text{ as a } 2n \times 2n \text{ (skew) matrix}} (z) dz^I \wedge dz^J$

- $\omega^\flat : TP \longrightarrow T^*P$

$$(z, X) \longmapsto (z, i_X \omega(z))$$

ω is non degenerate iff $\omega^\flat$ is injective

($\Rightarrow$ surjective $\Rightarrow$ bijective)

(finite dim)

②

$\omega(z))$

$\omega^\flat$ is injective

)

DEF [Liouville]

$$d\mu_L := \frac{1}{n!} \underbrace{\omega \wedge \cdots \wedge \omega}_{n\text{-times}} \in \Omega^{2n}_{\uparrow \text{top}}(P)$$

(vol form on P)

DEF [sympl. potential]

$\theta \in \Omega^1(P)$ s.t. $\omega := d\theta$ is sympl.

(existence not guaranteed)

THM $P = T^*Q$ is canonically symplectic

$\omega := d\theta =$

Pf: i) T_q^*Q v. sp. with basis $\{dq^i\}$

p_i are coords on T_q^*Q in this basis.

$\left(\text{i.e. } \alpha \in T_q^*Q \,, \quad \alpha = \sum_i p_i(\alpha)\, dq^i \right)$

ii) $\{p_i, q^i\}$ coords on $T^*Q = P$

$$\Rightarrow \theta_{(p,q)} = \sum_i p_i\, dq^i \in \Omega^1(P)$$

CANONICAL (TAUTOLOGICAL) 1-form

nonically symplectic

with basis $\{dq^i\}$

on T_q^*Q in this basis.

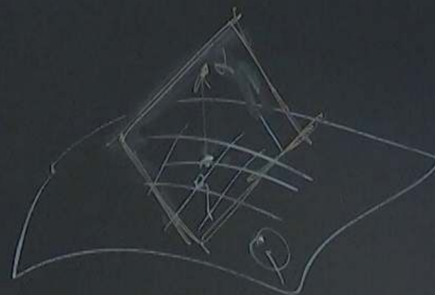
$Q \quad , \quad \alpha = \sum_i p_i(\alpha)\, dq^i \;)$

rds on $T^*Q = P$

$dq^i \in \Omega^1(P)$ CANONICAL (TAUTOLOGICAL) 1-form

$\omega := d\theta = \sum_i dp_i \wedge dq^i$ SYMPL.

$n=1 \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$



THM [Darboux]

Let (P, ω) sympl.

Then, $\boxed{\text{locally}}$ $\exists$ coords $z \stackrel{!}{=} (q^1, \dots, q^n, p_1, \dots, p_n)$

such that $\omega = \sum_i dp_i \wedge dq^i$

("canonical" or Darboux coords)

in an open neighborhood

(of every pt)

Cor: $d\mu_L = \pm\, dq^1 \wedge \dots \wedge dq^n \wedge dp_1 \wedge \dots \wedge dp_n$

ω = Area 2-form

$= d\cos\theta \wedge d\phi$

$\vartheta = \underbrace{\cos\theta}_{p}\ \underbrace{d\phi}_{q}$

Back to P.B.

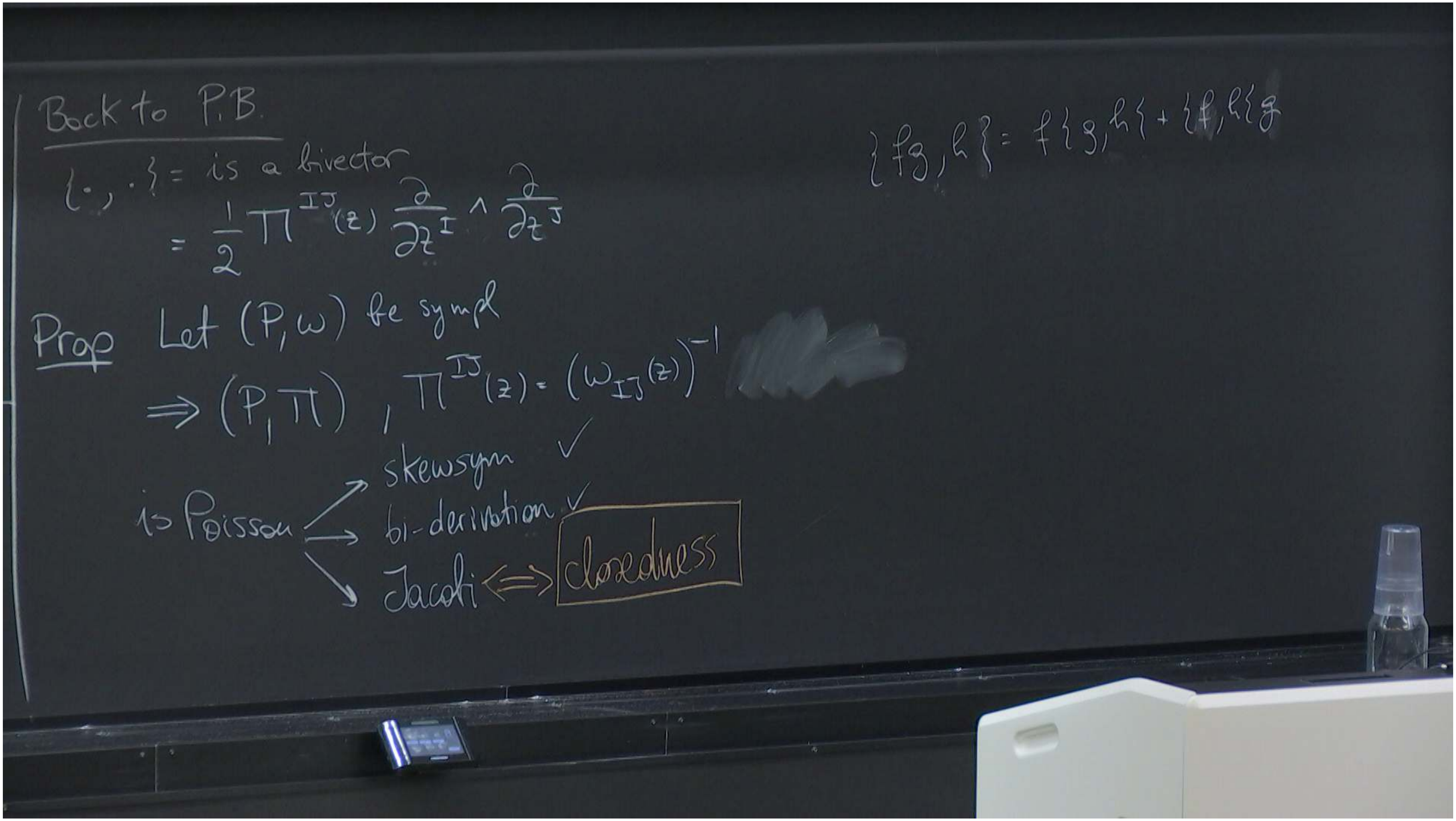
$\{\cdot,\cdot\}$ = is a bivector

$= \frac{1}{2}\Pi^{IJ}(z)\frac{\partial}{\partial z^I}\wedge\frac{\partial}{\partial z^J}$

Prop Let (P,ω) be sympl

$\Rightarrow (P,\Pi)$, $\Pi^{IJ}(z) = (\omega_{IJ}(z))^{-1}$

is Poisson
- skewsym ✓
- bi-derivation ✓
- Jacobi $\Longleftrightarrow$ $\boxed{\text{closedness}}$


Back to P.B.
$\{\cdot,\cdot\}$ = is a bivector
$= \frac{1}{2}\Pi^{IJ}(z)\,\frac{\partial}{\partial z^I}\wedge\frac{\partial}{\partial z^J}$
$\{fg,h\} = f\{g,h\} + \{f,h\}g$
Prop Let (P,ω) be sympl
$\Rightarrow (P,\Pi)$, $\Pi^{IJ}(z) = (\omega_{IJ}(z))^{-1}$
is Poisson
skewsym ✓
bi-derivation ✓
Jacobi $\Leftrightarrow$ closedness