Title: Quantum Gravity Lecture (230420)

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Collection: Quantum Gravity (2022/2023)

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Recap

$\underline{L} \in \Omega^{top,0}(M \times \mathcal{F})$ Lagrangian density

$\mathbb{d}\underline{L} =: \underline{E}_I \mathbb{d}\varphi^I + d\,\underline{\Theta}$

EOM

cov. sympl. pot. current

$\iota_{EL}: \bar{\mathcal{F}} = \{\varphi : E_I|_\varphi = 0\} \hookrightarrow \mathcal{F}$ Euler-Lagrange locus a.k.a. "the shell"

$\underline{\Omega} := \mathbb{d}\underline{\Theta}$ cov. sympl. pot. current

$\rightsquigarrow \Sigma \hookrightarrow M$

$\Omega_\Sigma := \int_\Sigma \underline{\Omega}$

THM

COR

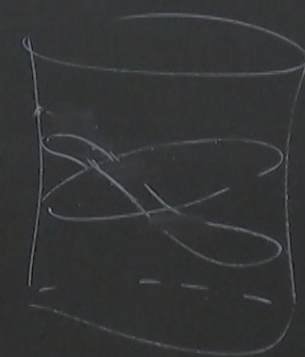
$\Rightarrow$

THM $d\underline{\Omega} \approx 0$

on shell
(after pullback to $\underline{\mathcal{F}} \hookrightarrow \mathcal{F}$) .

COR: $M = \Sigma \times \mathbb{R}$, $\partial\Sigma = \phi \Rightarrow \Sigma$ Cauchy surf

and $\imath_{EL}^{*}\Omega_{\Sigma}$ does <u>not</u>
depend on choice of $\Sigma \hookrightarrow M$.



$\Rightarrow \left(\underline{\mathcal{F}}, \imath_{EL}^{*}\Omega_{\Sigma}\right)$ is a candidate
(Covariant) phase space
$\Rightarrow$ indep. of choice of Σ .

$$\text{SYMMETRIES} \xrightarrow{(N1)} \text{Conserved charges}$$

Sym Generators [?]

$$\rho(\xi) = \int \delta_\xi \varphi \frac{\delta}{\delta \varphi}$$

off shell

DEF (g, ρ) Lagrangian sym iff $L_{\rho(\xi)} \underline{L} = d \underline{R}(\xi)$

DEF $\underline{J}(\xi) := i_{\rho(\xi)} \underline{\Theta} - \underline{R}(\xi)$ Noether current

THM (Noether 1) $d \underline{J}(\xi) = - E_I \delta_\xi \varphi^I \approx 0$.

$\partial \Sigma = \emptyset$, $Q_\Sigma(\xi) = \int_\Sigma \underline{J}(\xi)$ is indep of $\Sigma \hookrightarrow M$.

THM

$\rho(\xi)$ is tangent to $\bar{\mathcal{F}} \hookrightarrow \mathcal{F}$

$\mathbb{L}_{\rho(\xi)} E_I \approx 0$

Generators

"Best case scenario"

PROP: if $\underline{R}(\xi) = 0$, $\mathbb{L}_{\rho(\xi)} \underline{L} = 0$

then: $\mathbb{I}_{\rho(\xi)} \underline{\Omega} = - \mathbb{d} \underline{J}(\xi)$.

...t to $\mathcal{F} \hookrightarrow \mathcal{F}$

...rio"

$(\xi) = 0, \quad \mathbb{L}_{\rho(\xi)} \underline{\Theta} = 0$

$\xi)\,\underline{\Omega} = -\, d\, \underline{J}(\xi)$

Ex particle (1d field th.)

$M = \mathbb{R} \times \{pt\}$

$\mathcal{F} = C^\infty(M, \mathbb{R}^3) \ni q(t)$

$\Theta = m\dot{\vec{q}} \cdot d\vec{q} = m\dot{q}^i\, dq_i$

$\mathfrak{g}$ rotations $\rho(\xi) = \int_{\mathbb{R}} \epsilon_{ij}{}^{k}\, \xi^i\, q(t)^j \frac{\delta}{\delta q^k(t)}$

$\mathbb{L}_{\rho(\xi)} \Theta = 0$ ✓

$J(\xi) = \iota_{\rho(\xi)} \Theta = m\dot{q}^i \delta_\xi q^i = \underbrace{m\dot{q}^k \epsilon_{ijk}\, \xi^i q^j}_{\text{ang. mom.}}$

THM

$\rho(\xi)$ is tangent to $\bar{\mathcal{F}} \hookrightarrow \mathcal{F}$

$\mathbb{L}_{\rho(\xi)} E_I \approx 0$

then $\underline{J}(\xi)$ cov. sym. generator

Generators

"Best case scenario"

PROP: if $\underline{R}(\xi) = 0$, $\mathbb{L}_{\rho(\xi)} \underline{\Theta} = 0$

then: $\mathbb{i}_{\rho(\xi)} \underline{\Omega} = - \mathbb{d} \underline{J}(\xi)$

Ex pert

$M = \mathbb{R} \times$

$\mathcal{F} = C^\infty($

$\Theta = m$

$\mathfrak{g}$ rotat

$\mathbb{L}_{\rho(\xi)} \Theta$

$J(\xi) =$

Lemma $(\mathfrak{g}, \rho)$ Lagn. sym of $\mathcal{L}$, then

$$\mathring{\mathbb{i}}_{\rho(\xi)} \underline{\Omega} = -\mathbb{d}\, \underline{J}(\xi) + \underline{r}(\xi)$$

$$\underline{r}(\xi) := \mathbb{L}_{\rho(\xi)} \underline{L} - \mathbb{d}\, \underline{R}(\xi)$$

Thm 1.) OFF-SHELL if either

(test case scenario)

i) $\underline{r}(\xi) = 0$, or

ii) $\underline{r}(\xi) = \mathbb{d}\underline{s}(\xi)$, $\partial\Sigma = \phi$

then

$$\mathring{\mathbb{i}}_{\rho(\xi)} \Omega_\Sigma = -\mathbb{d}\, Q_\Sigma(\xi)$$

2.) ON-S

then

2) ON-SHELL if either

i) $\underline{\Gamma}(\xi) \simeq 0$, or

ii) $\underline{\Gamma}(\xi) \simeq d\underline{S}(\xi)$, $\partial\Sigma = \phi$ ← (time-translations OR "time" diffeos in GR)

then

$$\mathbb{I}_{\varphi(\xi)} \Omega_\Sigma \simeq -\delta\!\!\!/ Q_\Sigma(\xi)$$

case

cenario

Ex scalar field + time transl. sym.

$\underline{L} = \frac{1}{2} \nabla_a \varphi \nabla^a \varphi$ on (M, g_{ab})

χ is a killing vf.

$L_\chi g_{ab} = \nabla_a \chi_b + \nabla_b \chi_a = 0$

translations
OR
diffeos in GR

g vf.

$X_b + \nabla_b X_a = 0$

$\Theta^a = \nabla^a \varphi \, d\varphi$

$$R^a(\chi) = \chi^a \mathcal{L} = \chi^a \left(\tfrac{1}{2} \nabla_c \varphi \nabla^c \varphi\right)$$

$$J^a(\chi) = \chi^b \left(\nabla_a \varphi \nabla_b \varphi - \tfrac{1}{2} \delta^a_b \nabla_c \varphi \nabla^c \varphi\right)$$

$$r^a(\chi) = \chi^a \underbrace{\Box \varphi}_{E} \, d\varphi + \nabla_b \big(\underbrace{2 \chi^{[a} \nabla^{b]} \varphi \, d\varphi}_{S^{ab}}\big)$$

$\Longrightarrow$ case 2ii

off shell obstruction on $\partial \Sigma = \phi$

$$r_\Sigma(\xi) = \int_\Sigma \sqrt{h} \, n_a \chi^a \, \Box \varphi \, d\varphi$$

nonzero only if
$n \cdot \chi \neq 0$
i.e. χ transverse
to Σ.

Local Sym & Noether 2

- Results so far hold also for local sym's
- In this section we change notation

$\mathfrak{g}$ = finite dim. (semisimple) Lie alg.

$$\mathcal{G} = C^\infty(M, \mathfrak{g}) \longleftarrow \text{local Lie alg. of sym.}$$

$$\xi, \eta \in \mathcal{G}, \quad [\xi, \eta]_{\mathcal{G}}(x) = [\xi(x), \eta(x)]_{\mathfrak{g}}$$

& Noether 2

for hold also for

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ite dim. (semisimple) Lie alg.

$^\infty(M, \mathfrak{g}) \leftarrow$ local Lie alg. of sym.

, $[\xi, \eta]_{G}(x) = [\xi(x), \eta(x)]_{\mathfrak{g}}$

Recall Noether 1 (off shell)

$$d\underline{J}(\xi) = -E_I \underbrace{\delta_\xi \varphi^I}$$

$$\delta_\xi \varphi^I = \underbrace{\mathcal{D}^I_\alpha}\, \xi^\alpha$$

some φ-dep diff operator

$\underline{\text{Ex}}$: YM $\quad \delta_\xi A_a = \partial_a \xi + [A_a, \xi]$

$$\delta_\xi A^\alpha_a = \Big(\underbrace{\delta^\alpha_\gamma \partial_a + f^\alpha{}_{\beta\gamma} A^\beta_a}_{\mathcal{D}^I_\alpha}\Big)\xi^\gamma$$

Noether 1 (off shell)

$$\ldots(\xi) = -E_I \, \delta_\xi \varphi^I$$

$$\delta_\xi \varphi^I = \mathcal{D}^I_\alpha \, \xi^\alpha$$

some φ-dep diff operator

YM

$$\delta_\xi A_a = \partial_a \xi + [A_a, \xi]$$

$$\delta_\xi A_a^\alpha = \underbrace{\left(\delta^\alpha_\gamma \partial_a + f^\alpha{}_{\beta\gamma} A^\beta_a\right)}_{\mathcal{D}^I_\alpha} \xi^\gamma$$

$$\partial\Sigma = \emptyset$$

adjoint op.

$$\int_\Sigma E_I \mathcal{D}^I_\alpha \xi^\alpha \underset{ibp}{=} \int_\Sigma \left(\mathcal{D}^{\dagger I}_\alpha E_I\right) \xi^\alpha$$

THM (Noether 2)

$$\left(\mathcal{D}^{\dagger I}_\alpha E_I\right) = 0$$