Title: Quantum Gravity Lecture (230418)

Speakers: Aldo Riello

Collection: Quantum Gravity (2022/2023)

Date: April 18, 2023 - 9:00 AM

URL: https://pirsa.org/23040022

RECAP

Lagrangian density $\underline{L} \in \Omega^{top,0}(M \times \mathcal{F})$, $S = \int_M \underline{L} \in \Omega^0(\mathcal{F})$

qualifier meaning: "not-quite symplectic"

$$\mathbb{d}\underline{L} = \underline{E}_I \, \mathbb{d}\varphi^I + d \underline{\Theta}$$

(Takens, or i.b.p.)

- $\underline{\Theta} \in \Omega^{\boxed{top-1},0}(M \times \mathcal{F})$ $\boxed{\text{covariant}}$ sympl. $\boxed{\text{potential}}$ $\boxed{\text{current}}$
- $\underline{\Omega} := \boxed{\mathbb{d}} \underline{\Theta} \in \Omega^{\boxed{top-1},2}(M \times \mathcal{F})$ $\boxed{\text{covariant}}$ symplectic form $\boxed{\text{current}}$
- $\approx$ "going on-shell" $\equiv$ upon pullback by

$$\iota_{EL} : \mathcal{F} = \{E_I = 0\} \hookrightarrow$$

RECAP

Lagrangian density $\underline{\mathcal{L}} \in \Omega^{top,0}(M \times \mathcal{F})$, $S = \int_M \underline{\mathcal{L}} \in \Omega^0(\mathcal{F})$

$$\mathbb{d}\underline{\mathcal{L}} = \underline{E}_I \, \mathbb{d}\varphi^I + d\,\underline{\Theta}$$

(Takens, or i.b.p.)

qualifier meaning: "not-quite symplectic"

- $\underline{\Theta} \in \Omega^{\boxed{top-1},0}(M \times \mathcal{F})$ $\boxed{\text{covariant}}$ sympl. $\boxed{\text{potential}}$ $\boxed{\text{current}}$
- $\underline{\Omega} := \boxed{\mathbb{d}}\,\underline{\Theta} \in \Omega^{\boxed{top-1},2}(M \times \mathcal{F})$ $\boxed{\text{covariant}}$ symplectic form $\boxed{\text{current}}$
- $\approx$ "going on-shell" $\equiv$ upon pullback by $\iota_{EL} : \mathcal{F} = \{E_I = 0\} \hookrightarrow \mathcal{F}$

THM

$$d\,\underline{\Omega} \approx 0$$

Pf: $0 \equiv \mathbb{d}^2 \underline{L} = \mathbb{d}\left(E_I\, \mathbb{d}\varphi^I + d\,\underline{\Theta}\right)$

$$\approx 0 + \mathbb{d}\, d\, \Theta$$

$$\approx d\,\underline{\Omega} \qquad \boxtimes$$

meaning:
"symplectic"

rient
lectic form
urrent

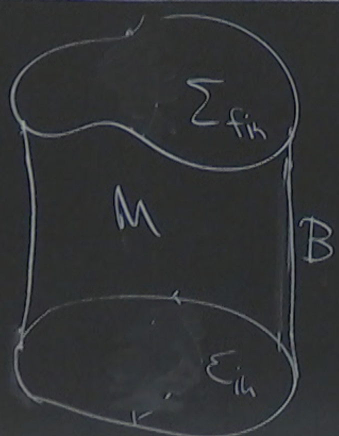
k by
$E_I = 0\} \hookrightarrow F$

$d\underline{\Omega} \approx 0$

$\underline{\text{Pf}}$: $0 \equiv d^2 \underline{L} = d\left(E_I\, d\varphi^I + d\underline{\Theta}\right)$

$\approx 0 + d\,d\,\Theta$

$\approx d\underline{\Omega}$ ∎



Corollary

$$0 \approx \int_M d\underline{\Omega} = \int_{\Sigma_f} \underline{\Omega} - \int_{\Sigma_i} \underline{\Omega} + \int_B \underline{\Omega}$$

$0 \approx \Omega_{fin} - \Omega_{in} +$ sympl. flux if $\partial\Sigma = \phi$

$\hookrightarrow \Omega_{fin} \simeq \Omega_{in} \simeq \Omega_{\Sigma}$

Σ "any Cauchy surf" of $M = \Sigma \times \mathbb{R}$

$$\rightsquigarrow \left(\overline{\mathcal{F}} , \underbrace{\iota^*_{EL} \Omega_{\Sigma}}_{\in \Omega^2(\overline{\mathcal{F}})} \right)$$

if everything goes well
this might be symplectic
(COVARIANT PH. SPACE)

Ex N-particles as a 1d Field Th

$M = \mathbb{R}$ (time), $\Sigma = \{1pt\}$,

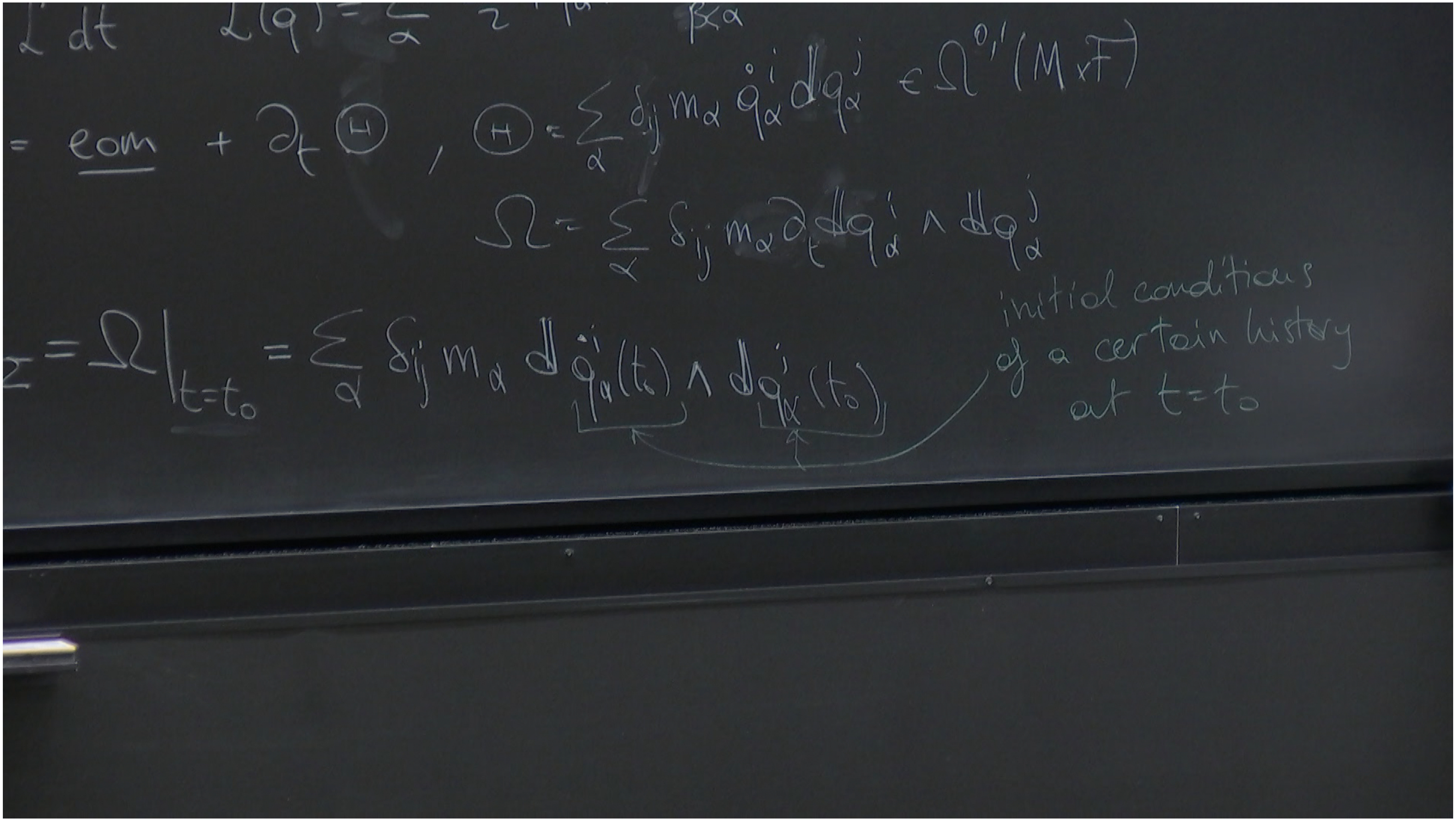
$\mathcal{F} = C^\infty(M, \mathbb{R}^{3N}) \ni q$ "histories"

$\underline{L} = \mathcal{L}\, dt \qquad \mathcal{L}(q) = \sum_\alpha \frac{m_\alpha}{2} |\dot{q}_\alpha|^2 - \sum_{\beta<\alpha} V(|q_\alpha - q_\beta|)$

$\mathbb{d} \underline{L} = \underline{eom} + \partial_t \Theta, \quad \Theta = \sum_\alpha \delta_{ij} m_\alpha \dot{q}^i_\alpha \mathbb{d} q^j_\alpha \in \Omega^{0,1}(M \times \mathcal{F})$

$\Omega = \sum_\alpha \delta_{ij} m_\alpha \partial_t \mathbb{d} q^i_\alpha \wedge \mathbb{d} q^j_\alpha$

$\Omega_\Sigma = \Omega\big|_{t=t_0} = \sum_\alpha \delta_{ij} m_\alpha \mathbb{d}\dot{q}^i_\alpha(t_0) \wedge \mathbb{d} q^j_\alpha(t_0)$


$= \underline{eom} + \partial_t \Theta \,, \quad \Theta = \sum_\alpha \delta_{ij} m_\alpha \dot{q}^i_\alpha \, dq^j_\alpha \in \Omega^{0,1}(M \times F)$
$\Omega = \sum_\alpha \delta_{ij} m_\alpha \partial_t dq^i_\alpha \wedge dq^j_\alpha$
$= \Omega|_{t=t_0} = \sum_\alpha \delta_{ij} m_\alpha \, d\dot{q}^i_\alpha(t_0) \wedge dq^j_\alpha(t_0)$
initial conditions
of a certain history
at $t=t_0$

$({}^s\mathcal{F}, \iota_{EL}^* \Omega_\Sigma)$ is isomorphic to the canonical ph. space.

$\rightsquigarrow$ symplectic

Ex : scalar field.

("same result")

shell:
conditions
tain history

$\mathbb{d}\varphi^I + d\,\Theta)$

$\mathbb{d}\, d\, \Theta$

$-\int_{\Sigma_i} \Omega + \int_{B} \Omega$

in + sympl. flux

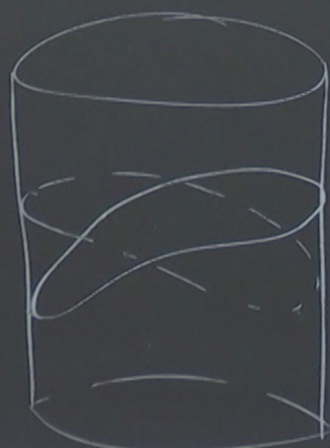
if $\partial\Sigma = \emptyset$

$\hookrightarrow \Omega_{fin} \simeq \Omega_{in} \simeq \Omega_{\Sigma}$

$\llcorner$ "any Cauchy surf" of $M = \Sigma \times \mathbb{R}$

$\rightsquigarrow \left(\mathcal{F},\ \underbrace{\iota_{EL}^{*}\Omega_{\Sigma}}_{\in\, \Omega^{2}(\mathcal{F})} \right)$

if everything goes well this might be symplectic (COVARIANT PH. SPACE)



→ $({}^{S}\mathcal{F}, \iota_{EL}^{*}\Omega_{\Sigma})$ is isomorphic to the canonical ph. space.

⇝ symplectic

<u>Ex</u>: scalar field

("same result")

$$\left.\begin{array}{l} \frac{1}{2}\nabla_a\varphi\nabla^a\varphi \\ -\frac{1}{2}\varphi\Box\varphi \end{array}\right]$$ different $\underline{\Theta}$'s, same $\underline{\Omega}$

NOETHER 1

$$
\begin{tikzcd}
\text{Symmetries} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow, bend right] & & \text{Conserved Charges} \arrow[dl, leftrightarrow, bend left] \\
& \text{Symmetry generators} &
\end{tikzcd}
$$

DEF [Lagrangian sym]

action $\rho : \mathfrak{g} \longrightarrow \mathfrak{X}'(\mathcal{F})$ such that

$$\mathbb{L}_{\rho(\xi)} \underline{L} = \boxed{d} \underline{R}(\xi), \text{ for a } \underline{R} : \mathfrak{g} \to \Omega^{top-1,0}(M \times \mathcal{F})$$

THM (Noether 1)

$(\mathfrak{g}, \rho)$ a Lagrangian sym of $\underline{L}$

$$\Rightarrow \forall \xi \in \mathfrak{g} \qquad \underline{J}(\xi) := \mathbb{i}^{\circ}_{\rho(\xi)} \underline{\Theta} - \underline{R}(\xi) \in \Omega^{\text{top}-1,0}(M \times \mathcal{F}) \quad \text{current}$$

(NOETHER CURRENT)

is conserved on-shell, more precisely:

$$d\,\underline{J}(\xi) = -\underline{E}_I\, \delta_\xi \varphi^I \approx 0$$

Pf: $\mathbb{L}_{\rho(\xi)} \underline{L} \underset{\uparrow\,\text{hyp.}}{=} d\,\underline{R}(\xi)$

$$\mathbb{i}^{\circ}_{\rho(\xi)} \mathbb{d} \underline{L} = \mathbb{i}^{\circ}_{\rho(\xi)} \left(E_I\, \mathbb{d}\varphi^I + d\,\Theta \right) = E_I\, \delta_\xi \varphi^I + d\, \mathbb{i}^{\circ}_{\rho(\xi)} \Theta$$

$0 \approx \Omega_{fin} - \Omega_{in} + \begin{smallmatrix}\text{sympl.}\\ \text{flux}\end{smallmatrix}$

THM (Noether 1)

$(\mathfrak{g}, \rho)$ a Lagrangian sym of $\underline{L}$

$\Rightarrow \forall \xi \in \mathfrak{g} \quad \underline{J}(\xi) := \mathring{\mathbb{I}}_{\rho(\xi)} \underline{\Theta} - \underline{R}(\xi) \in \Omega^{(top-1,0)}(M \times \mathcal{F})$ current

(NOETHER CURRENT)

is conserved on-shell, more precisely:

$$d\,\underline{J}(\xi) = -\underline{E}_I\, \delta_\xi \varphi^I \approx 0$$

Pf: $\mathbb{L}_{\rho(\xi)} \underline{L} \underset{hyp.}{=} d\,\underline{R}(\xi)$

$$\| $$

$d\,\mathring{\mathbb{I}}_{\rho(\xi)} \underline{L} + \mathring{\mathbb{I}}_{\rho(\xi)} \mathbb{d} \underline{L} = \mathring{\mathbb{I}}_{\rho(\xi)} \left(E_I \mathbb{d}\varphi^I + d\,\Theta \right) = E_I\, \delta_\xi \varphi^I + d\, \mathring{\mathbb{I}}_{\rho(\xi)} \Theta$

Rmk $\underline{J}(\xi) \in \Omega^{top-1,0}(M \times F)$

$$\underline{J} : \mathfrak{g} \longrightarrow \Omega^{top-1}(M \times F)$$

$\mathbb{R}$-linear

Rmk $* \underline{J} = g_{ab} J^a \, dx^b$

$$d\underline{J} \approx 0 \quad \text{iff} \quad \nabla_a J^a \approx 0$$

COROLLARY

$$0 \approx \int_M \underline{J}(\xi) = \int_{\Sigma_{fin}} J(\xi) - \int_{\Sigma_{in}} \underline{J}(\xi) + \int_B \underline{J}(\xi)$$

if $\partial \Sigma = \phi$

$$Q_\Sigma(\xi) = \int_\Sigma \underline{J}(\xi)$$ does not depend on choice of Cauchy surface.

$$\Sigma \hookrightarrow M$$

Ex: N-particle system

$\mathfrak{g}$ = rotations $\quad Q_\Sigma(\xi) = \overrightarrow{\text{angular mom}} \cdot \vec{\xi}$

$\times F)$

$M_\times F)$

≈ 0

COROLLARY

$$0 \simeq \int_M d\,\underline{J}(\xi) = \int_{\Sigma_{fin}} \underline{J}(\xi) - \int_{\Sigma_{in}} \underline{J}(\xi) + \int_B \underline{J}(\xi)$$

if $\partial\Sigma = \phi$

$$Q_\Sigma(\xi) = \int_\Sigma \underline{J}(\xi)$$

does not depend on choice of Cauchy surfa[ce]

$$\Sigma \hookrightarrow M$$

Ex: N-particle system

$\mathfrak{g}$ = rotations $\quad Q_\Sigma(\xi) = \overrightarrow{\text{angular mom}} \cdot \vec{\xi}$