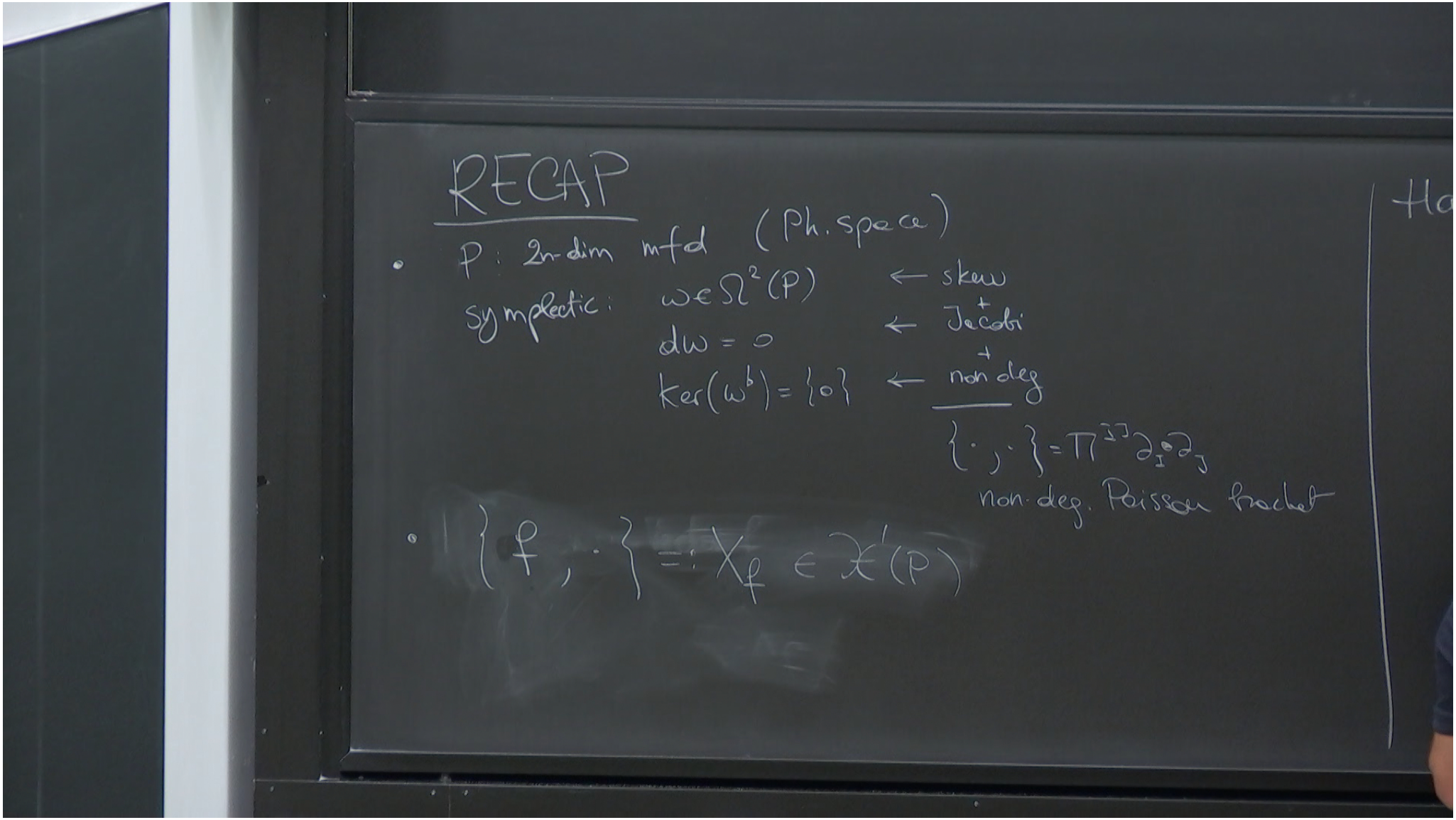
Title: Quantum Gravity Lecture (230411)

Speakers: Aldo Riello

Collection: Quantum Gravity (2022/2023)

Date: April 11, 2023 - 9:00 AM

URL: https://pirsa.org/23040020


RECAP
• P: 2n-dim mfd (Ph. space)
symplectic: $\omega \in \Omega^2(P)$ ← skew
$d\omega = 0$ ← + Jacobi
$\ker(\omega^\flat) = \{0\}$ ← + non deg
$\{\cdot,\cdot\} = \Pi^{IJ}\partial_I \cdot \partial_J$
non-deg. Poisson bracket
• $\{f, \cdot\} =: X_f \in \mathfrak{X}(P)$
Ha

Hamiltonian VF

$X \in \mathfrak{X}(P)$: $\exists f_X \in C^\infty(P)$

such that $i_X \omega = -df_X$

PROP $L_X \omega = 0$ if X is HVF

Converse almost true:

$0 = L_X \omega = i_X d\omega + d i_X \omega$

$\Rightarrow d\omega = 0$

PROP $[X_f, X_g]_{TP} = X_{\{f,g\}}$

$i \circ j$

sson bracket

$\in C^\infty(P)$

X

$= -df_X$

X is HVF

$= X_{\{f,g\}}$

DEF [Lie alg. action on P]

$\mathfrak{g}$ real Lie algebra

$\rho : \mathfrak{g} \to \mathfrak{X}^1(P)$ (linear)

homomorphism of Lie alg

$\rho([\xi,\eta]_{\mathfrak{g}}) = [\rho(\xi), \rho(\eta)]_{TP}$

DEF $G \times P \to P$ gp action on P

$\rho : \mathfrak{g} \to \mathfrak{X}^1(P)$ infinitesimal action

$z \in P \rightsquigarrow O_z = \{g \triangleright z,\ g \in G\}$ submfd.

ction on P]

gebra

$\mathfrak{X}'(P)$ (linear)

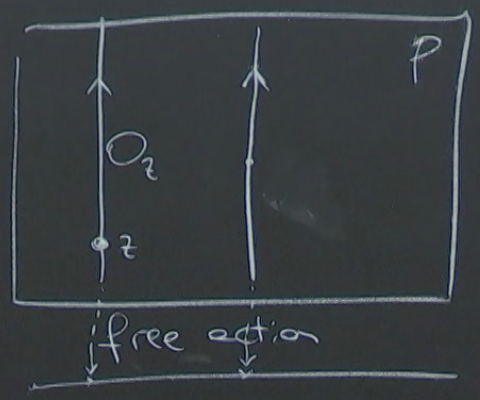
of Lie alg

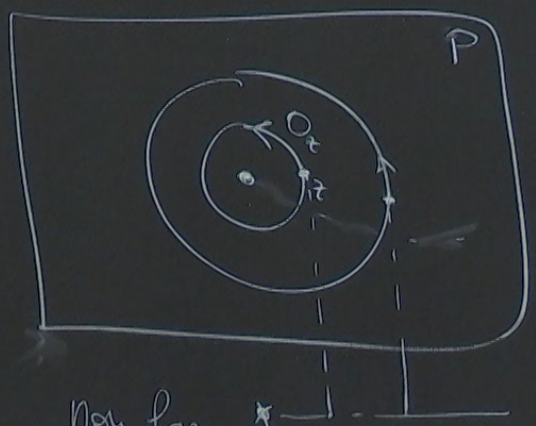
$[\rho(\xi), \rho(\eta)]_{TP}$

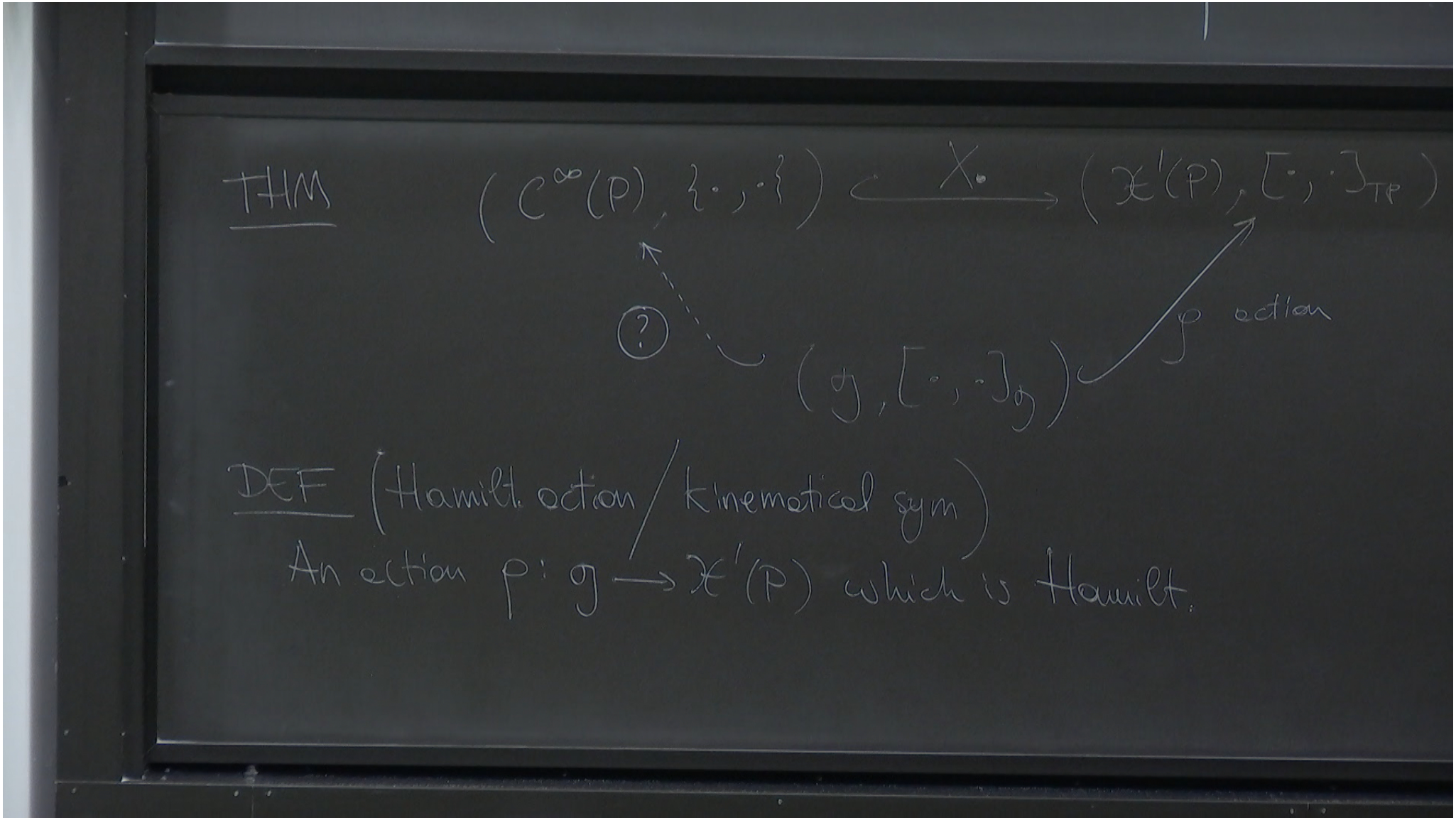
$\to P$ grp action on P

$\to \mathfrak{X}'(P)$ infinitesimal action

$O_z = \{g \triangleright z, g \in G\}$ submfd.






THM
$(C^\infty(P), \{\cdot,\cdot\})$
$X_\bullet$
$(\mathfrak{X}'(P), [\cdot,\cdot]_{TP})$
?
$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$
ρ action
DEF (Hamilt. action / kinematical sym)
An action $\rho: \mathfrak{g} \to \mathfrak{X}'(P)$ which is Hamilt.

That is:

$\tilde{J} : \mathfrak{g} \longrightarrow C^\infty(P)$ comomentum map

s.t. $\forall \xi \quad i_{\rho(\xi)}\omega = -d\tilde{J}(\xi) \qquad (*)$

Rmk $\tilde{J}$ must be linear $\tilde{J}(a\xi + b\eta) = a\tilde{J}(\xi) + b\tilde{J}(\eta)$

$a, b \in \mathbb{R}$, $\xi, \eta \in \mathfrak{g}$

bc. lhs of $(*)$ is linear in ξ.

$\Rightarrow \exists\, J : P \longrightarrow \mathfrak{g}^*$ momentum map

s.t. $\check{J}(\xi)(z) = \langle J(z), \xi \rangle$

Examples

Translations

$Q = (\mathbb{R}^3)^N$ N-particle config. sp.

$P = T^*Q \simeq (\mathbb{R}^6)^N \ni (\vec{p}_\alpha, \vec{q}_\alpha)$

$\omega = \sum_{\alpha=1}^{N} d\vec{p}_\alpha \wedge d\vec{q}_\alpha$

$\mathcal{G} = (\mathbb{R}^3, +)$

$\rho(\xi) = \sum_{\alpha=1}^{N} \xi^i \frac{\partial}{\partial q^i_\alpha}$

$i_{\rho(\xi)}\omega = -\sum_{\alpha=1}^{N} d\vec{p}_\alpha \cdot \vec{\xi} = -d\left(\sum_{\alpha=1}^{N} \vec{\xi} \cdot \vec{p}_\alpha\right)$

$\tilde{J}(\xi) \equiv \langle J, \xi \rangle$

$J : P$

$\to J(\vec{q}_\alpha$

$\mathfrak{g}^* \simeq \mathbb{R}^3$

$) = \sum_\alpha \vec{p}_\alpha$

Rot

$\mathfrak{g} = \mathfrak{so}(3) \simeq \mathbb{R}^3$

$\vec{J}(p_\alpha, q_\alpha) = \sum_\alpha \vec{p}_\alpha \times \vec{q}_\alpha$

$\mathfrak{g} = \mathfrak{so}(3) \simeq \mathbb{R}^3$ basis $\{e_i\}$

$\mathfrak{g} \ni \xi = \xi^i e_i$

$\mathfrak{g}^* \simeq \mathbb{R}^3$ basis $\{f^i\}$ $\langle f^i, e_j \rangle = \delta^i_j$

$\mathfrak{g}^* \ni J = J_i f^i$ $\langle J, \xi \rangle = J^i \xi_i$

$= \mathfrak{so}(3) \simeq \mathbb{R}^3$

$(p_\alpha, q_\alpha) = \sum_\alpha \vec{p}_\alpha \times \vec{q}_\alpha$

$\int$

$\tilde{J}(\xi)(z) = \tilde{J}(\xi)(p_\alpha, q_\alpha)$

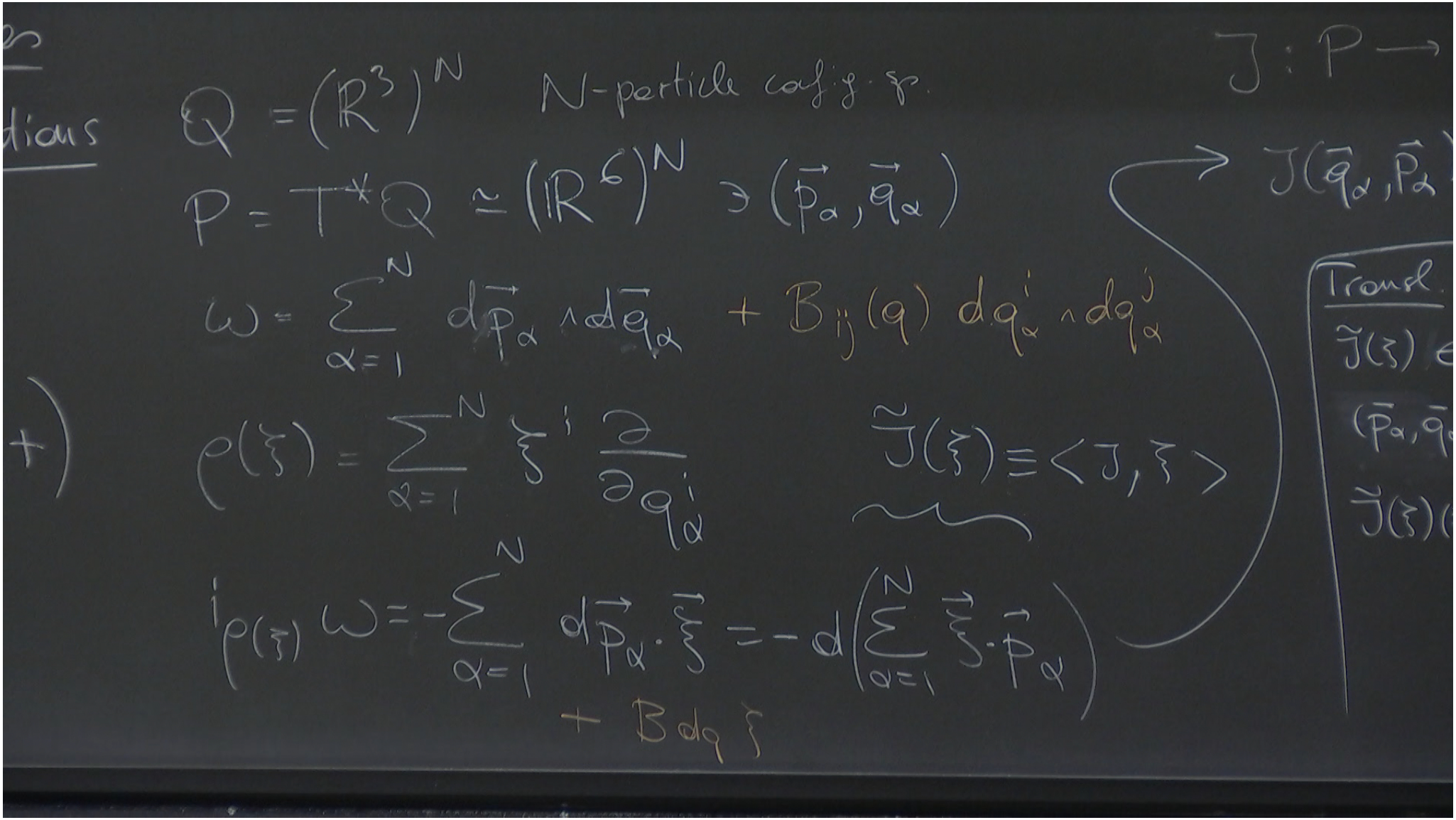
$= \xi^i \epsilon_i{}^j{}_k \sum_\alpha (p_\alpha)_j (q_\alpha)^k$

$= \mathfrak{so}(3) \simeq \mathbb{R}^3$ basis $\{e_i\}$

$\Rightarrow \xi = \xi^i e_i$

$^* \simeq \mathbb{R}^3$ basis $\{f^i\}$ $\langle f^i, e_j \rangle = \delta^i_j$

$J^* \ni J = J_i f^i$ $\langle J, \xi \rangle = J^i \xi_i$


Q = (R^3)^N N-particle config. sp.
P = T*Q ≃ (R^6)^N ∋ (p_α, q_α)
ω = Σ_{α=1}^N dp_α ∧ dq_α + B_ij(q) dq^i_α ∧ dq^j_α
ρ(ξ) = Σ_{α=1}^N ξ^i ∂/∂q^i_α
J̃(ξ) ≡ ⟨J, ξ⟩
i_{ρ(ξ)} ω = -Σ_{α=1}^N dp_α · ξ = -d(Σ_{α=1}^N ξ · p_α)
+ B dq ξ
J : P →
J(q_α, p_α)
Transl.
J̃(ξ) ∈
(p_α, q
J̃(ξ)(

$$\tilde{J}(\xi)(z) = \tilde{J}(\xi)(p_\alpha, q_\alpha)$$

$$= \xi^i \epsilon_i{}^j{}_k \sum_\alpha (p_\alpha)_j (q_\alpha)^k$$

$\times \vec{q}_\alpha$

basis $\{e_i\}$

$\{f^i\}$ $\langle f^i, e_j \rangle = \delta^i_j$

$\langle J, \xi \rangle = J^i \xi_i$

Ex

$P = S^2 \ni (\theta, \varphi)$

ω = Area 2-form on round sph

$= d\cos\theta \wedge d\varphi$

$G = SO(2)$ = rotations around z-axis

$\rho(\xi) = \xi \frac{\partial}{\partial \varphi}$

Rmk J is momentum map; $P \to \mathfrak{g}^*$

and $\lambda \in \mathfrak{g}^*$ (does not depend on $z \in P$)

then $J(z) + \lambda$ is also a momentum map.

Pf: $d(J(z)+\lambda) = dJ(z)$ ☑

$-L_{\rho(\eta)}$

EQUIVARIANCE ("covariance")

$L_{\rho(\eta)}(*) = 0$

$$-L_{\rho(\eta)}\, d\tilde{J}(\xi) = -d L_{\rho(\eta)}\tilde{J}(\xi) \equiv -d\langle L_{\rho(\eta)} J, \xi\rangle$$

$$\| $$

$$L_{\rho(\eta)} i_{\rho(\xi)}\omega = i_{[\rho(\eta),\rho(\xi)]}\omega + i_{\rho(\xi)} \underbrace{L_{\rho(\eta)}\omega}_{=0}$$

$$= i_{\rho([\eta,\xi])}\omega \overset{(*)}{=} -d\tilde{J}([\eta,\xi]) \equiv -d\langle J, [\eta,$$

$$\Rightarrow d\left(L_{\rho(\eta)}\tilde{J}(\xi) - \tilde{J}([\eta,\xi])\right) = 0$$

$$\text{iff } d\left(L_{\rho(\eta)} J + \mathrm{ad}^*_\eta J\right) = 0$$

DEF (Equivariance)

J mom. map is equiv.

iff. $L_{\rho(\gamma)} J = -\mathrm{ad}_\gamma^* J$

Ex rot. $L_{\rho(\gamma)} \vec{J} = \vec{\gamma} \times \vec{J}$

Rmk if J is equiv. $J(z) + \lambda$ is not

iff $d\left(L_{\rho(\eta)} J + \mathrm{ad}^*_\eta J\right) = 0$

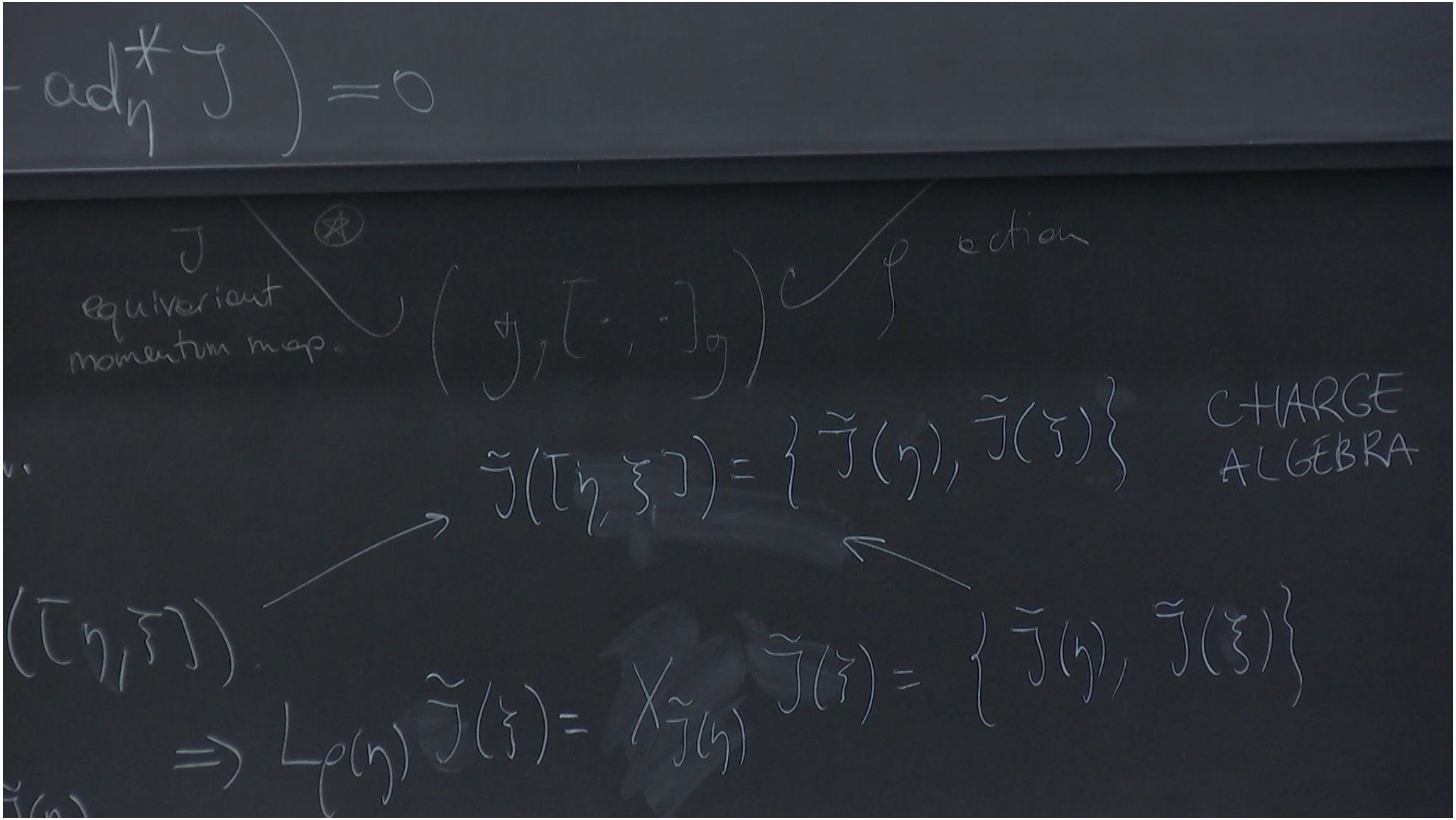
J equivariant momentum map. ⊛

$(g, [\cdot,\cdot]_g)$

Pf ⊛:

$L_{\rho(\eta)} \tilde{J}(\xi) \overset{\text{equiv.}}{=} \tilde{J}([\eta, \xi])$

since $\rho(\eta) = X_{\tilde{J}(\eta)} \Rightarrow L_{\rho(\eta)} \tilde{J}(\xi) = X_{\tilde{J}(\eta)} \tilde{J}(\xi) = \{$


$-\mathrm{ad}^*_\eta J) = 0$
J
equivariant
momentum map.
$(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$
action
$\tilde{J}([\eta,\xi]) = \{\tilde{J}(\eta), \tilde{J}(\xi)\}$
CHARGE
ALGEBRA
$([\eta,\xi])$
$\Rightarrow L_{\rho(\eta)} \tilde{J}(\xi) = X_{\tilde{J}(\eta)}$
$\tilde{J}(\xi) = \{\tilde{J}(\eta), \tilde{J}(\xi)\}$