Title: Quantum Gravity Lecture (230404)

Speakers: Aldo Riello

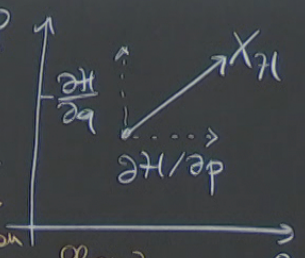
Collection: Quantum Gravity (2022/2023)

Date: April 04, 2023 - 9:00 AM

URL: https://pirsa.org/23040018

RECAP [w/ Errata (*)]

- Review Lagrangian + Hamiltonian Mech
- Recalled def of Poisson bracket (in canonical coords)



$$\{f,g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} \quad (*)$$

- Defined Symplectic mfd (aka "phase space")
  - P: $2n$-dim mfd
  - $\omega \in \Omega^2(P)$: $\begin{cases} \text{non-deg.} & (\ker \omega^\flat = \{0\}) \\ \text{closed} & (d\omega = 0) \end{cases}$

- $P = T^*Q$ canon
  $\Theta = \sum_i p_i dq$
  $\omega = d\Theta =$
- Darboux thm
  (P, ω) sympl
  "looks like"
  i.e. locally $\exists$
  s.t. $\omega = \frac{1}{2}$
- Related sym
  back to Pois

- $P = T^*Q$ canonically sympl
  $\theta = \sum_i p_i dq^i$
  $\omega = d\theta = \sum_i dp_i \wedge dq^i$
- Darboux thm (in OPEN neighborhoods)
  (P, ω) sympl **locally**
  "looks like" a cotangent bdle
  i.e. locally $\exists$ coords $z^I = (q^i, p_i)$
  s.t. $\omega = \frac{1}{2} \omega_{IJ} dz^I \wedge dz^J = dp_i \wedge dq^i$
- Related symplectic geometry back to Poisson bracket

(*) $\pi^{IJ}(z) = -(\omega_{IJ}(z))^{-1}$

$\pi = \frac{1}{2} \pi^{IJ} \partial_I \wedge \partial_J \in \mathfrak{X}^2(P)$

$\{f, g\}_\pi := \pi^{IJ} \frac{\partial f}{\partial z^I} \frac{\partial g}{\partial z^J}$

is Poisson $\Leftrightarrow$ skew, biderivation, Jacobi $\Leftrightarrow$ ($d\omega = 0$)

(Errata)

Prop if $H^2_{\text{deRham}}(P)$ is trivial,
then $\exists \theta \in \Omega^1(P) : \omega = d\theta$

RMK1 θ not unique

RMK2 counterex: $(P, \omega) = (S^2, \text{vol}_{S^2})$

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HAMILTONIAN V.F.

$\{f,\cdot\} = (\pi^{IJ}\partial_I f)\partial_J =: X_f$

$X_f \in \mathfrak{X}^1(P)$ HVF of $f \in C^\infty(P)$

$\rightarrow$ $\boxed{i_{X_f}\omega = -df}$ $(*)$

Def $X \in \mathfrak{X}^1(P)$ is Hamilt.

iff $\exists f_X \in C^\infty(P)$ s.t. $(*)$

(Restrictive!)

(closed $(d\omega = 0)$

PROP X HVF, then

$$L_X \omega = 0$$

Pf: $L_X \omega = i_X d\omega + d i_X \omega$

(Cartan's)

$= 0 \quad - d^2 f = 0$

ω closed $\quad d^2 = 0$

Corollary X HVF $\Rightarrow L_X d\mu_L = 0$

(Liouville thm)

(closed $(d\omega = 0)$

RMK1 θ not uni
RMK2 counterex

F, then

0

$i_X d\omega + d i_X \omega$

tan's)

$0 \quad - d^2 f = 0$

closed $d^2 = 0$

$F \Rightarrow L_X d\mu_L = 0$

Examples

$\mathcal{L}(q,\dot{q}) = \frac{1}{2}\dot{\vec{q}} - \vec{A}(q)\cdot\dot{\vec{q}} - V(q)$

$(V, \vec{A})$ = external EM field

EL eqn $\ddot{\vec{q}} + \underbrace{\dot{\vec{q}} \times \vec{B} + \vec{\nabla} V = 0}_{\text{Lorentz f.}} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$

Canonical mom: $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{\vec{q}} - \vec{A}(q)$

$\mathcal{H}(p,q) = \frac{1}{2}(\vec{p} + \vec{A}(q))^2 + V(\vec{q})$

H. eqn

$\begin{cases} \dot{\vec{q}} = \vec{p} + \\ \dot{\vec{p}} = -\vec{\nabla} \end{cases}$

$\omega = d\vec{p}$

Change of

$\vec{v} = \vec{p}$

$\tilde{\mathcal{H}}(\vec{v},$

RMK1 ... not unique

RMK2 counterex: $(P,\omega)=(S^2, \mathrm{vol}_{S^2})$ | (Restrictive!)

H. eom:

$$\begin{cases} \dot{\vec{q}} = \vec{p} + \vec{A} \\ \dot{\vec{p}} = -\vec{\nabla} V - \sum_j (p^j + A^j(q)) \vec{\nabla} A_j(q) \end{cases}$$

$$\omega = d\vec{p} \wedge d\vec{q}$$

$\vec{A}$

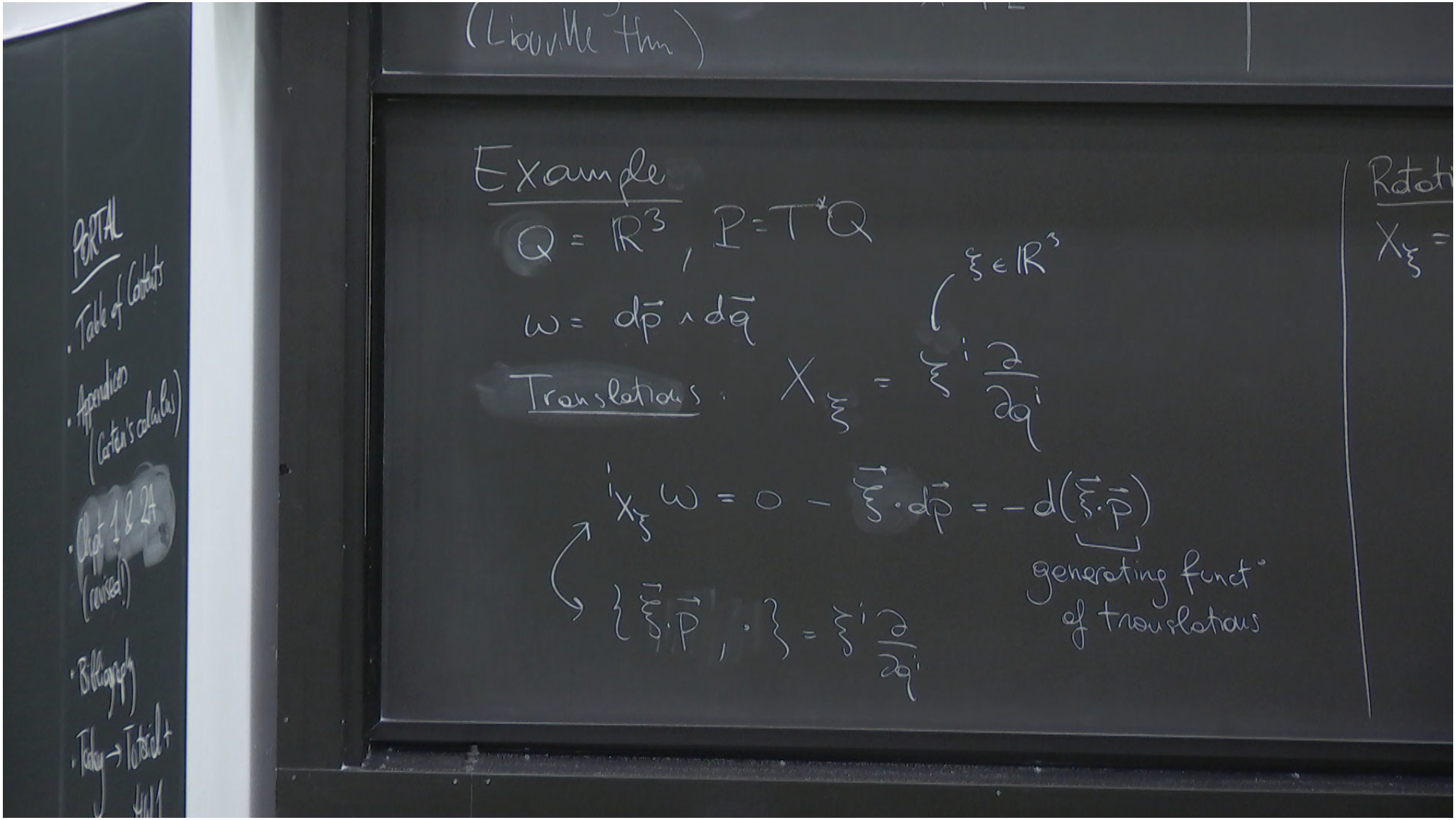
Change of coords on P:

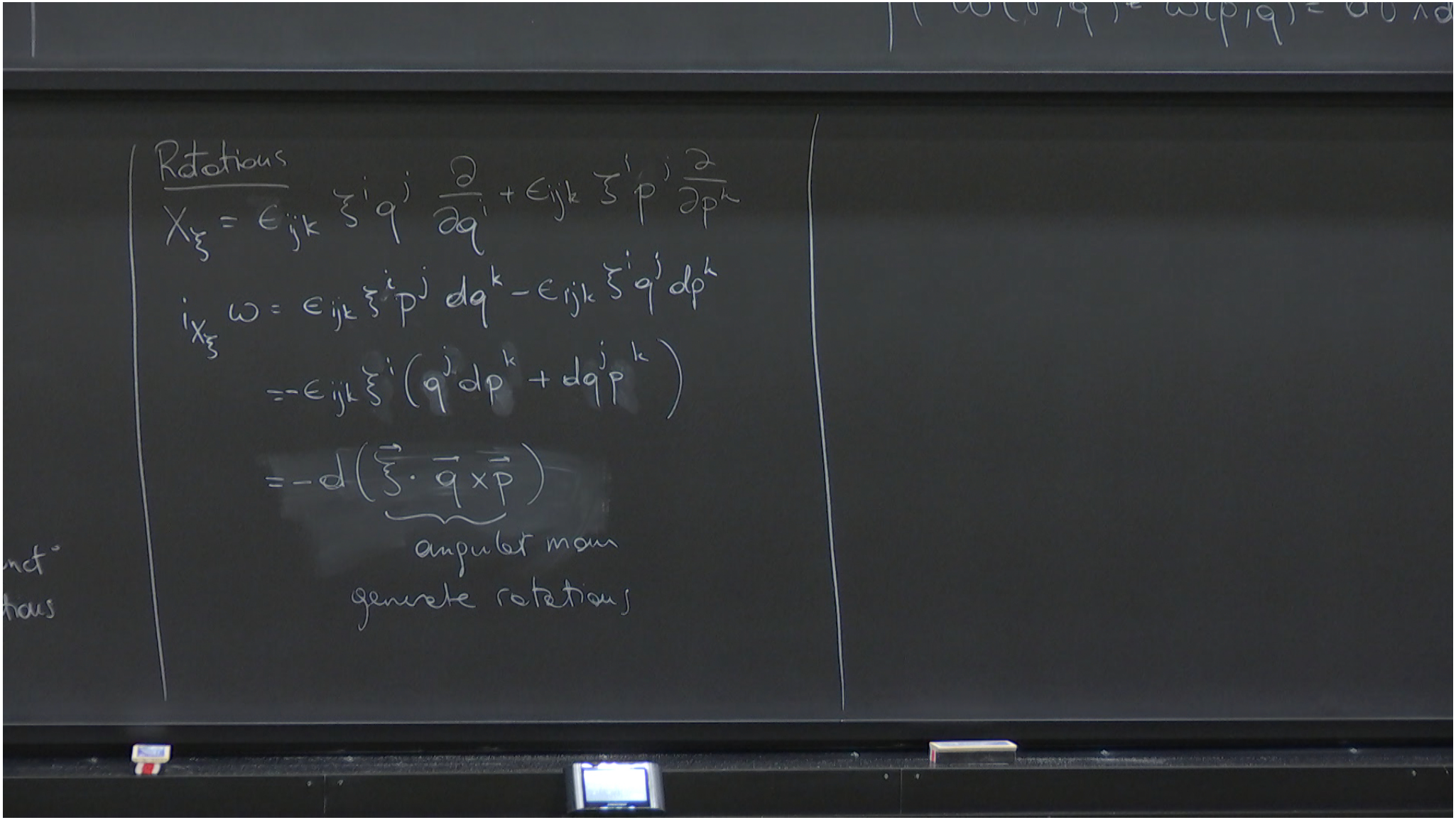
$$\vec{v} = \vec{p} + \vec{A}(q)$$

no magnetic potential

$$i_{\tilde{X}_{\tilde{H}}} \tilde{\omega} = -d\tilde{H}$$

$$\begin{cases} \tilde{H}(\vec{v}, \vec{q}) = H(p,q) = \frac{1}{2}\vec{v}^2 + V(\vec{q}) \\ \tilde{\omega}(\vec{v}, \vec{q}) = \omega(p,q) = d\vec{v} \wedge d\vec{q} - \frac{1}{2} B_{ij}(q)\, dq^i \wedge dq^j \end{cases}$$


(Liouville thm)
Example
$Q = \mathbb{R}^3$, $P = T^*Q$
$\omega = d\vec{p} \wedge d\vec{q}$
Translations: $X_\xi = \xi^i \frac{\partial}{\partial q^i}$ $(\xi \in \mathbb{R}^3)$
$i_{X_\xi}\omega = 0 - \vec{\xi} \cdot d\vec{p} = -d(\vec{\xi} \cdot \vec{p})$
generating funct° of translations
$\{\vec{\xi} \cdot \vec{p}, \cdot\} = \xi^i \frac{\partial}{\partial q^i}$
PORTAL
• Table of Contents
• Appendices (Cartan's calculus)
• Chpt 1 & 2A (revised)
• Bibliography
• Today → Tutorial
Rotati
$X_\xi =$


Rotations
$X_\xi = \epsilon_{ijk}\,\xi^i q^j \frac{\partial}{\partial q^i} + \epsilon_{ijk}\,\xi^i p^j \frac{\partial}{\partial p^k}$
$i_{X_\xi}\omega = \epsilon_{ijk}\,\xi^i p^j\, dq^k - \epsilon_{ijk}\,\xi^i q^j\, dp^k$
$= -\epsilon_{ijk}\,\xi^i \left(q^j dp^k + dq^j p^k\right)$
$= -d\left(\vec{\xi}\cdot\vec{q}\times\vec{p}\right)$
angular mom
generate rotations

SYMMETRIES

PROP $[X_f, X_g] = X_{\{f,g\}}$

↑ {Lie bracket btw VF on ph sp. P (Homomorph of Lie alg.)

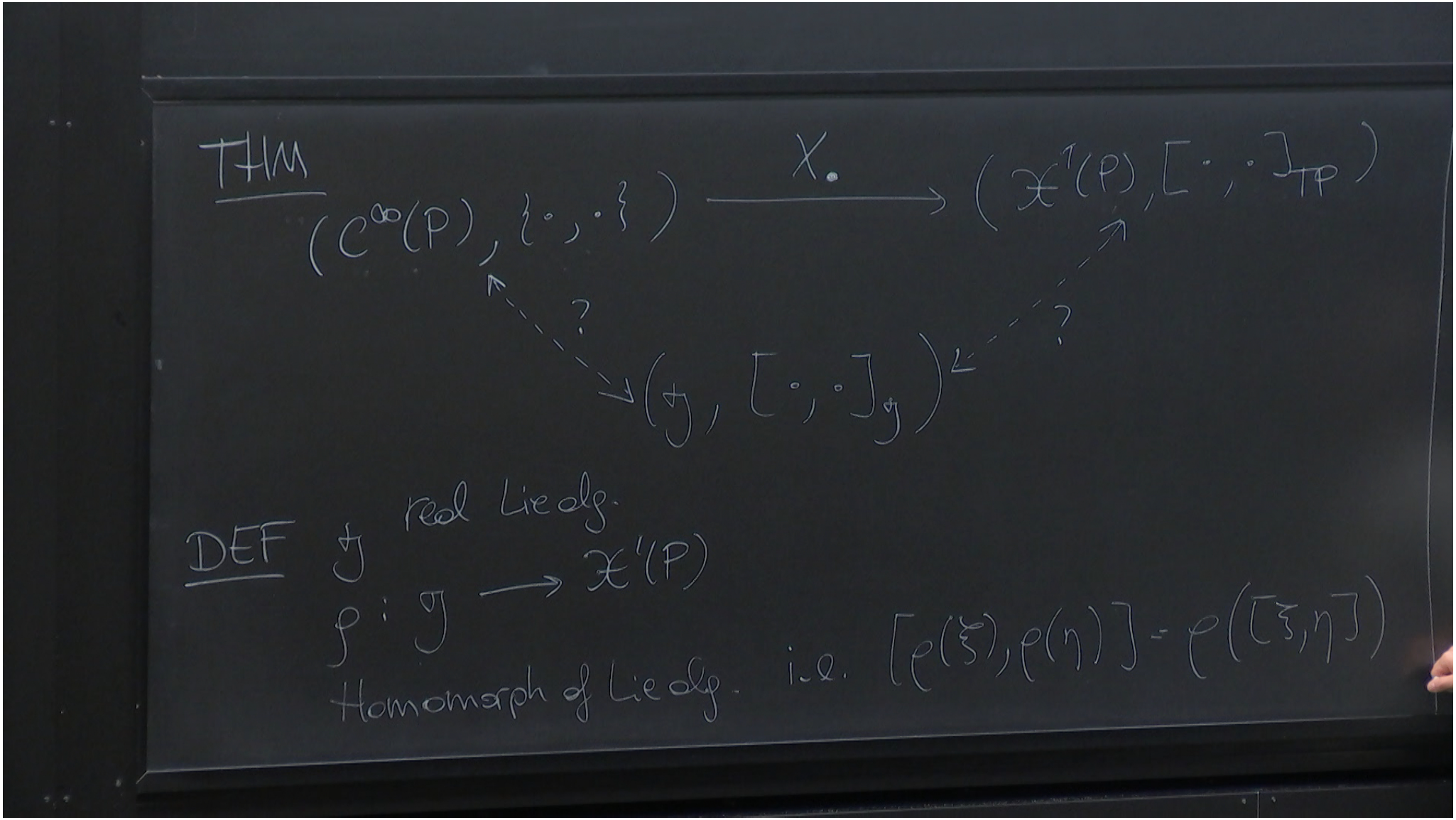
$L_{X_f} g = X_f(g) = \{f,g\}$

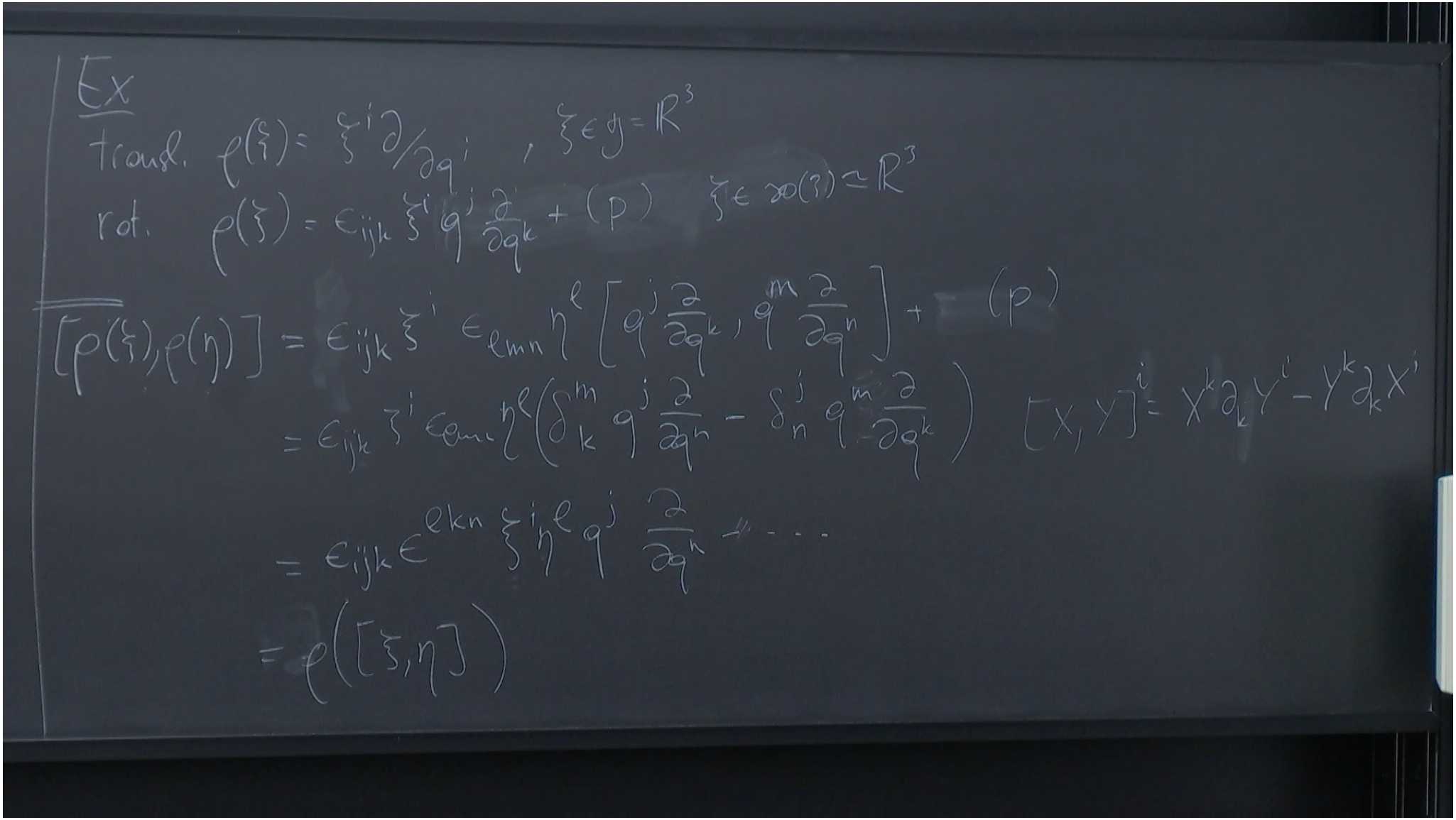
Pf: $L_{X_f}\left(i_{X_g}\omega + dg \right) = 0$

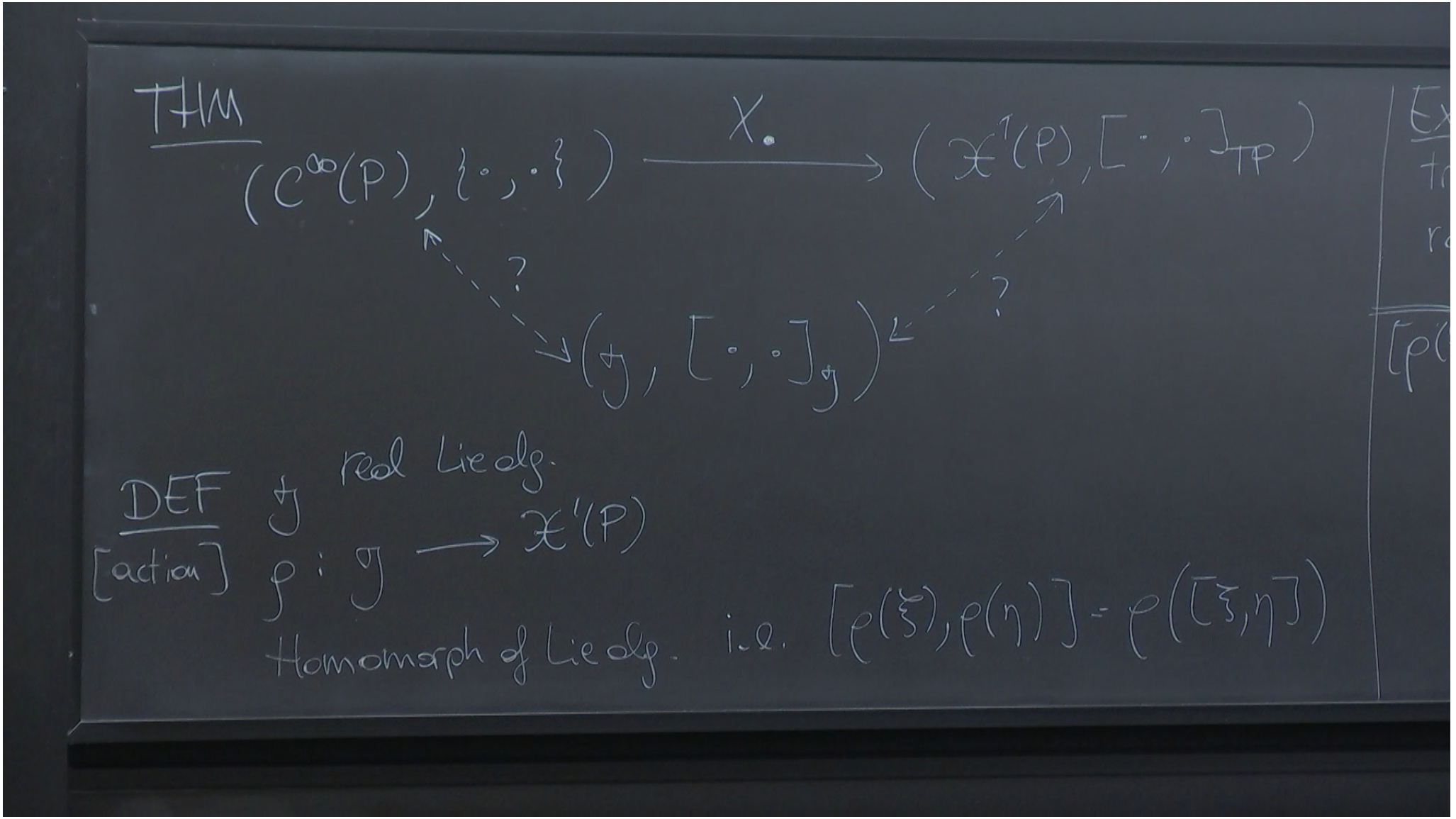
$i_{[X_f,X_g]}\omega + i_{X_g} \underbrace{L_{X_f}\omega}_{=0} + d(L_{X_f} g) = 0$

$i_{[X_f,X_g]}\omega = -d\{f,g\}$




THM
$(C^\infty(P), \{\cdot,\cdot\})$
$X_\bullet$
$(\mathfrak{X}^1(P), [\cdot,\cdot]_{TP})$
?
$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$
?
DEF $\mathfrak{g}$ real Lie alg.
$\rho : \mathfrak{g} \longrightarrow \mathfrak{X}^1(P)$
Homomorph of Lie alg. i.e. $[\rho(\xi), \rho(\eta)] = \rho([\xi,\eta])$


Ex
transl. $\rho(\xi) = \xi^i \partial/\partial q^i$, $\xi \in \mathfrak{g} = \mathbb{R}^3$
rot. $\rho(\xi) = \epsilon_{ijk} \xi^i q^j \frac{\partial}{\partial q^k} + (p)$ $\xi \in \mathfrak{so}(3) \simeq \mathbb{R}^3$
$[\rho(\xi), \rho(\eta)] = \epsilon_{ijk} \xi^i \epsilon_{\ell mn} \eta^\ell \left[q^j \frac{\partial}{\partial q^k}, q^m \frac{\partial}{\partial q^n} \right] + (p)$
$= \epsilon_{ijk} \xi^i \epsilon_{\ell mn} \eta^\ell \left(\delta^m_k q^j \frac{\partial}{\partial q^n} - \delta^j_n q^m \frac{\partial}{\partial q^k} \right)$
$[X, Y]^i = X^k \partial_k Y^i - Y^k \partial_k X^i$
$= \epsilon_{ijk} \epsilon^{\ell kn} \xi^i \eta^\ell q^j \frac{\partial}{\partial q^n} + \dots$
$= \rho([\xi, \eta])$


THM
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$(\mathfrak{X}^1(P), [\cdot,\cdot]_{TP})$
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?
$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$
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PROP

$(\mathfrak{g}, \rho)$ is an action, then

$\Rightarrow \mathrm{Im}(\rho)$ is integrable (Frobenius)

and therefore ρ is tangent to

submanifolds called "orbits" O_z

$= \{f, g\}$

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$\circledast$ $\rho : P \times \mathfrak{g} \longrightarrow TP$

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s called "orbits" O_z

$\rightarrow TP$

