Title: Chromatic aberrations of the geometric Satake equivalence

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Series: Mathematical Physics

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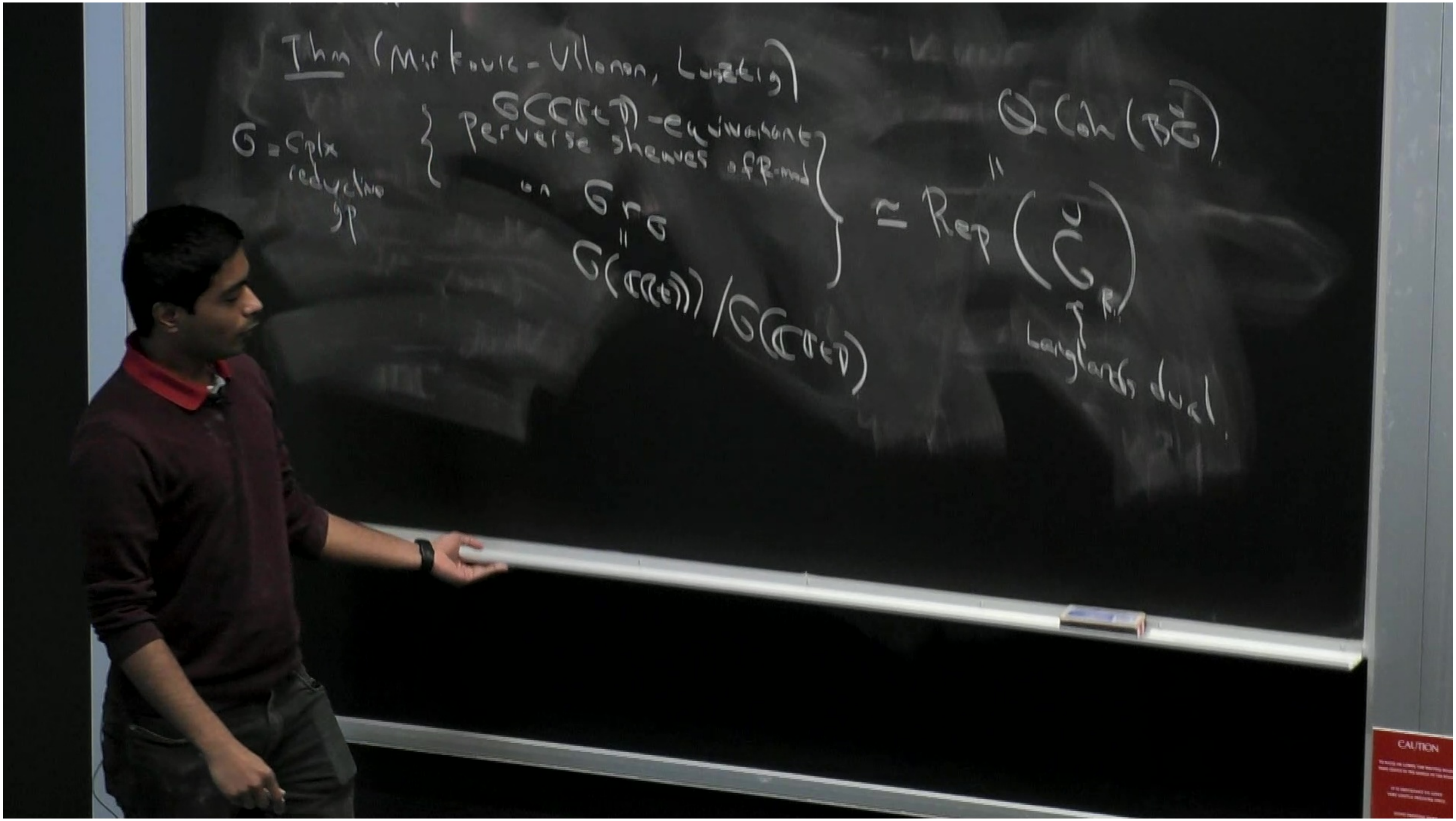
Abstract: The (derived) geometric Satake equivalence plays a central role in the geometric Langlands program: roughly, it describes the category of constructible sheaves of C-vector spaces on Bun_G(S^2) in terms of the Langlands dual group G^. In this talk, I will describe some ideas connecting chromatic homotopy theory to the derived geometric Satake equivalence. For example, we will describe the category of locally constant sheaves of A-modules on Bun_G(S^2), where A is complex K-theory or an elliptic cohomology theory, in Langlands
dual terms. Some of this work was motivated by considerations from physics, and I hope to say what little I know about this, as well as sketch its relationship to the Ben-Zvi-Sakellaridis-Venkatesh program.

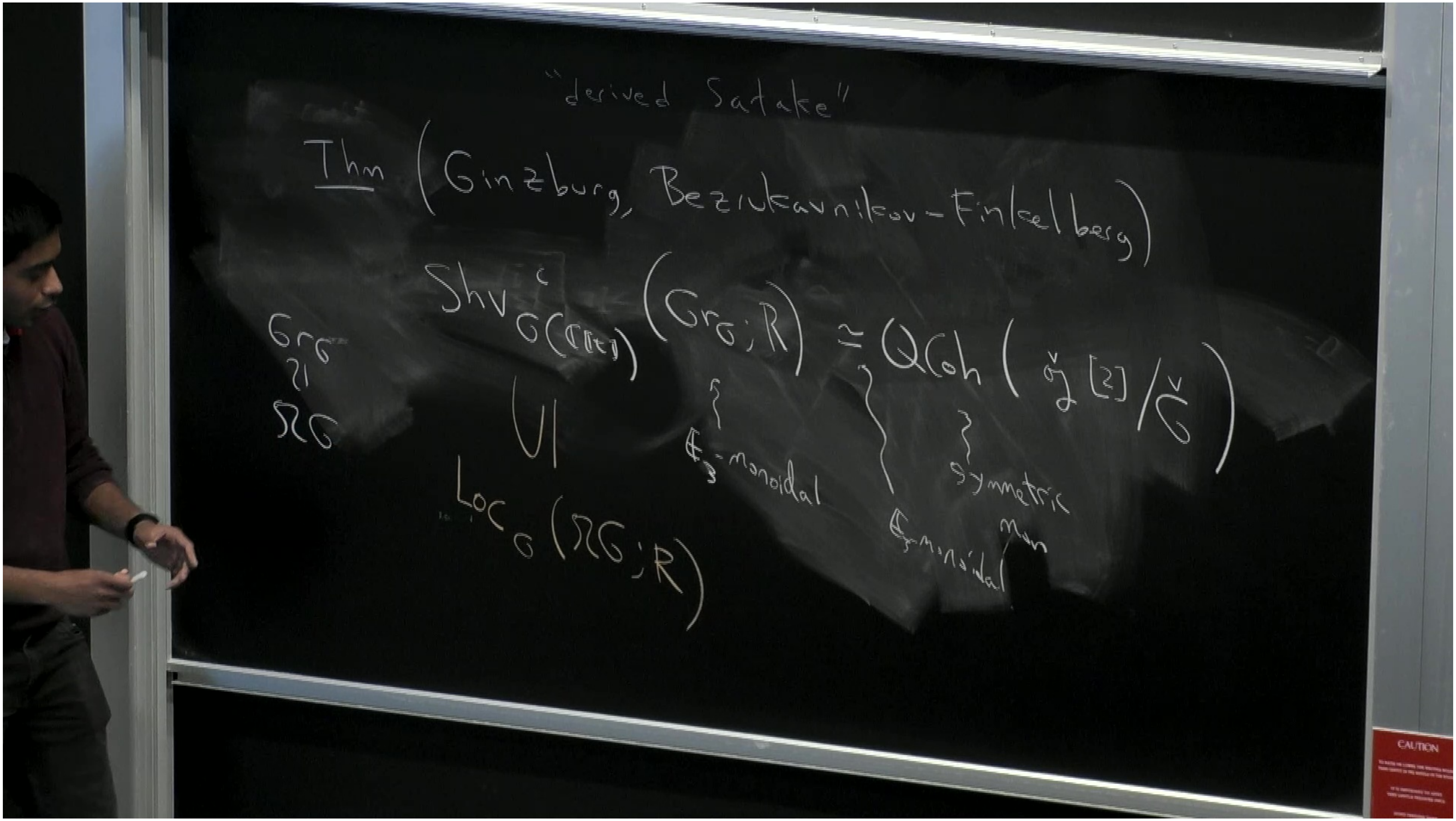
Zoom Link: https://pitp.zoom.us/j/94926220665?pwd=bVZFUFlvZGxVSG0xUFc1SGNaTDBKZz09


Thm (Mirkovic-Vilonen, Lusztig)
G = cplx reductive gp
G(C[[t]])-equivariant
Perverse sheaves of R-mod
on Gr_G
G(C((t)))/G(C[[t]])
≃ Rep
Coh
Langlands dual

"derived Satake"

<u>Thm</u> (Ginzburg, Bezrukavnikov–Finkelberg)

$$\underbrace{\mathrm{Shv}^c_{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G;\mathbb{R})}_{\mathbb{E}_3\text{-monoidal}} \underbrace{\simeq}_{\mathbb{E}_3\text{-monoidal}} \underbrace{\mathrm{QCoh}(\check{\mathfrak{g}}[2]/\check{G})}_{\text{symmetric mon}}$$


"derived Satake"
Thm (Ginzburg, Bezrukavnikov-Finkelberg)
Shv^c_{G(C[[t]])}(Gr_G; R) ≃ QCoh(ǧ[2]/Ǧ)
Gr_G ≃ ΩG
Loc_G(ΩG; R)
E_3-monoidal
E_2-monoidal
symmetric mon

"derived Satake"

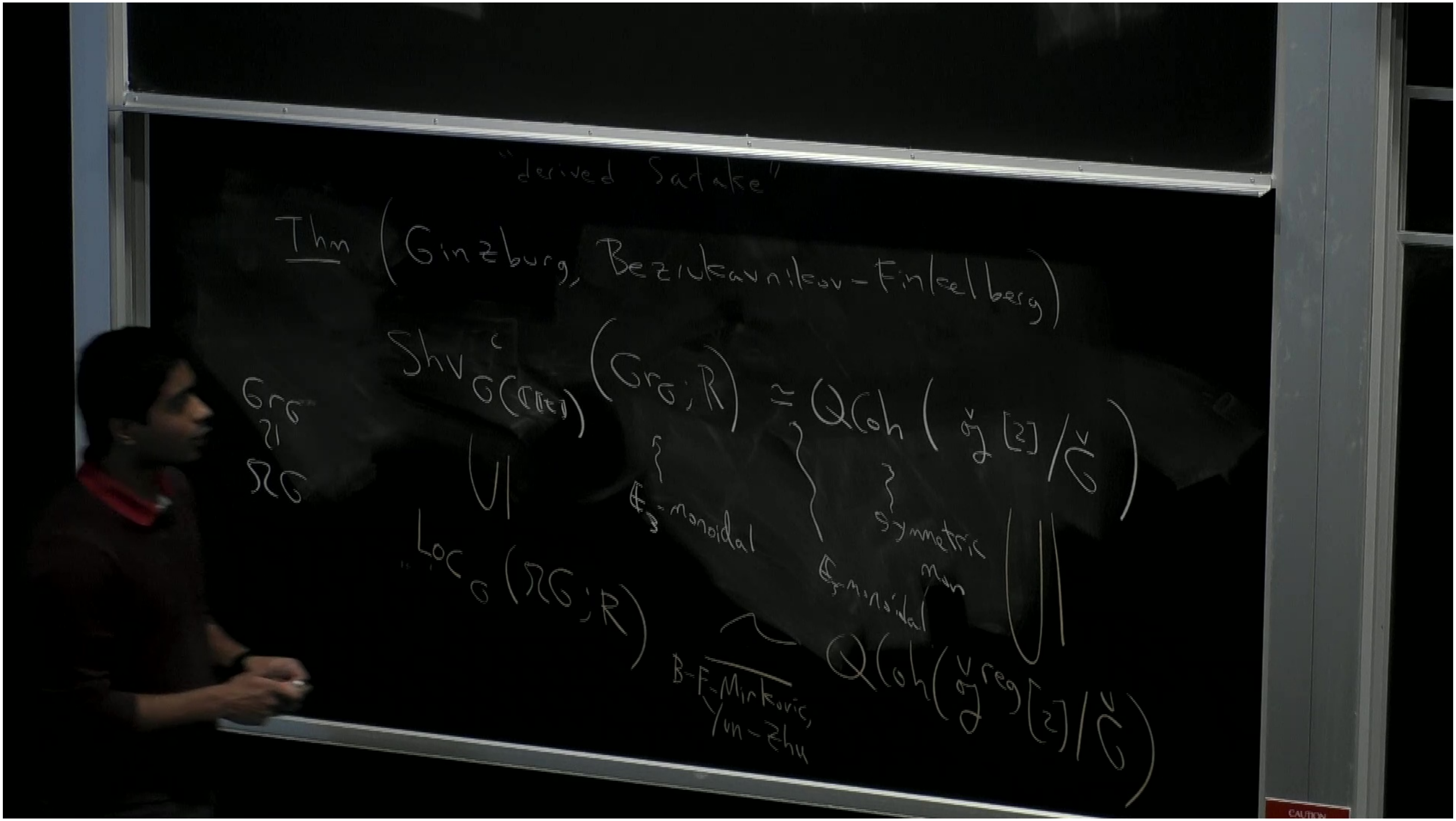
<u>Thm</u> (Ginzburg, Bezrukavnikov–Finkelberg)

$$Gr_G \simeq \Omega G$$

$$\underbrace{Shv^{c}_{G(\mathbb{C}[[t]])}(Gr_G; R)}_{E_3\text{-monoidal}} \simeq \underbrace{QCoh(\check{\mathfrak{g}}[2]/\check{G})}_{\text{symmetric mon}}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\underbrace{Loc_G(\Omega G; R)}_{} \simeq \underbrace{QCoh(\check{\mathfrak{g}}^{reg}[2]/\check{G})}_{E_3\text{-monoidal}}$$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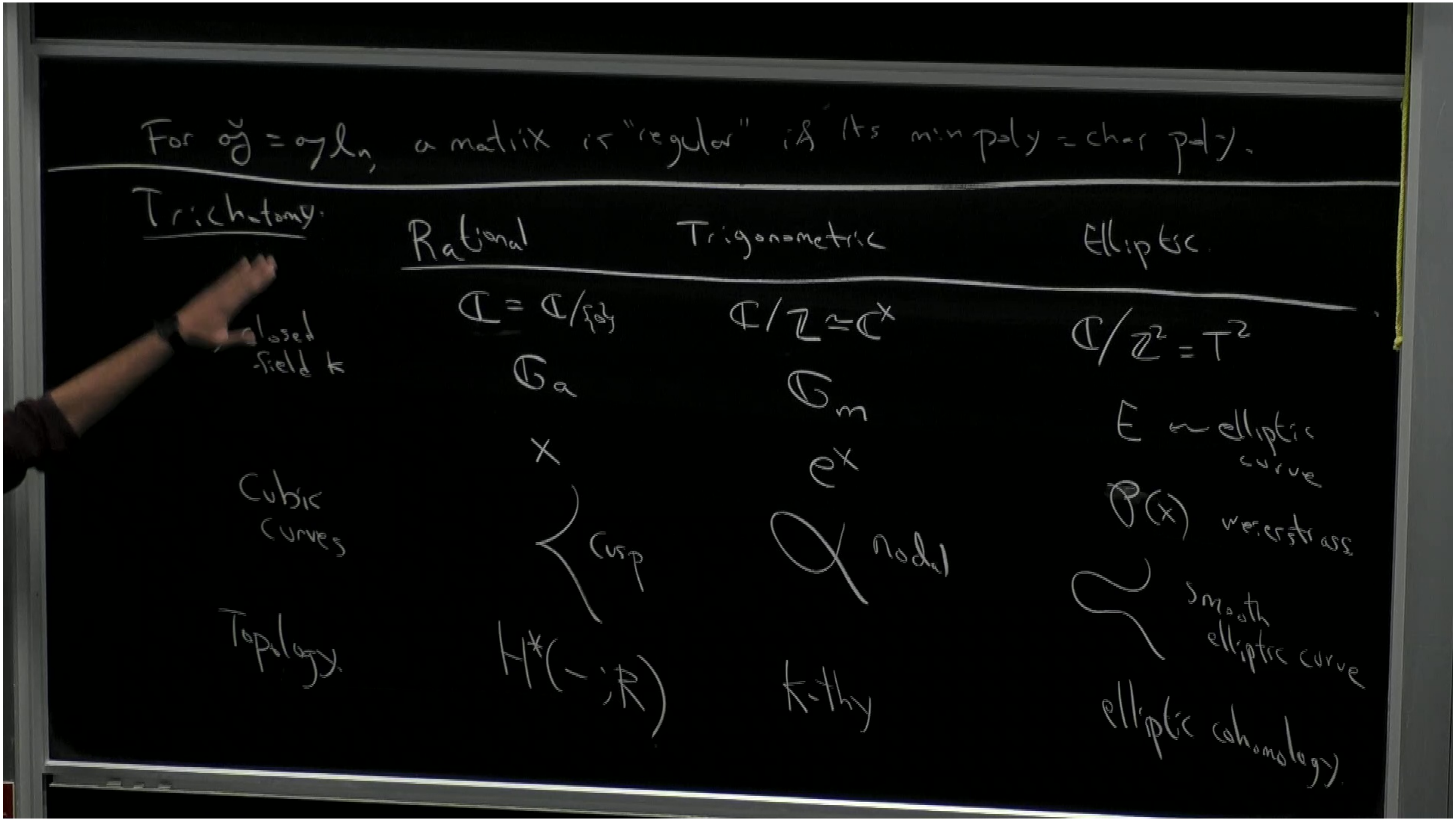
Loc_G(ΩG; R)
QCoh(ǧ^reg[2]/Ǧ)
B-F-Mirkovic, Yun-Zhu

For $\mathfrak{g} = \mathfrak{gl}_n$, a matrix is "regular" if its min poly = char poly.

Trichotomy.

	Rational	Trigonometric	Elliptic
	$\mathbb{C} = \mathbb{C}/\{0\}$	$\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^\times$	$\mathbb{C}/\mathbb{Z}^2 = T^2$
alg closed field k	$\mathbb{G}_a$	$\mathbb{G}_m$	E ~ elliptic curve
	x	e^x	$\wp(x)$ weierstrass
Cubic curves	cusp	nodal	smooth elliptic curve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Topology
$H^*(-;\mathbb{R})$
K-thy
elliptic cohomology

cohomology.

A cohomology theory.
w/ Chern classes
(cplx-oriented)

t = analogue of $c_1(\mathcal{O}(1))$

$$A^*(\mathbb{CP}^\infty) = A_*[[t]]$$

$$\begin{array}{ccc} \mathbb{CP}^\infty \times \mathbb{CP}^\infty & & A_*[[t_1, t_2]] \\ \downarrow \otimes & & \uparrow \\ \mathbb{CP}^\infty & & A_*[[t]] \end{array}$$

$$\mathrm{Spf}\, A^*(\mathbb{CP}^\infty) = \hat{\mathbb{G}}$$

$$\begin{array}{c} \hat{\mathbb{G}} \times \hat{\mathbb{G}} \\ \downarrow \\ \hat{\mathbb{G}} \end{array} \rightsquigarrow \text{addition law on } \hat{\mathbb{G}}$$

"formal group law".

- If $A = H^*(-;R)$,

$$\hat{\mathbb{G}} = (\mathbb{G}_a)^\wedge_0 \leftarrow \text{formal nbhd.}$$

- If $A = K\text{-thy}$,

$$\hat{\mathbb{G}} = (\mathbb{G}_m)^\wedge_1$$

- $A = Ell_E$, $\hat{\mathbb{G}} = E^\wedge_{zero}$

- If $A = H^*(-;R)$,
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- $A = Ell_E$, $\hat{\mathbb{G}} = \underline{E}^\wedge_{\text{zero}}$

In these cases, using $\mathbb{G}_a$, $\mathbb{G}_m$, E, $= \mathbb{G}$ one can define

"genuine S^1-equivariant A-cohomology"

$$A^*_{S^1}(-)$$

Analogue for hmlgy: $A^{S^1}_*(-)$.

$$A^{S^1}_*(pt) = \left\{ \begin{matrix} \text{fns on} \\ \mathbb{G} \end{matrix} \right\}$$

LHS of derived Satake

$$\mathrm{Shv}_{\sim}(\sim; R)$$

categorification of

$$H^*_{\sim}(\sim; R)$$

RHS of derived Satake

$$\mathrm{QCoh}(\underbrace{\check{\mathfrak{g}} / \check{G}}_{T})$$

moduli stack $\mathrm{Bun}^{\circ}_{\check{G}}(\ \ )$

<u>Thm</u> (D.) A coh thy = K-thy, or Ell_E

Then

$$\mathrm{Loc}_G(\Omega G$$

LHS of derived Satake

$$Shv_{\sim}(\sim; R)$$

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RHS of derived Satake

$$QCoh(\underbrace{\check{\mathfrak{g}}/\check{G}}_{T})$$

moduli stack $Bun^{0}_{\check{G}}(\prec)$

Thm (D.) A coh thy $= k$-thy or Ell_E

Then

$$\mathbb{Q} \otimes Loc_G(\Omega G; A) = \begin{cases} QCoh(\check{G}^{reg}/\check{G}) & A = k\text{-thy} \\ QCoh(Bun^{ss,0}_{\check{G}}(\check{E})) & A = Ell_E \end{cases}$$

$$\mathrm{Loc}_G(\Omega G; A) = \mathrm{coMod}_{A^G_*(\Omega G)}$$

· If $A = H^*(-;R)$, becomes $H^G_*(\Omega G; R)$.

Bezrukavnikov–Finkelberg–Mirković, Yun–Zhu

(If $R = \mathbb{C}$ Coulomb branch of 3d $N=4$ pure gauge theory)

· If $A = K$-theory, was also done by BFM.

BFM: $\operatorname{Spec} H^{G}_{*}(\Omega G;\mathbb{C})$

regular centralizers

$$\xrightarrow{\simeq} \left\{ (g,x) \text{ where } g \in \check{G},\ x \in \check{\mathfrak{g}}^{reg},\ \mathrm{Ad}_g(x) = x \right\} /\!\!/ \check{G}$$

BFM: $\operatorname{Spec} K^{G}_{*}(\Omega G)$

$$\simeq \left\{ (g,x), \text{ where } g \in \check{G},\ x \in \check{G}^{reg},\ \mathrm{Ad}_g(x) = x \right\} /\!\!/ \check{G}$$

Eg $\check{G} = SL_2$

$$\check{J} \cong \check{t}/\!/W \underset{\check{\mathfrak{g}}/\check{G}}{\times} \check{t}/\!/W$$

$\check{t} = \mathbb{A}^1 \circlearrowleft W = \mathbb{Z}/2 \rightsquigarrow \check{t}/\!/W = \mathbb{A}^1$

The map

$$\begin{tikzcd} \check{t}/\!/W \arrow[rr] \arrow[dr, "K"'] & & \check{\mathfrak{g}}/\check{G} \\ & \check{\mathfrak{g}} \arrow[ur] & \end{tikzcd}$$

K = Kostant slice

$$\mathbb{A}^1 \longrightarrow sl_2$$

$$\lambda \longmapsto \begin{pmatrix} 0 & -1 \\ \lambda & 0 \end{pmatrix}$$

In the elliptic case,

$\mathbb{Z}/2 \circlearrowright E$ by inversion

"$\operatorname{Spec} \operatorname{Ell}^{G}_{*}(\Omega G)$"

$\simeq \check{\mathfrak{J}}^{ell} = E/\!/(\mathbb{Z}/2) \times E/\!/(\mathbb{Z}/2)$

$E/\!/(\mathbb{Z}/2) \xrightarrow{k} \operatorname{Bun}^{0,ss}_{SL_2}(E)$

$k: E/\!/(\mathbb{Z}/2) \longrightarrow \operatorname{Bun}^{0}_{SL_2}(E)$

$$x \longmapsto \begin{cases} L_x \oplus L_{-x} & 2x \neq 0 \\ \mathcal{F}_2 \otimes L_x & 2x = 0 \end{cases}$$

$\mathcal{O}(x)$

Atiyah bdl

$\mathcal{O}_E \to \mathcal{F}_2 \to \mathcal{O}_E$

nonsplit

In the elliptic case,

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$\mathcal{O}_E \to \mathcal{F}_2 \to \mathcal{O}_E$

nonsplit

$\mathrm{Loc}_G(\Omega G; A) = \mathrm{coMod}\; A_*^G(\Omega G)$

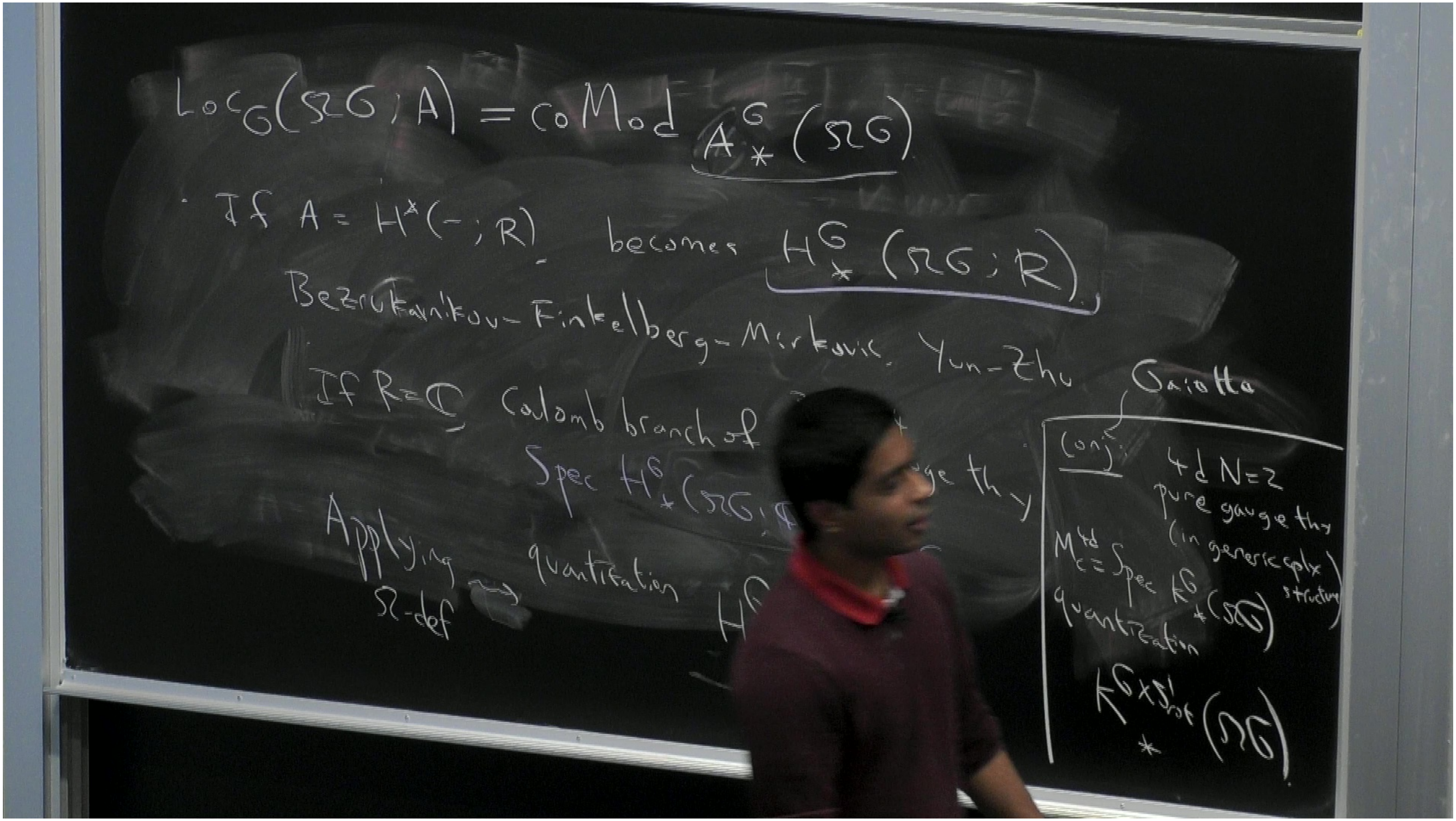
- If $A = H^*(-;R)$ becomes $H_*^G(\Omega G; R)$

Bezrukavnikov–Finkelberg–Mirkovic, Yun–Zhu

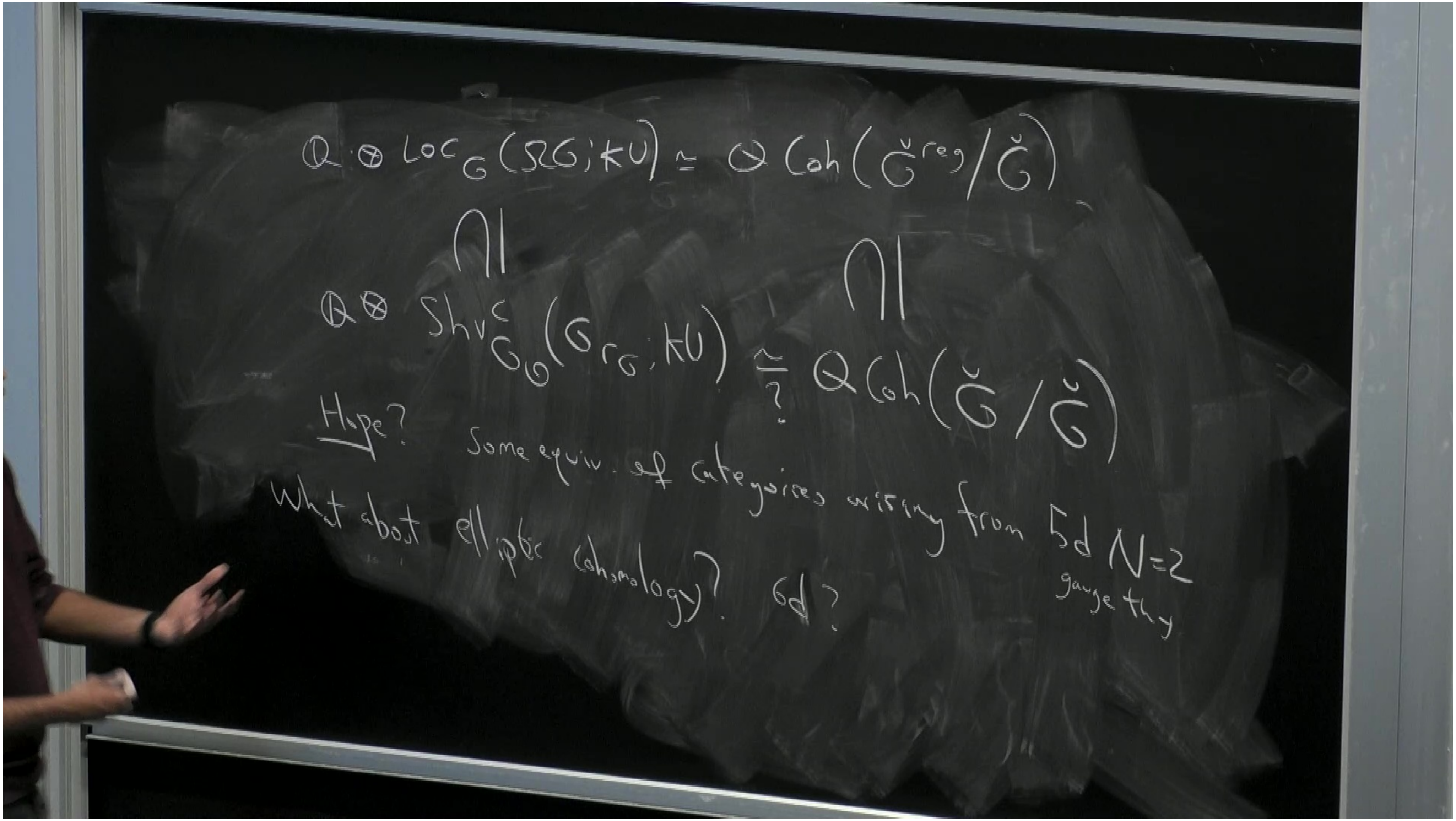
If $R=\mathbb{C}$ Coulomb branch of 3d $N=4$ pure gauge thy

$\mathrm{Spec}\; H_*^G(\Omega G;\mathbb{C}) = \mathcal{M}_C$

Applying Ω-def $\rightsquigarrow$ quantization $H_*^{G\times S^1_{rot}}(\Omega G;\mathbb{C})$



$Loc_G(\Omega G; A) = coMod\ A^G_*(\Omega G)$
If $A = H^*(-;R)$ becomes $H^G_*(\Omega G; R)$
Bezrukavnikov–Finkelberg–Mirkovic, Yun–Zhu
If $R = \mathbb{C}$ Coulomb branch of
Spec $H^G_*(\Omega G, \mathbb{C})$
Applying
Ω-def
quantization
Gaiotto
Conj:
4d N=2 pure gauge thy (in generic cplx structure)
$M^{4d}_C = \mathrm{Spec}\ K^G_*(\Omega G)$
quantization
$K^{G\times S^1_{rot}}_*(\Omega G)$



$\mathbb{Q} \otimes \mathrm{Loc}_G(\Omega G; KU) \simeq \mathbb{Q}\,\mathrm{Coh}(\check{G}^{reg}/\check{G})$
$\mathbb{Q} \otimes \mathrm{Shv}^c_{G_O}(\mathrm{Gr}_G; KU) \overset{?}{\simeq} \mathbb{Q}\,\mathrm{Coh}(\check{G}/\check{G})$
Hope? Some equiv. of categories arising from 5d N=2 gauge thy
What about elliptic cohomology? 6d?

$$\mathbb{Q}\otimes \mathrm{Loc}_G(\Omega G; KU) \simeq \mathbb{Q}\,\mathrm{Coh}(\check{G}^{reg}/\check{G})$$

$$\cap| \qquad\qquad \cap|$$

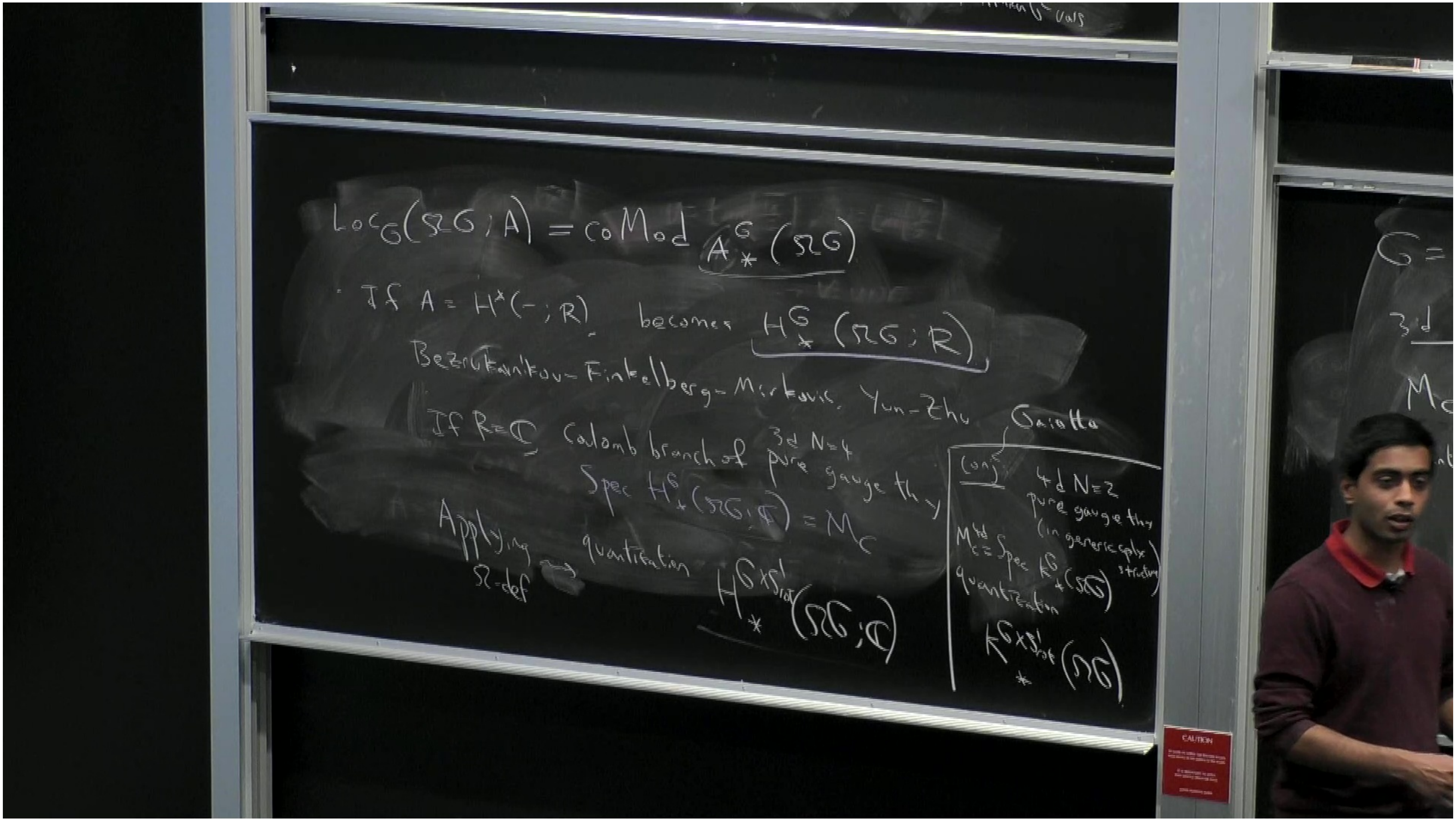
$$\mathbb{Q}\otimes \mathrm{Shv}^c_{G_\mathcal{O}}(\mathrm{Gr}_G; KU) \underset{?}{\simeq} \mathbb{Q}\,\mathrm{Coh}(\check{G}/\check{G})$$

Hope? Some equiv. of categories arising from 5d $N=2$ gauge thy.

— What about elliptic cohomology? 6d?

$\mathrm{Shv}^c_{G_\mathcal{O}}(\mathrm{Gr}_G;\mathbb{C}) = \mathbb{Q}\,\mathrm{Coh}(\check{\mathfrak{g}}/\check{G})$ from 4d $N=4$ $\rightsquigarrow$ duality between Hamiltonian G-vars and Hamiltonian $\check{G}$-vars

BZSV


Loc_G(ΩG; A) = coMod A^G_*(ΩG)
If A = H^*(−; R) becomes H^G_*(ΩG; R)
Bezrukavnikov–Finkelberg–Mirkovic, Yun–Zhu
If R = ℂ Coulomb branch of 3d N=4 pure gauge thy
Spec H^G_*(ΩG; ℂ) = M_C
Applying Ω-def quantization H^{G×S^1_rot}_*(ΩG; ℂ)
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Conj
4d N=2 pure gauge thy (in generic cplx structure)
M^{4d}_C = Spec K^G_*(ΩG)
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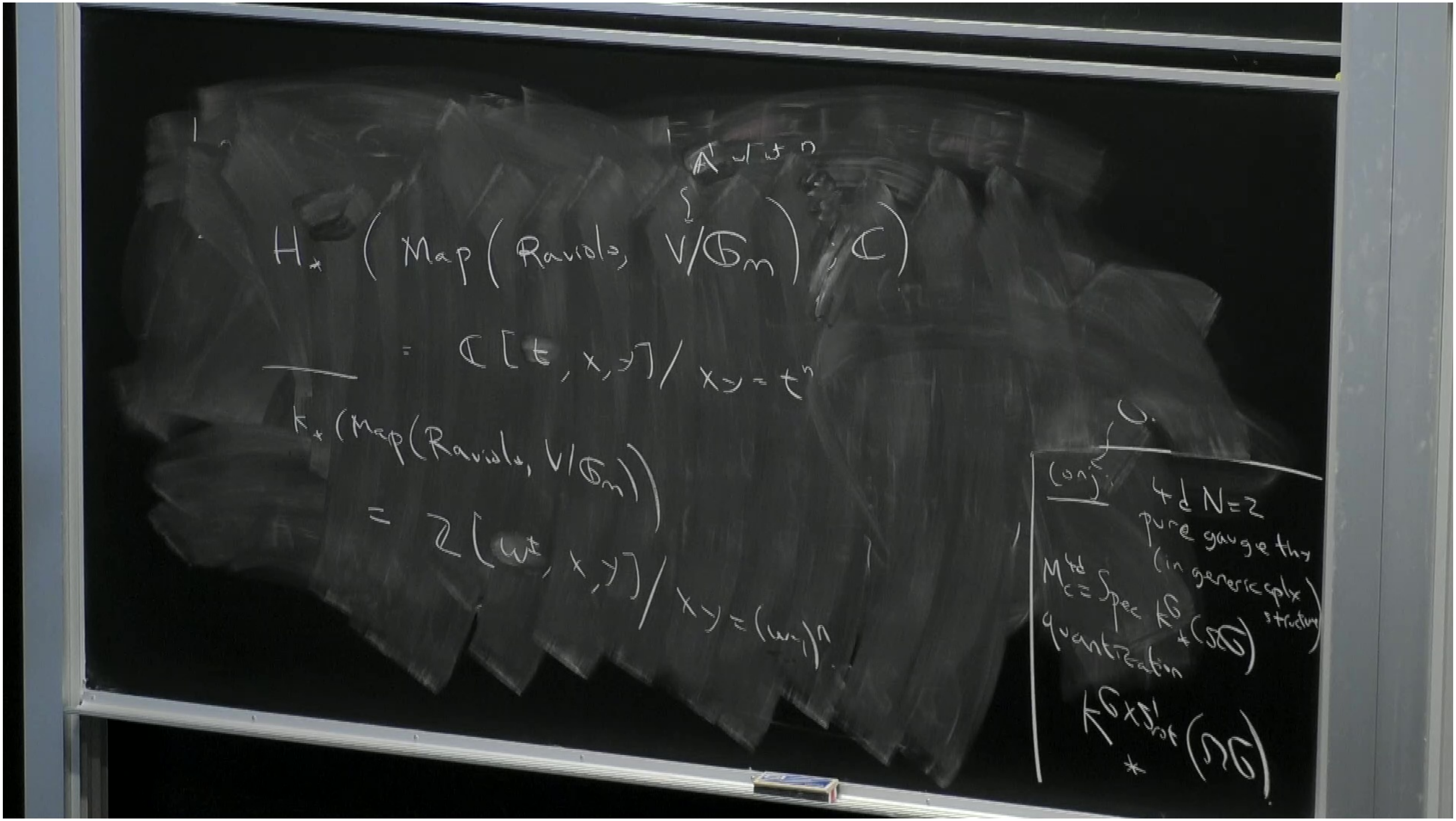

$H_*(\mathrm{Map}(\mathrm{Raviolo}, V/\mathbb{G}_m); \mathbb{C})$
$= \mathbb{C}[t, x, y] / xy = t^n$
$K_*(\mathrm{Map}(\mathrm{Raviolo}, V/\mathbb{G}_m))$
$= \mathbb{Z}[w^{\pm}, x, y] / xy = (w-1)^n$
Conj: 4d N=2
pure gauge thy
(in generic cplx structure)
$M^{4d}_{\mathbb{C}} = \mathrm{Spec}\, K^G_*(\Omega G)$
quantization
$K^{G \times S^1_{rot}}_*(\Omega G)$

K-thy

elliptic cohomology

$G = SO(3) \ (= PSU(2))$

Bullimore – Dimofte – Gaiotto

3d $\mathcal{N}=4$

$M_{\mathbb{C}} \subseteq \mathbb{A}^3_{\mathbb{C}}$ cut out by $U^2 - 4 = \Phi V^2$

Atiyah-Hitchin mfld.

quantized Coul. branch algebra:

$$[\Phi, V] = 2\hbar V - \hbar^2 V$$
$$[\Phi, U] = 2\hbar \Phi V - \hbar^2 U$$
$$[U, V] = \hbar V^2$$

$$U^2 - 2 = \Phi V^2 - \hbar U V$$

$X = PGL_n / GL_{n-1} \sim \check{G}_X = sl_2$

$T^*(\mathrm{std})$