

Title: Chromatic aberrations of the geometric Satake equivalence

Speakers: Sanath Devalapurkar

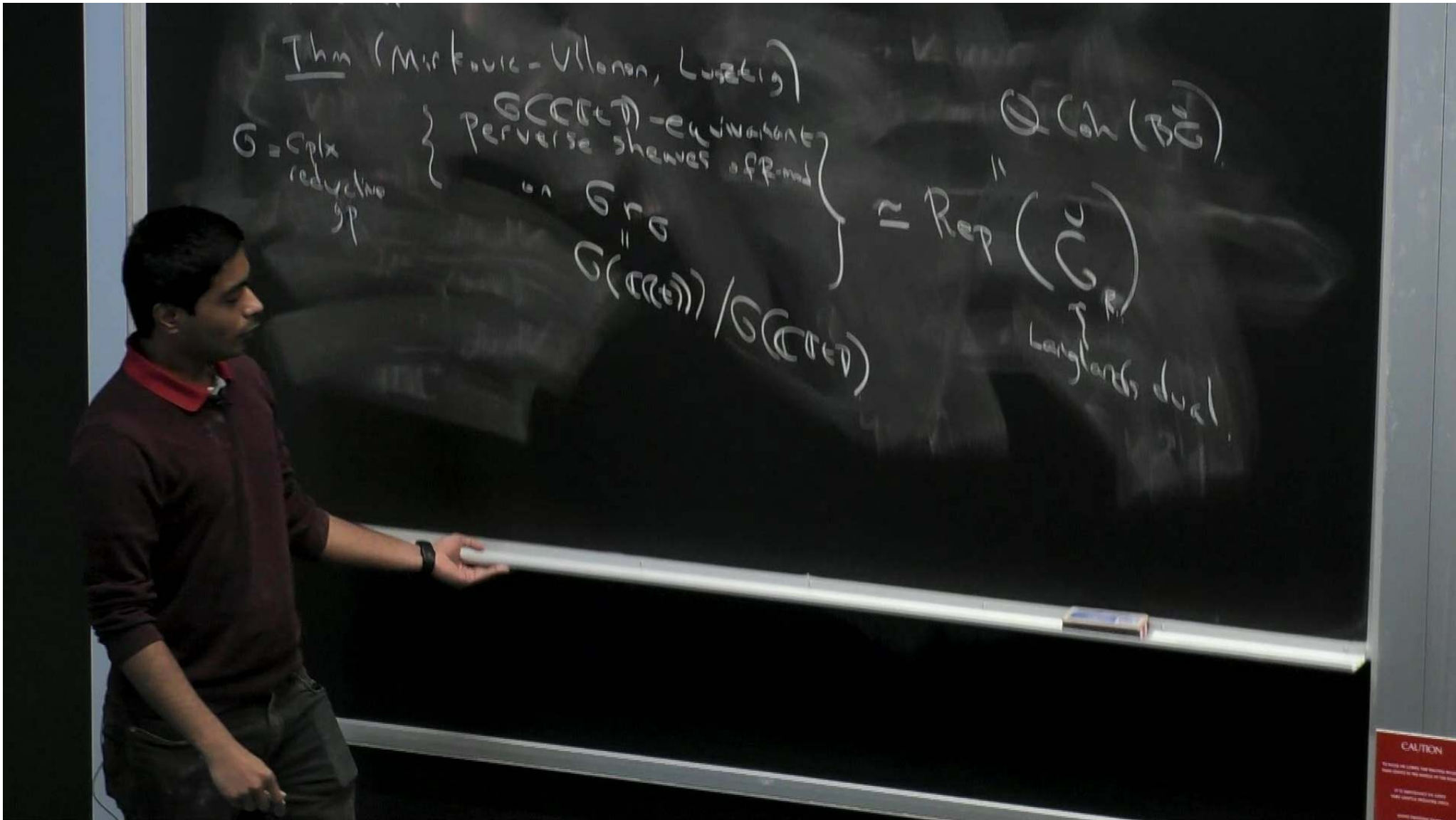
Series: Mathematical Physics

Date: April 14, 2023 - 1:30 PM

URL: <https://pirsa.org/23040005>

Abstract: The (derived) geometric Satake equivalence plays a central role in the geometric Langlands program: roughly, it describes the category of constructible sheaves of \mathbb{C} -vector spaces on $\text{Bun}_G(S^2)$ in terms of the Langlands dual group G^\vee . In this talk, I will describe some ideas connecting chromatic homotopy theory to the derived geometric Satake equivalence. For example, we will describe the category of locally constant sheaves of A -modules on $\text{Bun}_G(S^2)$, where A is complex K-theory or an elliptic cohomology theory, in Langlands dual terms. Some of this work was motivated by considerations from physics, and I hope to say what little I know about this, as well as sketch its relationship to the Ben-Zvi-Sakellaridis-Venkatesh program.

Zoom Link: <https://pitp.zoom.us/j/94926220665?pwd=bVZFUFFlvZGxVSG0xUFc1SGNaTDBKZz09>



Thm (Mirkovic-Vilonen, Lusztig)

$G = \text{Cplx reductive gp}$

$G(\mathbb{C}(F))$ -equivariant perverse sheaves of rank n

on $G \backslash G$

$G(\mathbb{C}(F)) / G(\mathbb{C}(F))$

$\cong \text{Rep}(G)$

$Q \text{ Coh}(BG)$

Langlands dual

CAUTION

"derived Satake"

Thm (Ginzburg, Beilinson-Kazhdan-Finkelberg)

$$\begin{array}{ccc}
 \text{Shv}_{\mathbb{G}}^c(\mathbb{A}^1) & \simeq & \text{QCoh}(\mathbb{A}^1_{\mathbb{G}} / \mathbb{G}) \\
 \downarrow \cup & & \downarrow \cup \\
 \text{Loc}_{\mathbb{G}}(\mathbb{A}^1; \mathbb{R}) & & \text{Sym}^{\infty}(\mathbb{A}^1_{\mathbb{G}} / \mathbb{G})
 \end{array}$$

\mathbb{G} is a reductive group, $\mathbb{A}^1_{\mathbb{G}} = \mathbb{A}^1 \times \mathbb{G} / \mathbb{G}$.

The top row is \mathbb{F}_3 -monoidal. The bottom row is symmetric monoidal.

\mathbb{G}
 \mathbb{A}^1
 $\mathbb{A}^1_{\mathbb{G}}$

CAUTION

"derived Satake"

Thm (Ginzburg, Beilinson-Kazhdan-Finkelberg)

$$\begin{array}{ccc}
 \text{Shv}_{G(\mathbb{A}^1)}^c(G_{\text{reg}}; R) & \simeq & \mathcal{Q}(\text{oh}(\mathfrak{g}^{\vee}[2]/\mathfrak{g}^{\vee})) \\
 \downarrow \cup & \left\{ \begin{array}{l} \mathbb{F}_3\text{-monoidal} \\ \text{symmetric mon} \\ \mathbb{F}_3\text{-bimodular} \end{array} \right\} & \downarrow \cup \\
 \text{Loc}_G(\Omega G; R) & \xrightarrow{\quad} & \mathcal{Q}(\text{oh}(\mathfrak{g}^{\vee}_{\text{reg}}[2]/\mathfrak{g}^{\vee}))
 \end{array}$$

G_{reg}
 ΩG

"derived Satake"

Thm (Ginzburg, Beilinson-Drinfeld-Finkelberg)

$$\begin{array}{ccc}
 \text{Shv}_{\mathbb{G}}^c(\mathbb{A}^1) & (\mathbb{G}_{\mathbb{G}}; R) & \simeq \mathcal{Q}(\text{oh}(\mathfrak{g}_{\mathbb{G}}[2]/\mathbb{G})) \\
 \cup & \left\{ \begin{array}{l} \mathbb{E}_3\text{-monoidal} \\ \text{symmetric mon} \end{array} \right\} & \cup \\
 \text{Loc}_{\mathbb{G}}(\mathbb{R}\mathbb{G}; R) & & \mathcal{Q}(\text{oh}(\mathfrak{g}_{\mathbb{G}}^{\text{reg}}[2]/\mathbb{G})) \\
 & \left\{ \begin{array}{l} \mathbb{E}_3\text{-monoidal} \\ \text{symmetric mon} \end{array} \right\} & \\
 & \text{B-F-Minkovic, Yun-Zhu} &
 \end{array}$$

For $\mathfrak{O}_K = \mathfrak{O}_K \otimes \mathbb{Z}$, a matrix is "regular" if its min poly = char poly.

Trichotomy.

Rational

Trigonometric

Elliptic.

alg closed field k

$$\mathbb{C} = \mathbb{C}/\langle f \rangle$$

$$\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$$

$$\mathbb{C}/\mathbb{Z}^2 = T^2$$

G_a

G_m

$E \sim$ elliptic curve

\times

e^x

$\mathcal{P}(x)$ Weierstrass

Cubic curves



For $\mathfrak{sl}_n = \mathfrak{so}_n$, a matrix is "regular" if its min poly = char poly.

Trichotomy:

Rational

Trigonometric

Elliptic

closed field k

$$\mathbb{C} = \mathbb{C}/\mathbb{Z}$$

$$\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$$

$$\mathbb{C}/\mathbb{Z}^2 = T^2$$

$$\mathbb{G}_a$$

$$\mathbb{G}_m$$

E = elliptic curve

x

e^x

Cubic curves



$\mathcal{P}(x)$ Weierstrass

Topology

$$H^*(-; \mathbb{R})$$

k -thy

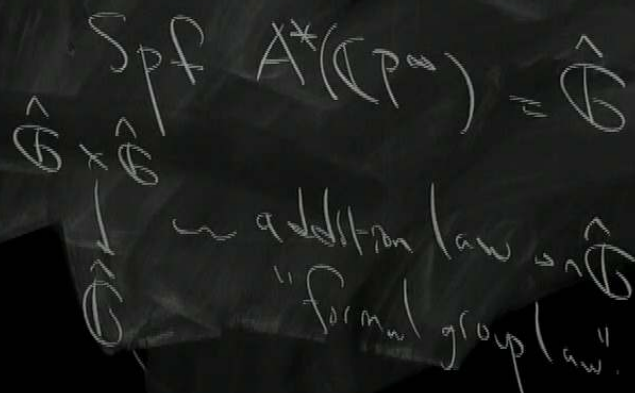
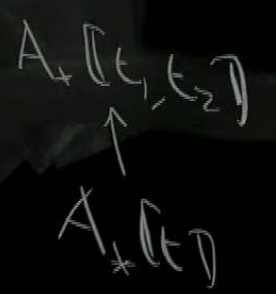
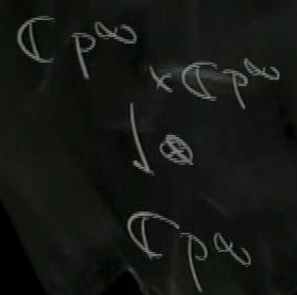


smooth elliptic curve
elliptic cohomology

A Cohomology theory
w/ Chern classes
(Cplx-oriented)

t = analogue
of $c_1(\mathcal{O}(1))$

$$A^*(\mathbb{C}P^\infty) = A_*[[t]]$$



• If $A = H^1(-; \mathbb{R})$,
 $\hat{B} = (\hat{B}_a)_0^1 \leftarrow \text{formal nbhd.}$

• If $A = k\text{-thy}$,

$$\hat{B} = (\hat{B}_3)^1$$

• $A = E \parallel E$, $\hat{B} = \hat{E}_{\text{zero}}$

CAUTION
 DO NOT TOUCH THE BOARD
 IT IS A PROPERTY OF THE BOARD
 AND IT IS NOT TO BE TOUCHED

• If $A = H^1(-; \mathbb{R})$,
 $\hat{\mathbb{G}} = (\mathbb{G}_a)_0^\wedge \leftarrow \text{formal nbhd.}$

• If $A = k\text{-thy}$,
 $\hat{\mathbb{G}} = (\mathbb{G}_m)_0^\wedge \rightarrow 1$

• $A = E // E$, $\hat{\mathbb{G}} = E_{\text{zero}}^\wedge$

In these cases, using
 $\mathbb{G}_a, \mathbb{G}_m, E, = \mathbb{G}$
 one can define

"genuine S^1 -equivariant A -cohomology"

$$A_{S^1}^*(-)$$

Analogue for hmg. $A_{S^1}^*(-)$

$$A_{S^1}^*(pt) = \left\{ \begin{array}{l} f_{\text{non}} \\ \mathbb{G} \end{array} \right\}$$

CAUTION

BE CAREFUL TO AVOID THE FOLLOWING SITUATION
 WHICH MAY BE DANGEROUS TO YOUR HEALTH
 IF YOU EXPERIMENTAL TO AVOID
 YOUR OWN DESTRUCTION

LHS of derived Satake

$$\mathrm{Shv}_{\mathbb{Z}}(\sim; R)$$

categorification of

$$H_n^*(\sim; R)$$

RHS of derived Satake

$$\mathrm{QCoh}(\mathfrak{g} / \mathfrak{G})$$

module stack

$$\mathrm{Bun}_{\mathfrak{G}}^0(\sim)$$

Thm (D.)

A cothy
= kthy on
- Ell E

Then

$$\mathrm{Loc}_{\mathfrak{G}/\mathfrak{PS}}$$



LHS of derived Satake

$$\mathrm{Shv}_*(\sim; R)$$

categorification of

$$H^*(\sim; R)$$

RHS of derived Satake

$$\mathrm{QCoh}(\mathfrak{a}_G / \mathfrak{G})$$

moduli stack

$$\mathrm{Bun}_G^{\mathrm{ss}, 0}(\mathbb{A}^1)$$

$\mathrm{Hom}(D)$

A coh thry
= k thry on
- Ell E

Then

$$\mathbb{Q} \otimes \mathrm{Loc}_G(\mathbb{P}^1; A)$$

$$= \begin{cases} \mathrm{QCoh}(\mathfrak{a}_G^{\mathrm{reg}} / \mathfrak{G}) & A = k\text{-thry} \\ \mathrm{QCoh}(\mathrm{Bun}_G^{\mathrm{ss}, 0}(\mathbb{A}^1)) & A = \mathrm{Ell}_E \end{cases}$$

$$\text{Loc}_G(\Omega G; A) = \text{coMod } \underline{A_*^G}(\Omega G)$$

· If $A = H^*(-; R)$, becomes $H_*^G(\Omega G; R)$

Bezrukavnikov-Finkelberg-Mirkovic, Yun-Zhu

(If $R = \mathbb{C}$, Calabi branch of $3d N=4$ pure gauge theory)

· If $A = k\text{-thy}$, was also done by BFM.

BFM

$$\text{Spec } H_*^G(\Omega G; \mathbb{C})$$

regular centralizers \cong $\left\{ (g, x) \text{ where } g \in \check{G}, \right.$
 $\left. x \in \check{\mathfrak{z}}^{\text{reg}}_g, \text{ Ad}_g(x) = x \right\} // \check{G}$

BFM

$$\text{Spec } K_*^G(\Omega G)$$

$$\cong \left\{ (g, x) \text{ where } g \in \check{G}, \right.$$

 $\left. x \in \check{\mathfrak{z}}^{\text{reg}}_g, \text{ Ad}_g(x) = x \right\} // \check{G}$

Ex $\check{G} = SL_2$

$$\check{Y} \cong \check{X}/W \times_{\check{G}/\check{G}} \check{X}/W$$

$$\check{X} = A' \xrightarrow{\check{G}} \check{X}/W = A'$$

$W = \mathbb{Z}/2$

The map $\check{X}/W \rightarrow \check{G}/\check{G}$
 $K \rightarrow \check{G}/\check{G}$

$K =$ constant slice

$$A' \rightarrow SL_2$$

$$\lambda \mapsto \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$$

In the elliptic case,

$$\text{"Spec Ell}_x^S(\Omega_S)\text{"}$$

$$\cong \check{Y}^{\text{ell}} =$$

$$\text{Ell}(\tau|_2) \times \text{Ell}(\tau|_2)$$

$\tau|_2 \hookrightarrow E$ by inversion

$$k: \text{Ell}(\tau|_2) \rightarrow \text{Bun}_{\text{SL}_2}^0(E)$$

$$\text{Bun}_{\text{SL}_2}^{\text{ors}}(E)$$

$$\mathcal{O}(x) \xrightarrow{x} \begin{cases} L_x \oplus L_{-x} & \tau x \neq 0 \\ F_2 \oplus L_{-x} & \tau x = 0 \end{cases}$$

Atiyah bell

$$\mathcal{O}_E \rightarrow F_2 \rightarrow \mathcal{O}_E$$

non-split

In the elliptic case,

$$\text{"Spec Ell}_x^G(\Omega G)\text{"}$$

$$\cong \int^{\vee} \text{ell} = \text{Ell}(\mathbb{Z}/2) \times \text{Ell}(\mathbb{Z}/2)$$

$\mathbb{Z}/2 \hookrightarrow E$ by inversion

$$k: \text{Ell}(\mathbb{Z}/2) \rightarrow \text{Bun}_{\text{SL}_2}^0(E)$$

$$\hookrightarrow \text{Bun}_{\text{SL}_2}^{\text{class}}(E)$$

$$x \longmapsto \begin{cases} L_x \oplus L_{-x} & \mathbb{Z}x \neq 0 \\ \mathcal{F}_2 \oplus L_{-x} & \mathbb{Z}x = 0 \end{cases}$$

Atiyah bell

$$\mathcal{O}_E \rightarrow \mathcal{F}_2 \rightarrow \mathcal{O}_E$$

non-split

$$\text{Loc}_G(\Sigma G; A) = \text{coMod } \underline{A_*^G(\Sigma G)}$$

If $A = H^*(-; R)$, becomes $\underline{H_*^G(\Sigma G; R)}$

Bezrukavnikov-Finkelberg-Mirkovic, Yun-Zhu

If $R = \mathbb{C}$, Coulomb branch of $3d N=4$ pure gauge theory
 $\text{Spec } H_*^G(\Sigma G; \mathbb{C}) = M_{\mathbb{C}}$

Applying Σ -def \rightarrow quantization $H_*^G(\Sigma G; \mathbb{C})$

$$\text{Loc}_G(\Sigma G; A) = \text{coMod } \underline{A_*^G(\Sigma G)}$$

If $A = H^*(-; R)$ becomes $\underline{H_*^G(\Sigma G; R)}$

Bezrukavnikov-Finkelberg-Mirkovic, Yun-Zhu, Gaiotto

If $R = \mathbb{C}$ Coulomb branch of

$\text{Spec } H_*^G(\Sigma G; \mathbb{C})$ gauge theory

Applying Σ -def \rightarrow quantization H^*

(con) $4d N=2$ pure gauge theory (in generic complex structure)
 $M^{\text{td}}_{\mathbb{C}} = \text{Spec } \underline{H_*^G(\Sigma G)}$
 quantization
 $\underline{H_*^G(\Sigma G)}$

$$\mathbb{Q} \otimes \text{Loc}_G(\text{SUG}; \text{KU}) \simeq \mathbb{Q} \text{ Coh}(\check{G}^{\text{reg}} / \check{G})$$

$$\mathbb{Q} \otimes \text{Shv}_{G_0}^c(\mathcal{G}_{G_0}, \text{KU}) \stackrel{?}{\simeq} \mathbb{Q} \text{ Coh}(\check{G} / \check{G})$$

Hope?

Some equiv. of categories arising from 5d N=2 gauge th

What about elliptic cohomology? 6d?

$$\mathbb{Q} \otimes \text{Loc}_G(\Omega G; KU) \simeq \mathbb{Q} \text{Coh}(\check{G}^{\text{reg}}/\check{G})$$

$$\mathbb{Q} \otimes \text{Shv}_{G_0}^c(G_{\text{reg}}, KU) \stackrel{?}{\simeq} \mathbb{Q} \text{Coh}(\check{G}/\check{G})$$

Hope?

Some equiv. of categories arising from 5d $N=2$ gauge theory

What about elliptic cohomology? 6d?

$$\text{Shv}_{G_0}^c(G_{\check{G}}; \mathbb{C}) = \mathbb{Q} \text{Coh}(\check{G}/\check{G})$$

from 4d $N=4$

BE-SV
 duality between Hamiltonian G -vars
 and Hamiltonian \check{G} -vars

$$\text{Loc}_G(\Sigma G; A) = \text{coMod } \underline{A^G_*}(\Sigma G)$$

If $A = H^*(-; R)$ becomes $\underline{H^G_*}(\Sigma G; R)$

Bezrukavnikov-Finkelberg-Mirkovic, Yun-Zhu, Gaiotto

If $R = \mathbb{C}$ Calabi branch of 3d $N=4$ pure gauge th.
 $\text{Spec } H^G_* (\Sigma G; \mathbb{C}) = M_C$

Applying Ω -def

quantization

$$\underline{H^G_*}(\Sigma G; \mathbb{C})$$

(LGA)

4d $N=2$ pure gauge th (in generic splx stratum)

$M_C^{\text{hd}} = \text{Spec } K^G_+(\Sigma G)$

quantization

$\underline{K^G_*}(\Sigma G)$



$$H_* \left(\text{Map}(\text{Raviolo}, V/\mathbb{G}_m) ; \mathbb{C} \right) \\ = \mathbb{C} [t, x, y] / x, y = t^n$$

$$K_* \left(\text{Map}(\text{Raviolo}, V/\mathbb{G}_m) \right) \\ = \mathbb{Z} [wt, x, y] / x, y = (wt)^n$$

(cont.)
 4d N=2
 pure gauge theory
 (in generic complex structure)
 $M^{\text{td}} = \text{Spec } \mathbb{K}G \times (\mathbb{G}_m)$
 quantization
 $\mathbb{K}G \times_{\text{dot}} (\mathbb{G}_m)$
 *

$$G = SO(3) (= PSU(2))$$

$$3d \quad N=4$$

$$M_C \cong \mathbb{A}^3_{\mathbb{C}} \text{ cut out by}$$

quantized Coul branch algebra.

$$[\phi, v] = 2\hbar v - \hbar^2 v$$

$$[\phi, u] = 2\hbar \phi v - \hbar^2 u$$

$$[u, v] = \hbar v^2$$

Bullimore - Dimofte - Gaiotto

$$U^2 - 4 = \Phi V^2$$

Atiyah-Hitchin m.f.d.

$$U^2 - 2 = \phi V^2 - \hbar UV$$

$$X = PGL_n / GL_n$$

$$\downarrow$$

$$G_X = SL_2$$

$$\uparrow$$

$$T^*(Std)$$