Title: Causal Inference Lecture - 230412

Speakers: Robert Spekkens

Collection: Causal Inference: Classical and Quantum

Date: April 12, 2023 - 10:00 AM

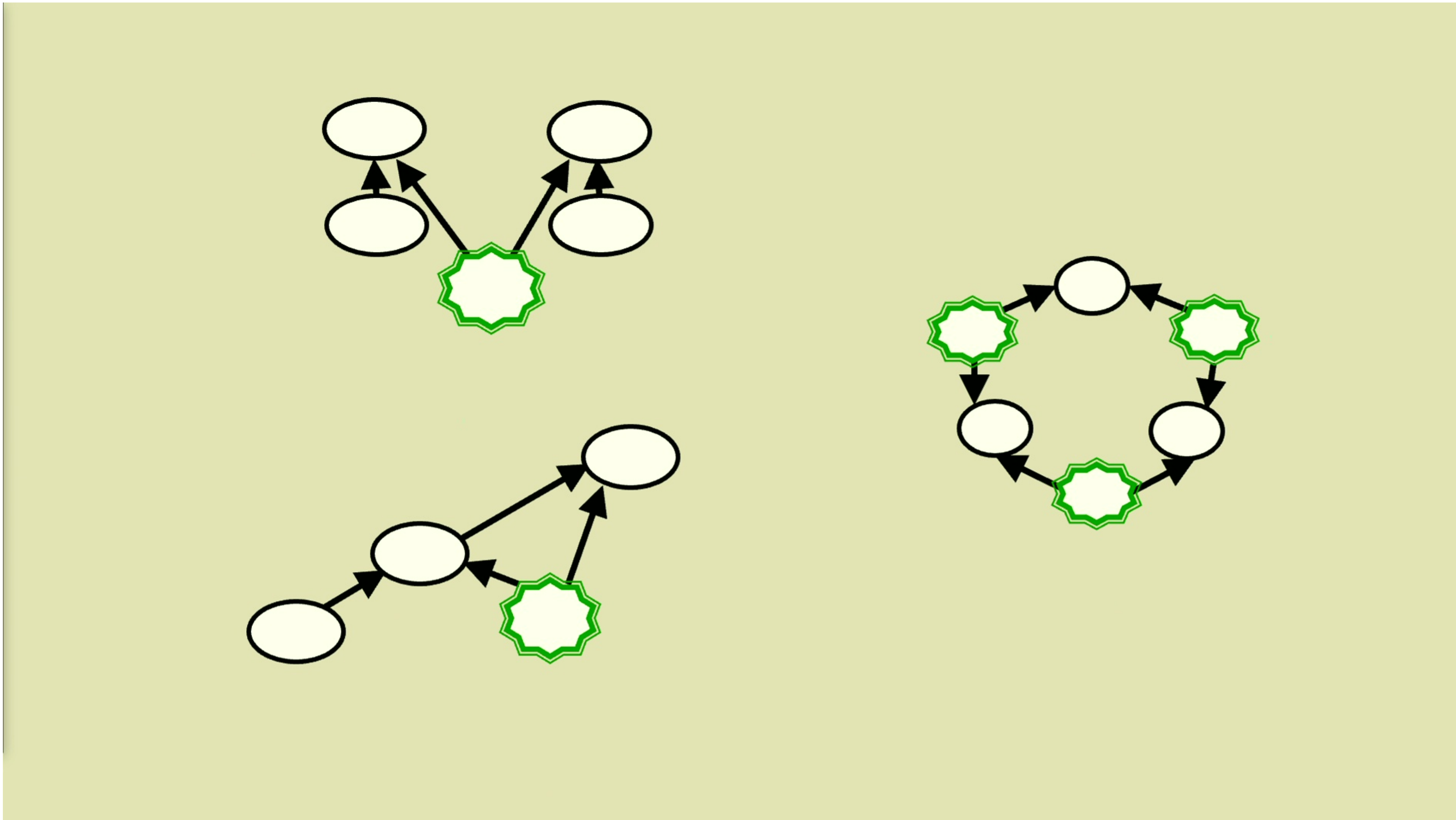
URL: https://pirsa.org/23040003

Abstract: zoom link: https://pitp.zoom.us/j/94143784665?pwd=VFJpajVIMEtvYmRabFYzYnNRSVAvZz09

Causal compatibility in quantum causal models

Quantum-classical hybrid causal models a.k.a. quantum-latent-permitting causal models

What probability distributions over classical variables are compatible with a given causal structure when the latent systems can be quantum?



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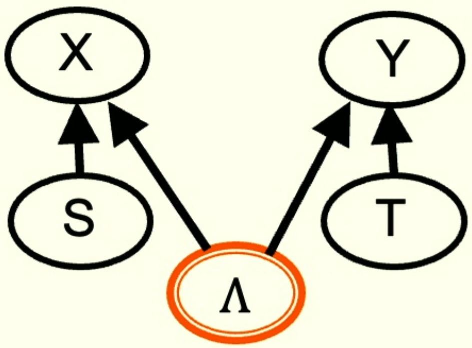
Classical Bell model



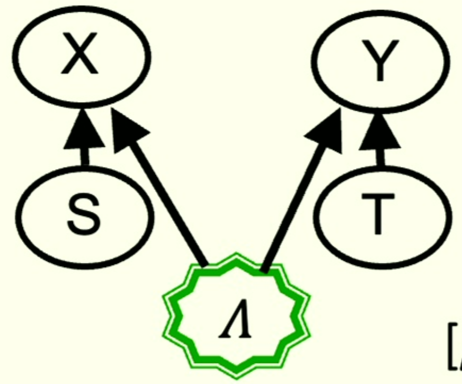
$P_{X|S\Lambda}$

$P_{Y|T\Lambda}$

P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda}$$

Quantum Bell model



$\rho_{X|S\Lambda}$

$\rho_{Y|T\Lambda}$

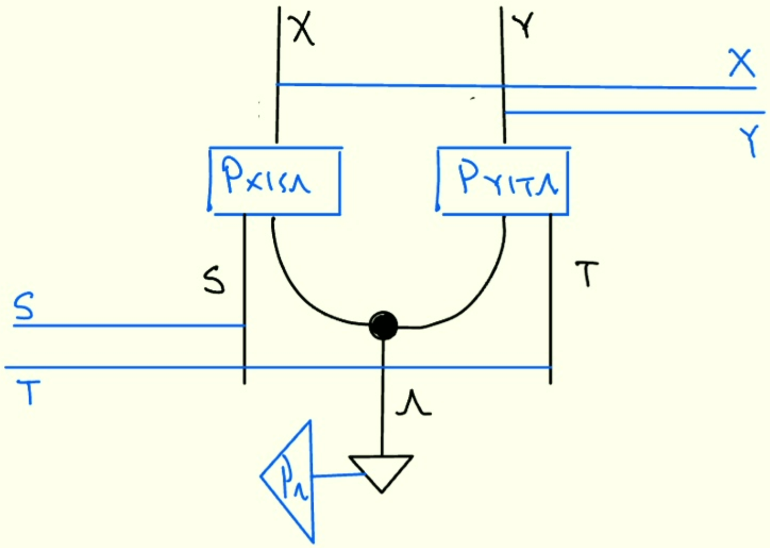
ρ_{Λ}

$$[\rho_{X|S\Lambda}, \rho_{Y|T\Lambda}] = 0$$

$$P_{XY|ST} = \text{Tr}_{\Lambda}(\rho_{X|S\Lambda} \rho_{Y|T\Lambda} \rho_{\Lambda})$$

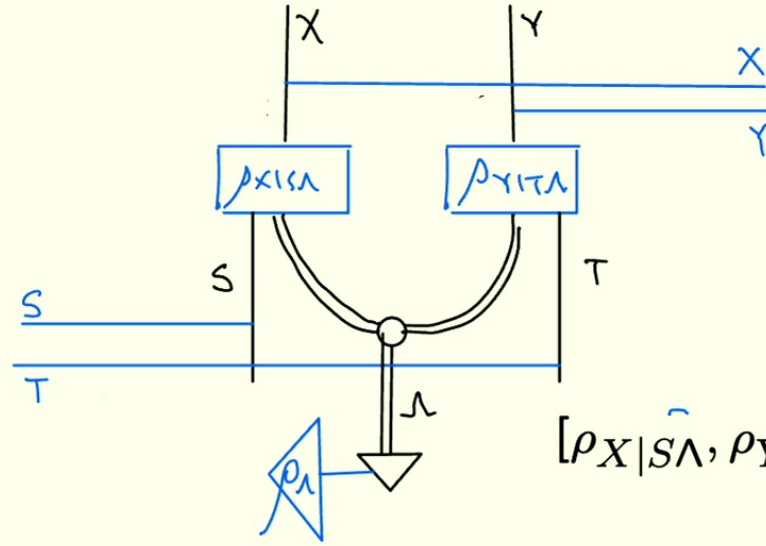
Recall: A distribution is compatible with a model if there exists parameter choices that yield it

Classical Bell model



$$P_{XY|ST} = \sum_\Lambda P_{X|S\Lambda} P_{Y|T\Lambda} P_\Lambda$$

Quantum Bell model



$$[\rho_{X|S\Lambda}, \rho_{Y|T\Lambda}] = 0$$

$$P_{XY|ST} = \mathrm{Tr}_\Lambda(\rho_{X|S\Lambda} \rho_{Y|T\Lambda} \rho_\Lambda)$$

Recall general form of
"factorization within subspaces"

$$\mathcal{H}_\Lambda = \bigoplus_i \mathcal{H}_{\Lambda_i^L} \otimes \mathcal{H}_{\Lambda_i^R}$$

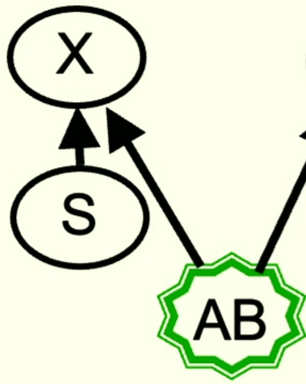
Special case of pure factorization

$$\mathcal{H}_\Lambda = \mathcal{H}_{\Lambda^L} \otimes \mathcal{H}_{\Lambda^R}$$

$$\dim(\mathcal{H}_{\Lambda^L}) \times \dim(\mathcal{H}_{\Lambda^R}) = \dim(\mathcal{H}_\Lambda)$$

For arbitrary dimension of Λ, there is no loss of generality in taking pure factorization

Quantum Bell model



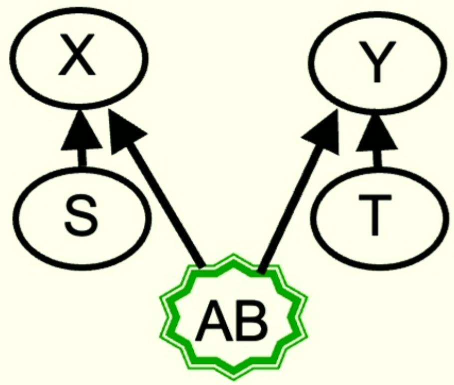
$\rho_{X|SA}$

$\rho_{Y|TB}$

ρ_{AB}

$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

Quantum Bell model



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$$P_{XY|ST} = \mathsf{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

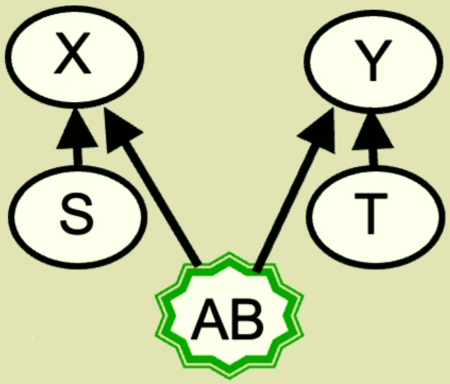
Causal compatibility constraints:

$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

Tsirelson, Lett. Math. Phys. 4, 93 (1980)

Quantum Bell model



$\rho_{X|SA}$
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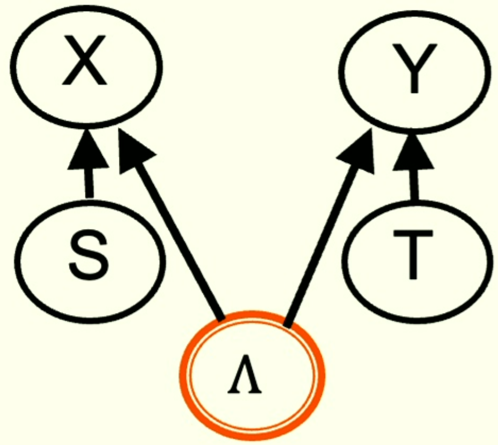
$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

One can sometimes determine an upper bound on the dimension of AB
But the necessary quantifier elimination problem is nonlinear and infeasible

The classical Bell model



$P_{X|S\Lambda}$

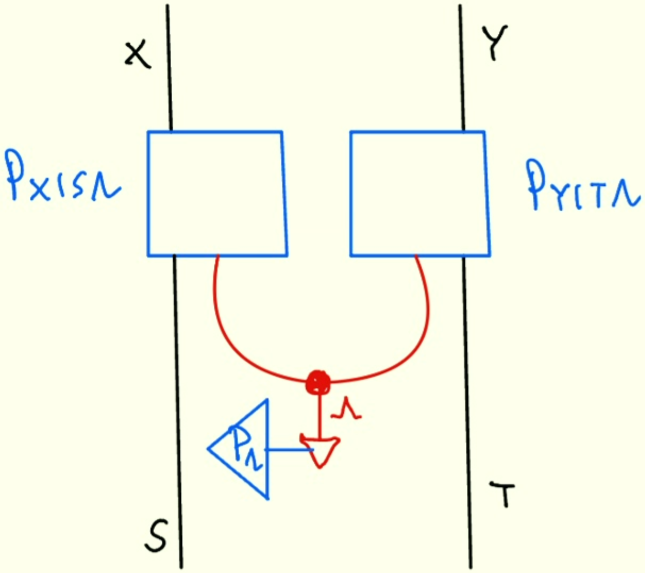
$P_{Y|T\Lambda}$

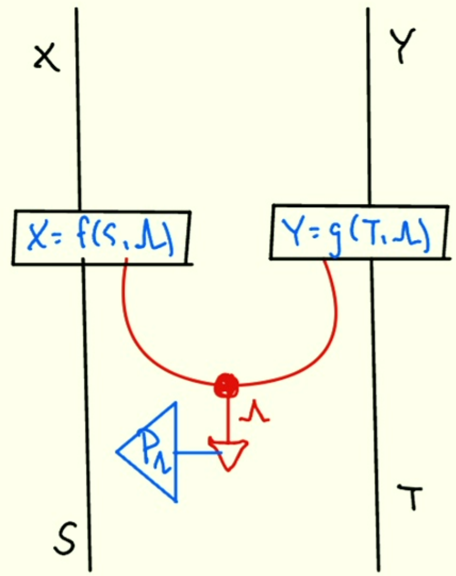
P_{Λ}

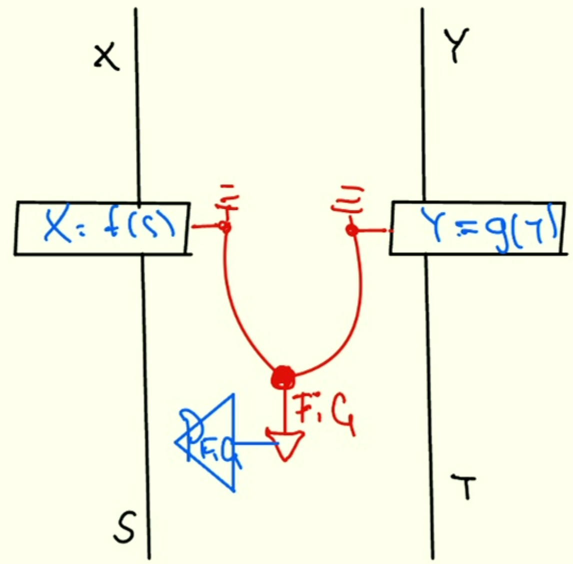
P_S

P_T

$$P_{XY|ST} = \sum_{\Lambda} P_{Y|T\Lambda} P_{X|S\Lambda} P_{\Lambda}$$


X
Y
P_{X|SΛ}
P_{Y|TΛ}
Λ
P_Λ
S
T


X
Y
X = f(S, Λ)
Y = g(T, Λ)
Λ
P_Λ
S
T


X
Y
X = f(S)
Y = g(T)
F, G
P_{F,G}
S
T

$$P_{XY|ST} = \sum_{f,g} \delta_{X,f(S)} \delta_{Y,g(T)} P_{F,G}(f,g)$$

If X,Y,S,T are binary, Λ can have cardinality 16

$$p_{xy|st} := P_{XY|ST}(xy|st) \qquad q_{fg} := P_{F,G}(f,g)$$

$$x,y,s,t \in \{0,1\} \qquad f,g \in \{\mathrm{id}, \mathrm{fp}, \mathrm{r}_0, \mathrm{r}_1\}$$

$$p_{00|00} = q_{\mathrm{r}_0,\mathrm{r}_0} + q_{\mathrm{r}_0,\mathrm{id}} + q_{\mathrm{id},\mathrm{r}_0} + q_{\mathrm{id},\mathrm{id}}$$

$$p_{00|01} = q_{\mathrm{r}_0,\mathrm{r}_1} + q_{\mathrm{r}_0,\mathrm{fp}} + q_{\mathrm{id},\mathrm{r}_1} + q_{\mathrm{id},\mathrm{fp}}$$

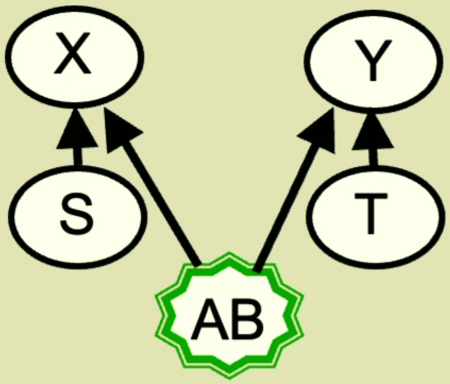
$$0 \leq q_{fg} \leq 1 \ \forall f,g$$

$$\vdots$$

16 linear equalities + inequalities

Do linear quantifier elimination on the 16 q's.

Quantum Bell model



$\rho_{X|SA}$

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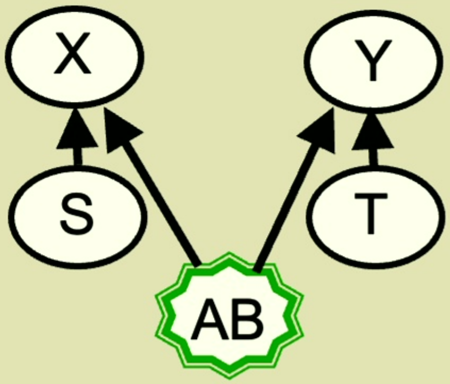
$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

Quantum Bell model



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$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

A feasible solution:

The NPA semidefinite programming hierarchy

Navascués, Pironio, and Acín, New J. Phys. 10, 073013 (2008)

Recall that conditioning on a quantum system that is a mediary within a causal structure is not meaningful given the impossibility of passive observation of a quantum system

Recall that conditioning on a quantum system that is a mediary within a causal structure is not meaningful given the impossibility of passive observation of a quantum system

But it is still meaningful to ask:
What conditional independence relations hold *among the classical variables* in the quantum-latent-permitting causal model?

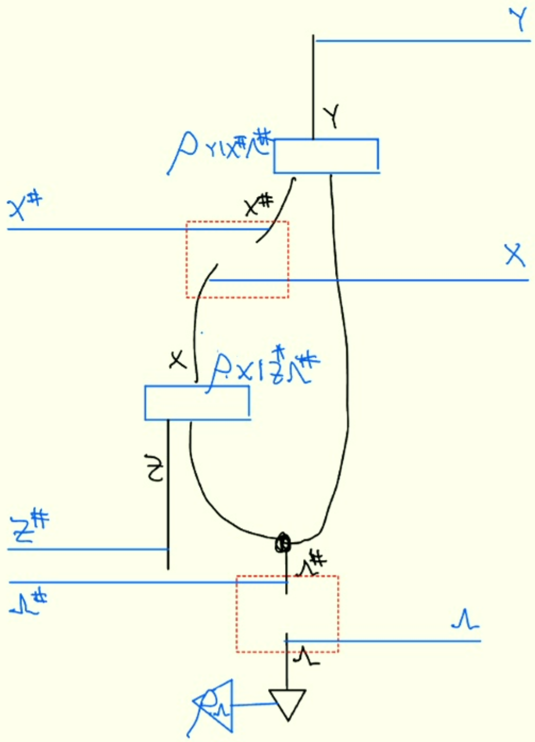
Definition (path blocking) A path between classical node X and classical node Y is blocked by a set of classical nodes **Z** if at least one of the following conditions holds:

1. The path contains a chain whose mediary node is in **Z**
2. The path contains a fork whose tail node is in **Z**

2. The path contains a collider whose head node is **not** in **Z** and no descendant of which is in **Z**.

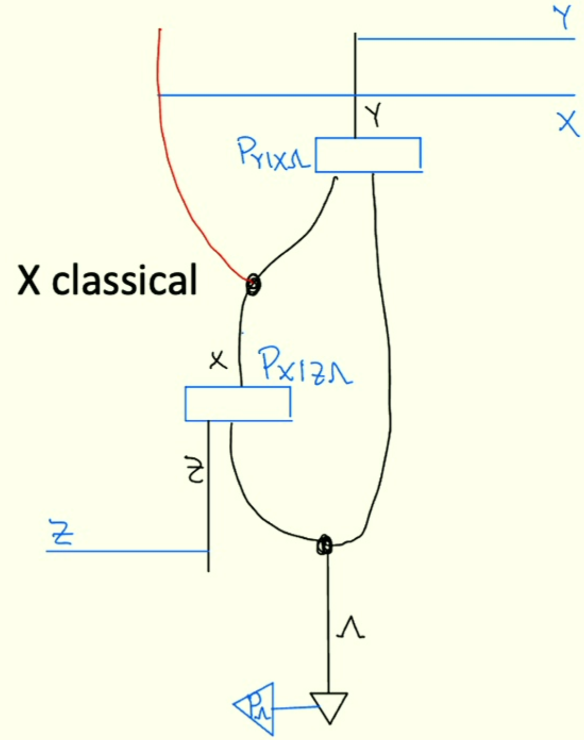
Definition (d-separation) Given a quantum-latent-permitting model M with observed classical nodes **V**, two disjoint set of classical observed nodes **X**, **Y** ⊂ **V** are d-separated by a set of classical observed nodes **Z** ⊂ **V** if and only if for every pair of nodes, X ∈ **X** and Y ∈ **Y**, every path between X and Y is blocked.

Henson, Lal, Pusey, New Journal of Physics 16, 113043 (2014)



Markov condition for split-node intervention probing schemes

$$\rho_{XY\Lambda|Z^\#X^\#\Lambda^\#} = \rho_{Y|X^\#\Lambda^\#}\rho_{X|Z^\#\Lambda^\#}\rho_{\Lambda}$$



Markov condition for passive observation of classical X, no intervention on Λ

$$\rho_{XY|Z} = \mathrm{Tr}_{\Lambda}(\rho_{Y|X\Lambda}\rho_{X|Z\Lambda}\rho_{\Lambda})$$

Extension of d-separation theorem to quantum-latent-permitting causal models:

Consider a quantum-latent-permitting causal model on DAG G and three disjoint subsets of observed variables **X**, **Y** and **Z**.

Soundness

$$\mathbf{X} \perp_d \mathbf{Y}|\mathbf{Z} \text{ in } G \quad \Longrightarrow \quad \forall P \in \mathsf{Comp}_M : \mathbf{X} \perp \mathbf{Y}|\mathbf{Z} \text{ in } P$$

Completeness

$$\forall P \in \mathsf{Comp}_M : \mathbf{X} \perp \mathbf{Y}|\mathbf{Z} \text{ in } P \quad \Longrightarrow \quad \mathbf{X} \perp_d \mathbf{Y}|\mathbf{Z} \text{ in } G$$

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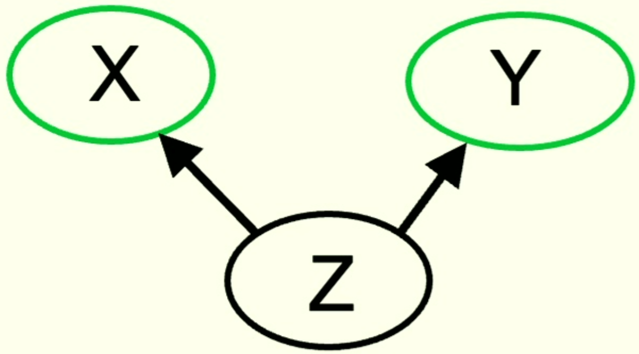
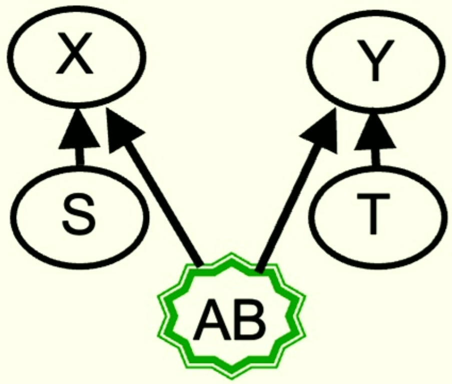


$$X \perp Y | Z$$

$$\rho_{XY|Z} = \rho_{X|Z}\rho_{Y|Z}$$

$$[\rho_{X|Z}, \rho_{Y|Z}] = 0$$

Quantum Bell model



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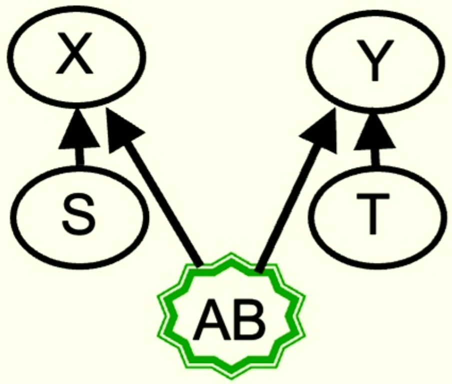

$\{E^A_{x|s}\}_x$ for each s

$\{E^B_{y|t}\}_y$ for each t

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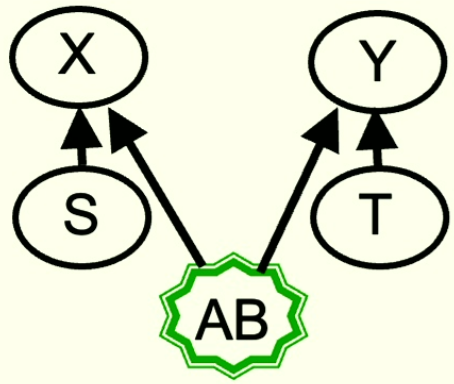
$$P_{XY|ST}(xy|st) = \mathrm{Tr}_{AB}((E^A_{x|s} \otimes E^B_{y|t})\rho_{AB})$$

Quantum Bell model



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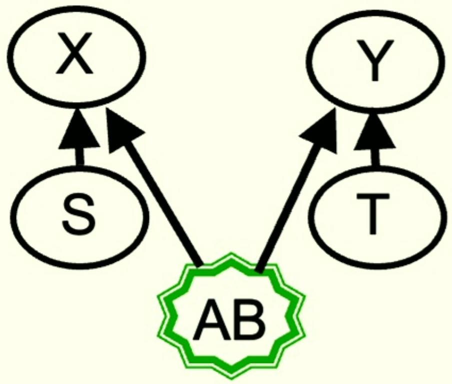

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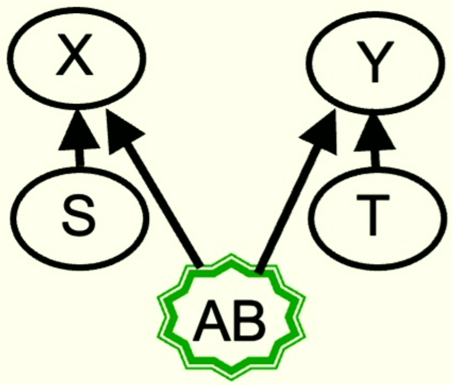
Quantum Bell model



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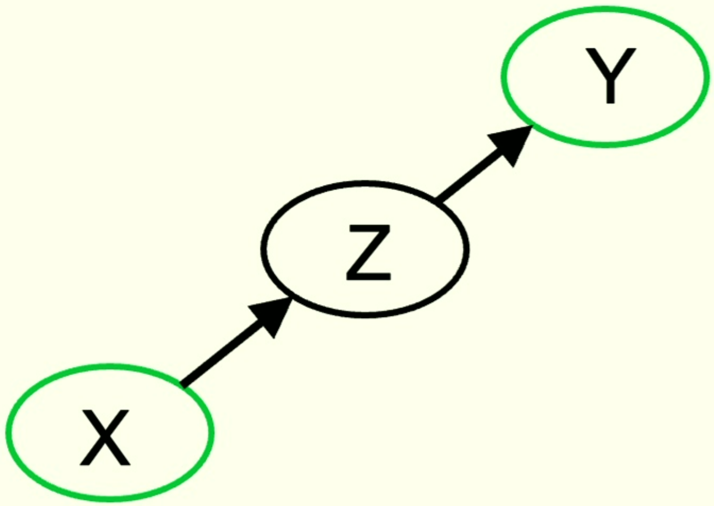
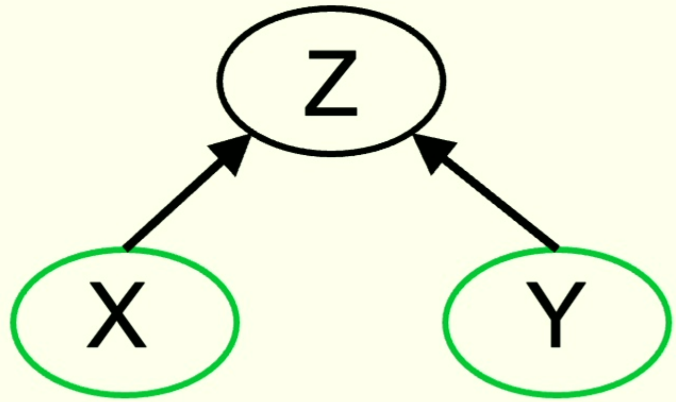


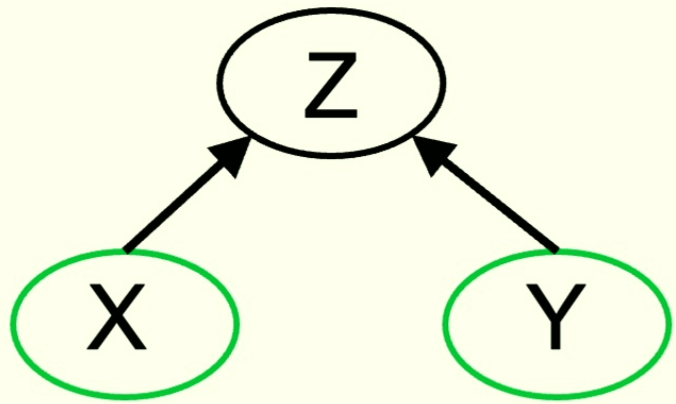
$$Y \perp X | Z$$
$$\rho_{YZ|X} = \rho_{Y|Z}\rho_{Z|X}$$



X and Y are independent when one marginalizes over Z

$$X \perp Y$$

$$\rho_{XY} = \rho_X \rho_Y$$



X and Y are independent when one marginalizes over Z

$$X \perp Y$$

$$\rho_{XY} = \rho_X \rho_Y$$

However, X and Y can become dependent if one conditions on Z

Extension of d-separation theorem to quantum-latent-permitting causal models:

Consider a quantum-latent-permitting causal model M and three disjoint subsets of observed variables **X**, **Y** and **Z**.

Soundness

$$\mathbf{X} \perp_d \mathbf{Y}|\mathbf{Z} \text{ in } G \quad \Longrightarrow \quad \forall P \in \text{Comp}_M : \mathbf{X} \perp \mathbf{Y}|\mathbf{Z} \text{ in } P$$

Completeness

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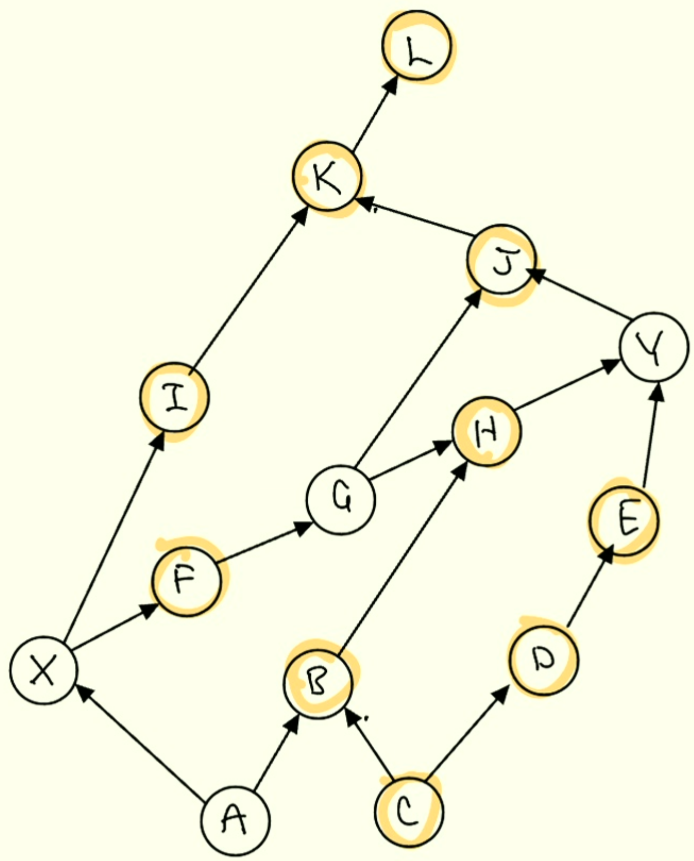
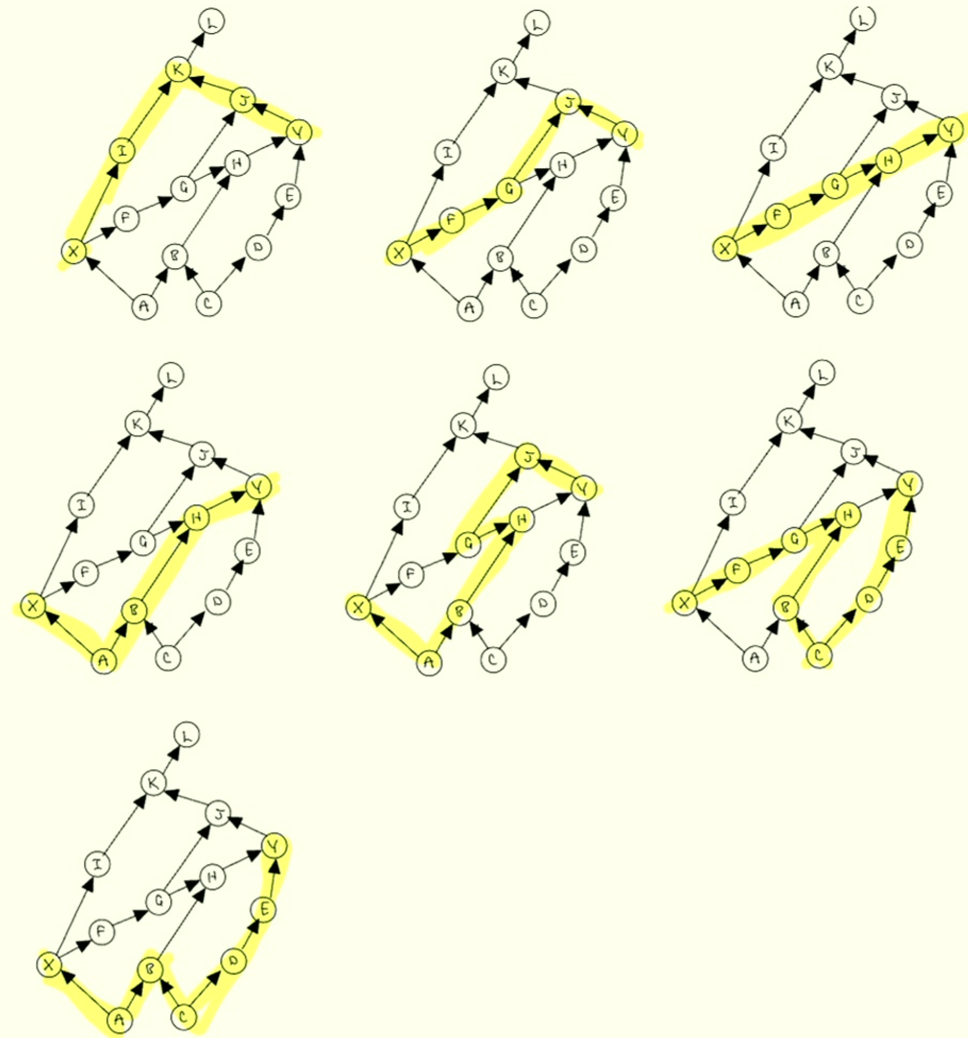
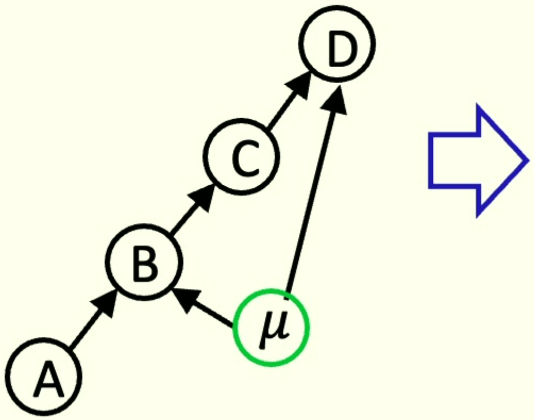


$$X \perp Y | GA$$



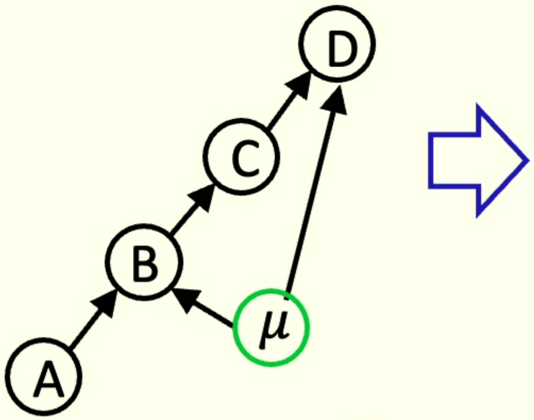
Nested Markov constraints for quantum-latent-permitting causal models



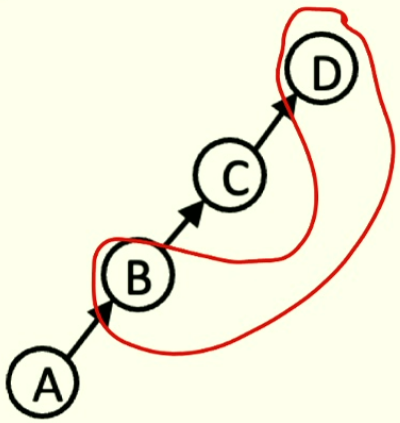
$$P_{ABCD} = \mathrm{Tr}_\mu \left(\rho_{D|\mu C} P_{C|B} \rho_{B|A\mu} P_A \rho_\mu\right)$$

$$= \mathrm{Tr}_\mu(\rho_{D|\mu C}\rho_{B|A\mu}\rho_\mu) P_{C|B} P_A$$

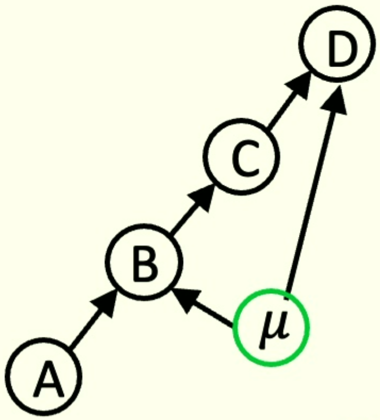
$$= Q_{BD|AC} P_{C|B} P_A$$



$$A \perp C \mid B$$

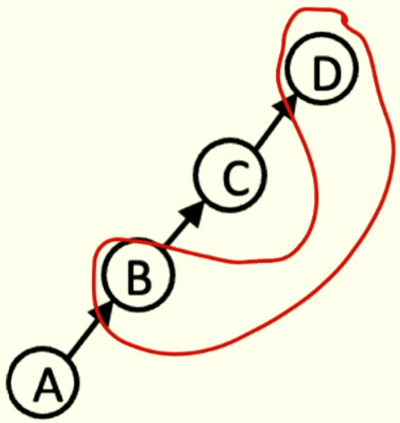


Verma graph



$$A \perp C \mid B$$

This implies equality constraints on P_{ABCD}



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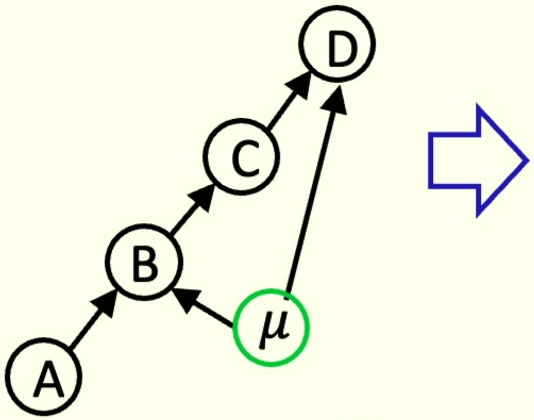

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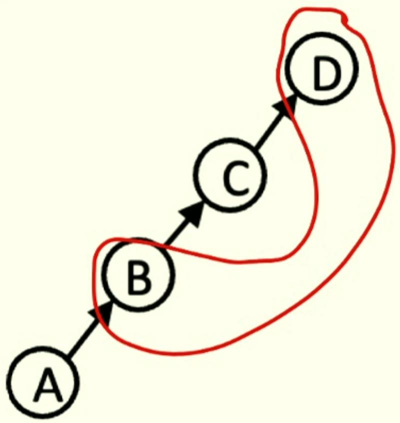
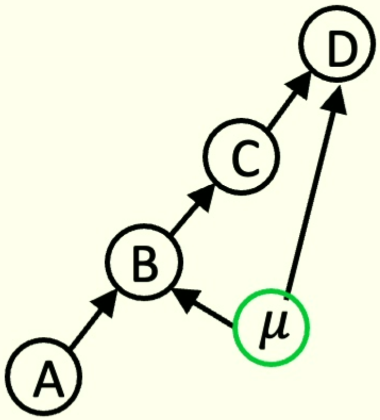


Verma graph

But there is also another type of equality constraint that arises here…



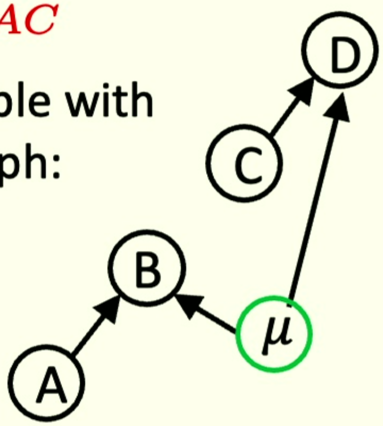
$$\mathrm{P}_{ABCD} = \mathrm{Tr}_\mu \left(\rho_{D|\mu C} \mathrm{P}_{C|B} \rho_{B|A\mu} \mathrm{P}_A \rho_\mu \right)$$

$$= \mathrm{Tr}_\mu(\rho_{D|\mu C} \rho_{B|A\mu} \rho_\mu) \mathrm{P}_{C|B} \mathrm{P}_A$$

$$= \mathrm{Q}_{BD|AC} \mathrm{P}_{C|B} \mathrm{P}_A$$

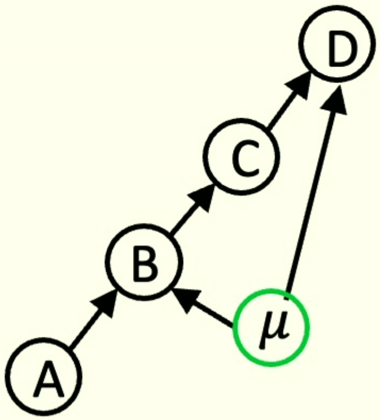
$\mathrm{Q}_{BD|AC}$

Is compatible with the subgraph:



This subgraph has d-separation relations implying

$D \perp A|C$ or equivalently, $\mathrm{Q}_{D|AC} = \mathrm{Q}_{D|C}$



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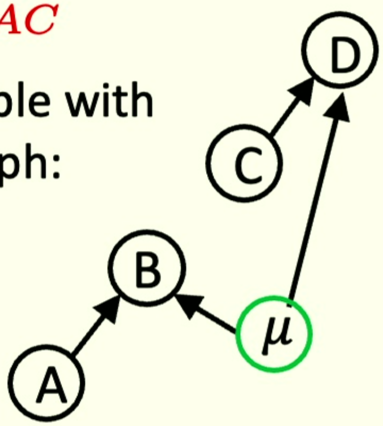
$$= \mathrm{Q}_{BD|AC}\mathrm{P}_{C|B}\mathrm{P}_A$$

$\mathrm{Q}_{BD|AC}$

Is compatible with the subgraph:

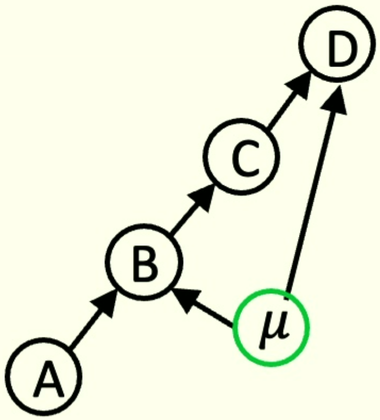


This subgraph has d-separation relations implying

$D \perp A|C$ or equivalently, $\mathrm{Q}_{D|AC} = \mathrm{Q}_{D|C}$

This implies equality constraints on $\mathrm{Q}_{BD|AC}$ and hence equality constraints on $\mathrm{P}_{ABCD}/\mathrm{P}_{C|B}\mathrm{P}_A$

Verma constraints



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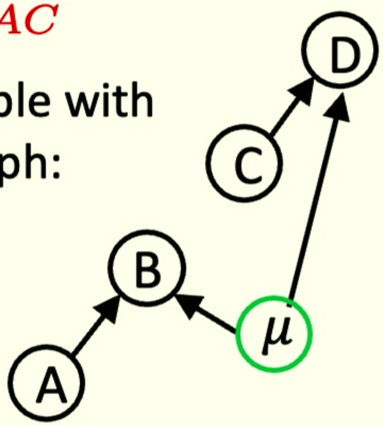
$$= Q_{BD|AC} P_{C|B} P_A$$

$Q_{BD|AC}$

Is compatible with the subgraph:



This subgraph has d-separation relations implying

$D \perp A|C$ or equivalently, $Q_{D|AC} = Q_{D|C}$

This implies equality constraints on $Q_{BD|AC}$ and hence equality constraints on $P_{ABCD}/P_{C|B}P_A$

Verma constraints

Note: *inequality* constraints on $Q_{BD|AC}$ (Tsirelson bound) also imply inequality constraints on $P_{ABCD}/P_{C|B}P_A$

Entropic techniques by quantifier elimination for quantum-latent-permitting causal models

R. Chaves and T. Fritz, Phys. Rev. A 85 (2012)
T. Fritz, New J. Phys. 14 103001 (2012)
R. Chaves, L. Luft, D. Gross, New J. Phys. 16, 043001 (2014)
R. Chaves, L. Luft, T. O. Maciel, D. Gross, D. Janzing, B. Schölkopf, Proceedings of UAI 2014
M. Weilenmann and R. Colbeck, Proc. Roy. Soc. A 473.2207 (2017): 20170483.

Entropy vector

For the joint distribution of the random variables $X_1, \ldots X_n$, the components of the entropy vector are the entropies of the marginals for all possible subsets of variables:

$$(H(X_1), H(X_2), \ldots, H(X_n), H(X_1X_2), H(X_1X_3), \ldots, H(X_1, X_2, \cdots, X_n))$$

An outer approximation to the entropy cone: the Shannon cone

Monotonicity

$H(\mathbf{X}A) \geq H(\mathbf{X})$

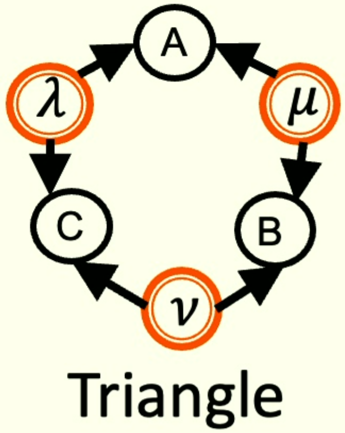
for every variable A and sets of variables **X**

Submodularity

$H(\mathbf{X}) + H(\mathbf{X}AB) \leq H(\mathbf{X}A) + H(\mathbf{X}B)$

where A and B are variables not in the set **X**

Entropic constraint for the triangle scenario

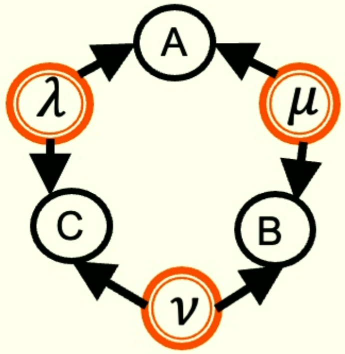


T. Fritz, 2012

$$A \perp B|\mu \quad \lambda \perp \mu\nu$$
$$A \perp C|\lambda \quad \mu \perp \lambda\nu$$
$$B \perp C|\nu \quad \nu \perp \lambda\mu$$

$$A \perp B|\mu \implies I(A:B|\mu) = 0 \implies I(A:B) \leq I(A:\mu)$$

By Shannon-type inequalities



Triangle

Entropic constraint for the triangle scenario

T. Fritz, 2012

$$A \perp B|\mu \quad \lambda \perp \mu\nu$$
$$A \perp C|\lambda \quad \mu \perp \lambda\nu$$
$$B \perp C|\nu \quad \nu \perp \lambda\mu$$

$$A \perp B|\mu \implies I(A:B|\mu) = 0 \implies I(A:B) \leq I(A:\mu)$$
$$A \perp C|\lambda \implies I(A:C|\lambda) = 0 \implies I(A:C) \leq I(A:\lambda)$$

By Shannon-type inequalities

$$I(A:B) + I(A:C) \leq I(A:\mu) + I(A:\lambda)$$
$$\leq H(A) + I(\mu:\lambda)$$

By Shannon-type inequalities

$$\mu \perp \lambda \implies I(\mu:\lambda) = 0$$

$$I(A:B) + I(A:C) \leq H(A)$$

von Neumann entropy obeys

Submodularity

H(**X**) + H(**X**AB) ≤ H(**X**A) + H(**X**B)

where A and B are systems not in the set **X**

But does not obey

Monotonicity

H(**X**A) ≥ H(**X**)

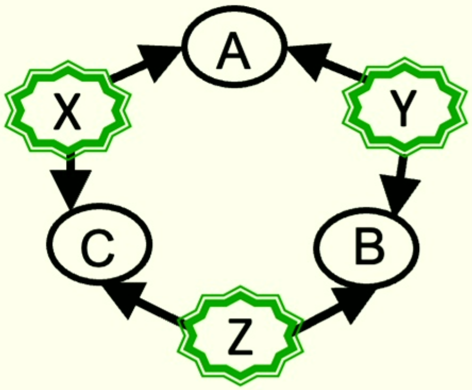
for every system A and sets of systems **X**

Instead, it obeys a weaker condition (weak monotonicity)

The constraints obtained by entropic techniques via quantifier elimination generally do not distinguish quantum and classical models

Inflation technique for quantum-latent-permitting causal models

Quantum triangle model



$\rho_{A|XY}$

$\rho_{B|YZ}$

$\rho_{C|XZ}$

ρ_X

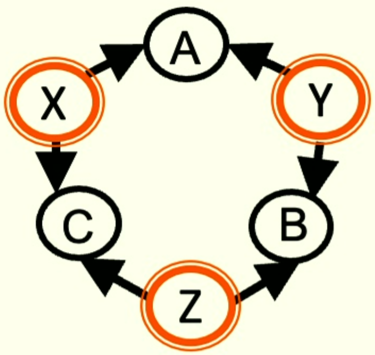
ρ_Y

ρ_Z

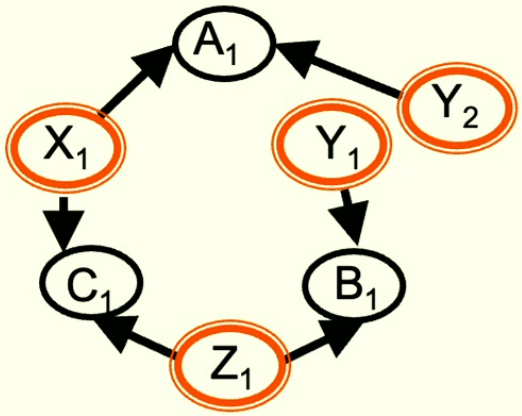
$$[\rho_{A|XY}, \rho_{B|YZ}] = 0$$
$$[\rho_{B|YZ}, \rho_{C|XZ}] = 0$$
$$[\rho_{A|XY}, \rho_{C|XZ}] = 0$$

$$P_{ABC} = \mathrm{Tr}_{XYZ}\left(\rho_{A|XY}\rho_{B|YZ}\rho_{C|XZ}\rho_X\rho_Y\rho_Z\right)$$

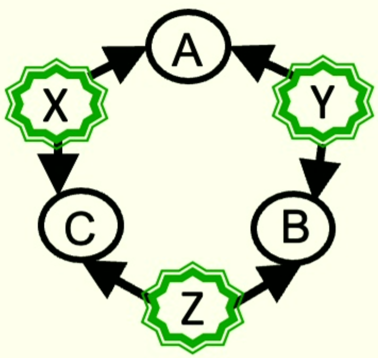
Recall the classical case:



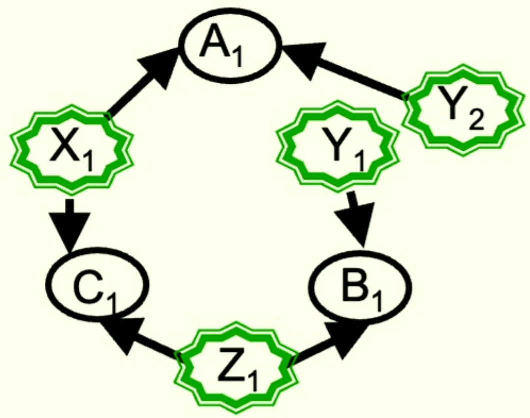
Triangle



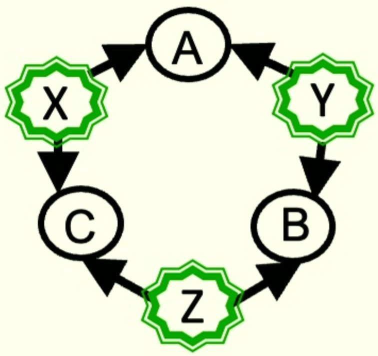
Cut inflation of Triangle



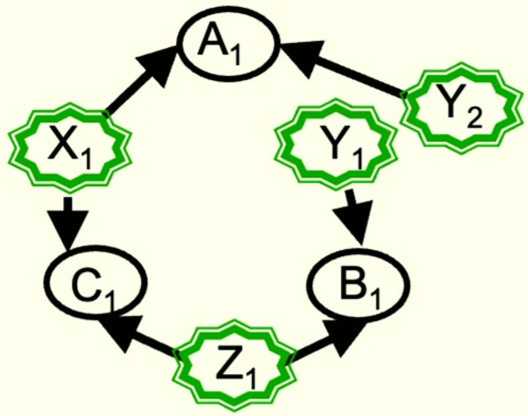
Quantum triangle



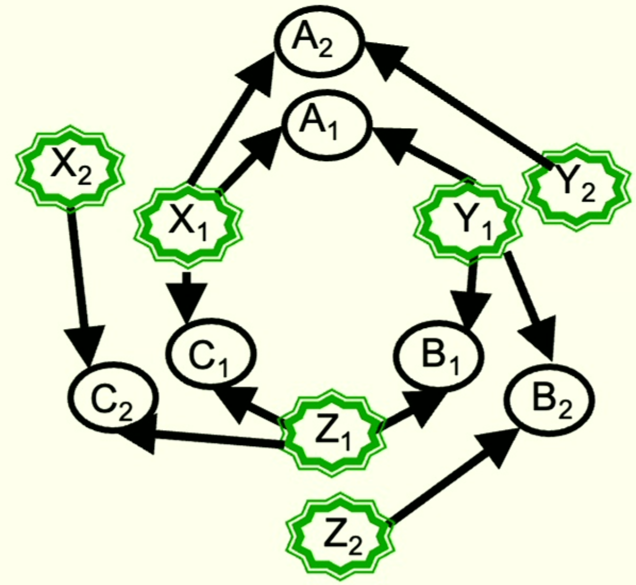
Cut inflation of Quantum Triangle



Quantum triangle



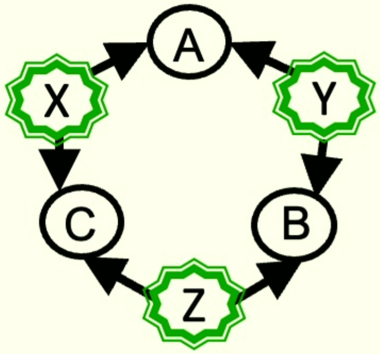
Cut inflation of Quantum Triangle



Spiral inflation of Quantum Triangle?

Not meaningful

Quantum model M on DAG G



$\rho_{A|XY}$

$\rho_{B|YZ}$

$\rho_{C|XZ}$

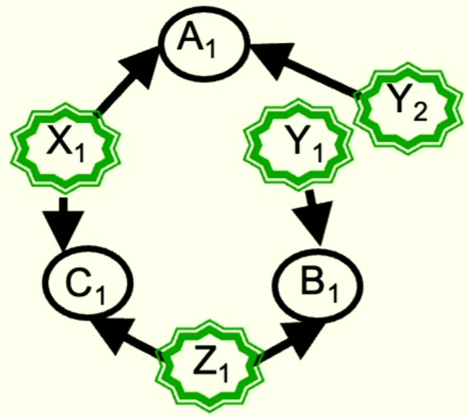
ρ_X

ρ_Y

ρ_Z

$$[\rho_{A|XY}, \rho_{B|YZ}] = 0$$
$$[\rho_{B|YZ}, \rho_{C|XZ}] = 0$$
$$[\rho_{A|XY}, \rho_{C|XZ}] = 0$$

Quantum model M' = G → G' Inflation of M



$\rho_{A_1|X_1Y_2}$

$\rho_{B_1|Y_1Z_1}$

$\rho_{C_1|X_1Z_1}$

ρ_{X_1}

ρ_{Y_1}

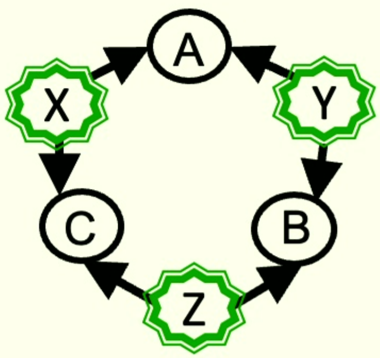
ρ_{Y_2}

ρ_{Z_1}

with symmetry constraint: $\rho_{Y_1} = \rho_{Y_2}$

$$[\rho_{B_1|Y_1Z_1}, \rho_{C_1|X_1Z_1}] = 0$$
$$[\rho_{A_1|X_1Y_2}, \rho_{C_1|X_1Z_1}] = 0$$

Quantum model M on DAG G



$\rho_{A|XY}$

$\rho_{B|YZ}$

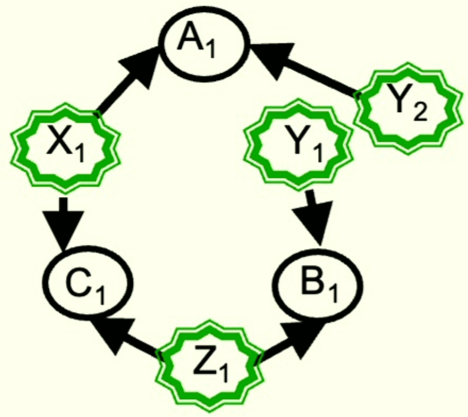
$\rho_{C|XZ}$

ρ_X

ρ_Y

ρ_Z

Quantum model M' = G → G' Inflation of M



$\rho_{A_1|X_1Y_2}$

$\rho_{B_1|Y_1Z_1}$

$\rho_{C_1|X_1Z_1}$

ρ_{X_1}

ρ_{Y_1}

ρ_{Y_2}

ρ_{Z_1}

with symmetry constraint:

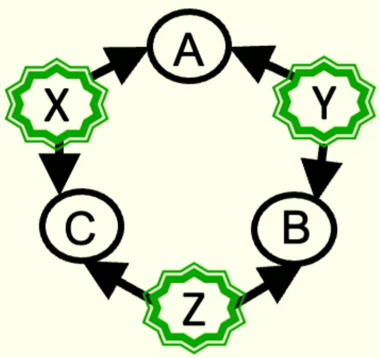
$\rho_{Y_1} = \rho_{Y_2}$

$\{A_1C_1\}$ is an injectable set

$$P_{A_1C_1} = \mathrm{Tr}_{X_1Y_2Z_1}\left(\rho_{A_1|X_1Y_2}\rho_{C_1|X_1Z_1}\rho_{X_1}\rho_{Y_2}\rho_{Z_1}\right)$$

$$P_{AC} = \mathrm{Tr}_{XYZ}\left(\rho_{A|XY}\rho_{C|XZ}\rho_X\rho_Y\rho_Z\right)$$

Quantum model M on DAG G



$\rho_{A|XY}$

$\rho_{B|YZ}$

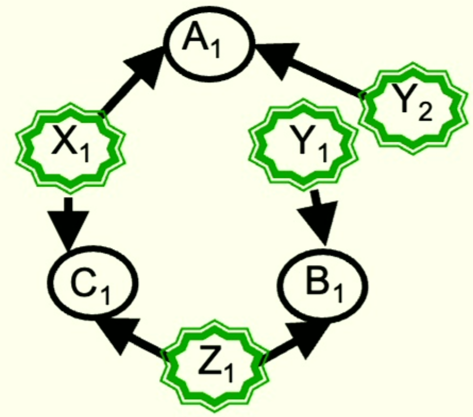
$\rho_{C|XZ}$

ρ_X

ρ_Y

ρ_Z

Quantum model M' = G → G' Inflation of M



$\rho_{A_1|X_1Y_2}$

$\rho_{B_1|Y_1Z_1}$

$\rho_{C_1|X_1Z_1}$

ρ_{X_1}

ρ_{Y_1}

ρ_{Y_2}

ρ_{Z_1}

with symmetry constraint:

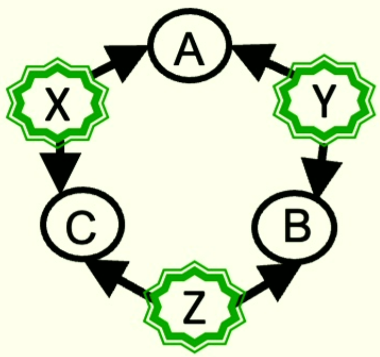
$\rho_{Y_1} = \rho_{Y_2}$

$\{A_1C_1\}$ is an injectable set

$$P_{A_1C_1} = \mathrm{Tr}_{X_1Y_2Z_1}\left(\rho_{A_1|X_1Y_2}\rho_{C_1|X_1Z_1}\rho_{X_1}\rho_{Y_2}\rho_{Z_1}\right)$$

$$P_{AC} = \mathrm{Tr}_{XYZ}\left(\rho_{A|XY}\rho_{C|XZ}\rho_X\rho_Y\rho_Z\right)$$

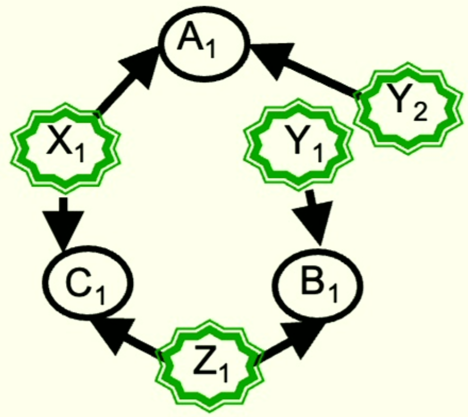
$$P_{AC} \text{ compatible with } M \quad \Longrightarrow \quad P_{A_1C_1} = P_{AC} \text{ compatible with } M'$$



is incompatible with

$$P_{ABC} = \frac{1}{2}[000] + \frac{1}{2}[111]$$

Quantum model M' = G → G' Inflation of M



$\rho_{A_1|X_1Y_2}$

$\rho_{B_1|Y_1Z_1}$

$\rho_{C_1|X_1Z_1}$

ρ_{X_1}

ρ_{Y_1}

ρ_{Y_2}

ρ_{Z_1}

with symmetry constraint:

$\rho_{Y_1} = \rho_{Y_2}$

$$(P_{A_1C_1}, P_{B_1C_1})$$

where $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is not compatible with M'

Quantum model M' = G → G' Inflation of M

Proof:

If $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

then

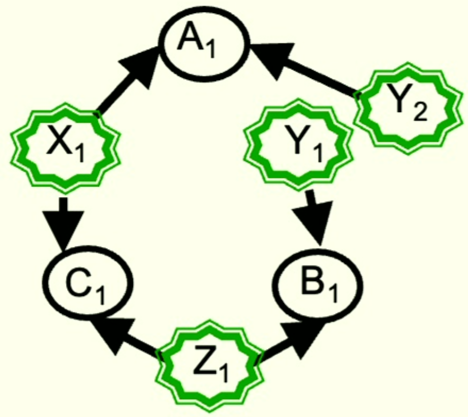
$$P_{A_1B_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$$

(recall example 3 of marginal problem)

But this violates $A_1 \perp B_1$

which is required by the d-separation relation $A_1 \perp_d B_1$

Which is quantum valid



$\rho_{A_1|X_1Y_2}$

$\rho_{B_1|Y_1Z_1}$

$\rho_{C_1|X_1Z_1}$

ρ_{X_1}

ρ_{Y_1}

ρ_{Y_2}

ρ_{Z_1}

with symmetry constraint: $\rho_{Y_1} = \rho_{Y_2}$

$$(P_{A_1C_1}, P_{B_1C_1})$$

where $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is not compatible with M'

M' = G → G' Inflation of M

$\mathcal{S} \subseteq \mathrm{ImagesInjectableSets}(G)$

$\mathcal{S}' \subseteq \mathrm{InjectableSets}(G')$

$$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$$
is compatible with M

$$\Longrightarrow$$

$$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is compatible with M'

Let $I_{\mathcal{S}}$ be an inequality that acts on the family of distributions $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

Whenever

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$ is compatible with M $\implies$ $I_{\mathcal{S}}$ is **satisfied** for $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

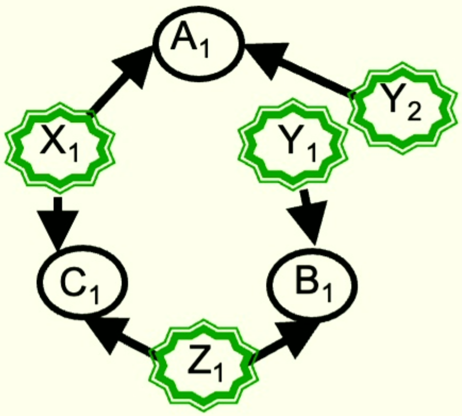
we say that

$I_{\mathcal{S}}$ is a **causal compatibility inequality** for model M

$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$$
is a valid set of marginals

$$\Longrightarrow$$

$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1}) \text{ satisfy}$$
$$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$$

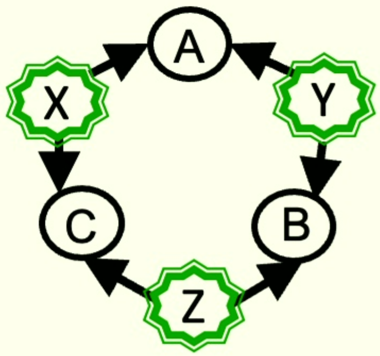


$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$$
is compatible with M'

$$\Longrightarrow A_1 \perp B_1 \Longrightarrow P_{A_1B_1} = P_{A_1}P_{B_1}$$

which is quantum valid

binary A, B and C

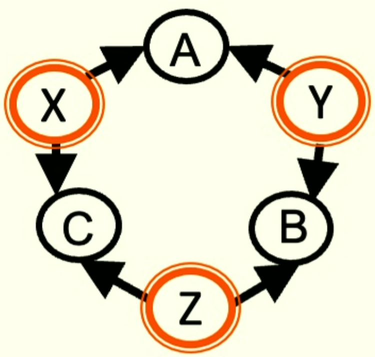


$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

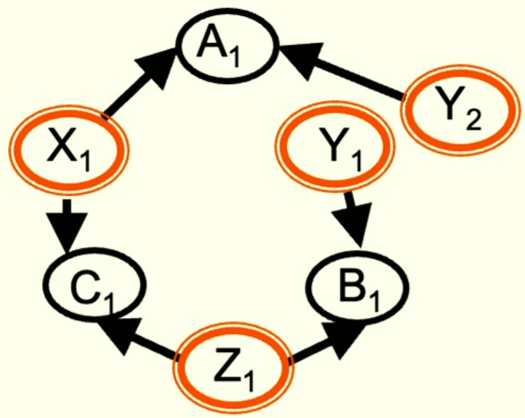
$$\langle AC \rangle + \langle BC \rangle \leq 1 + \langle A \rangle \langle B \rangle$$

$$I(A : C) + I(C : B) \leq H(C)$$

Quantum valid

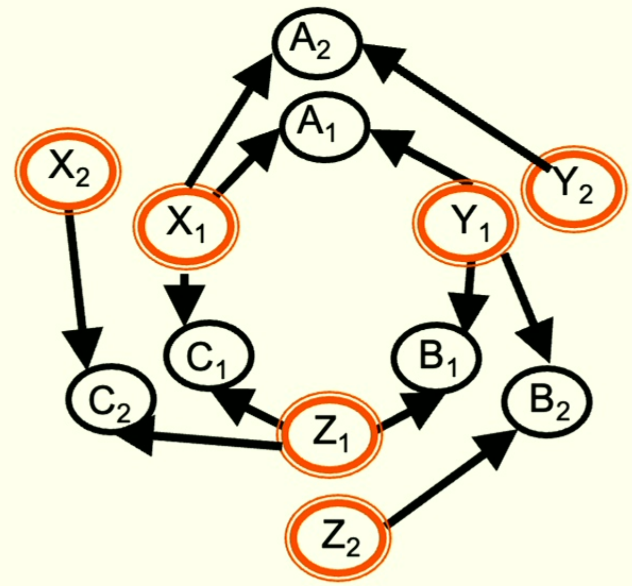


Triangle



Cut inflation of Triangle

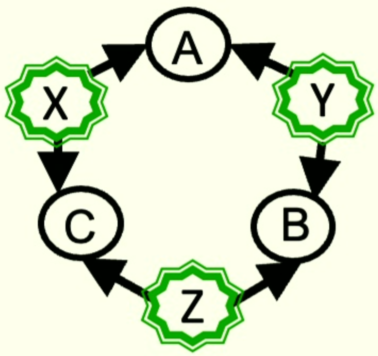
Non-fan-out



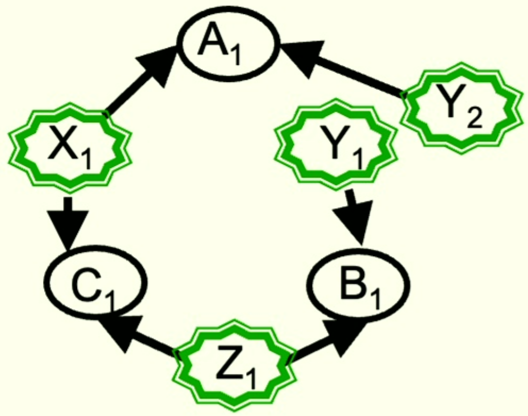
Spiral inflation of Triangle

Fan-out

Fan-out vs. non-fan-out inflations

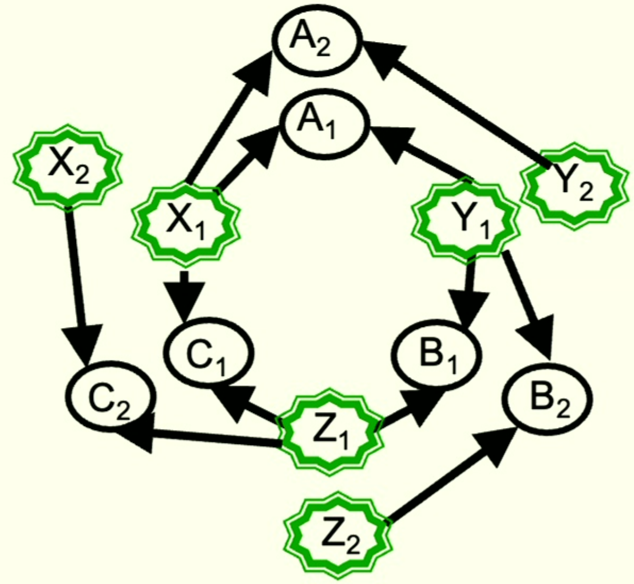


Quantum triangle



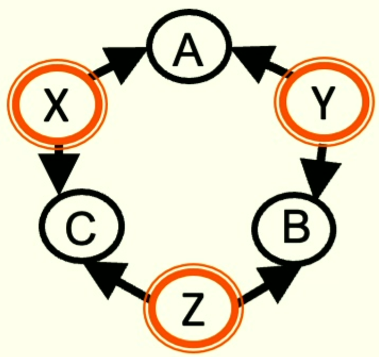
Cut inflation of Quantum Triangle

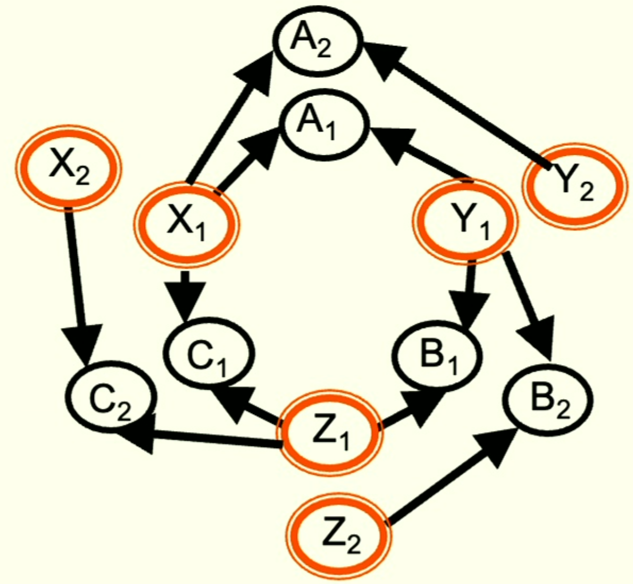
Non-fan-out



Spiral inflation of Quantum Triangle?

Fan-out

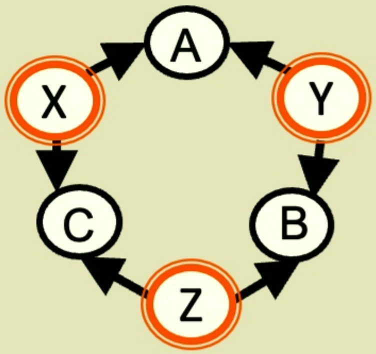




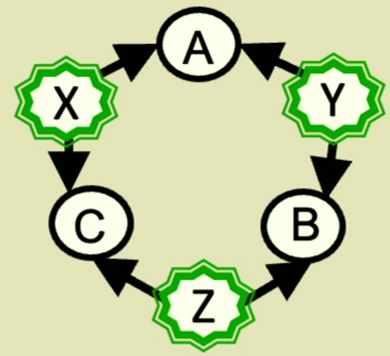
$$P_A(1)P_B(1)P_C(1)$$
$$\leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1)$$
$$+ P_{AC}(11)P_B(1) + P_{ABC}(000)$$

$$P_{A_2}(1)P_{B_2}(1)P_{C_2}(1)$$
$$\leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1)$$
$$+ P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)$$

Classical Triangle model



Quantum-latent-permitting Triangle model

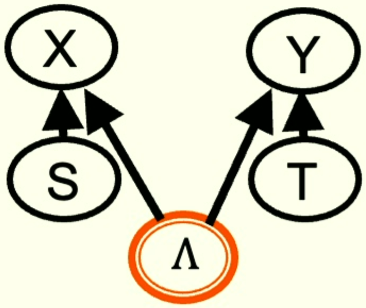


$$P_A(1)P_B(1)P_C(1)$$
$$\leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1)$$
$$+ P_{AC}(11)P_B(1) + P_{ABC}(000)$$

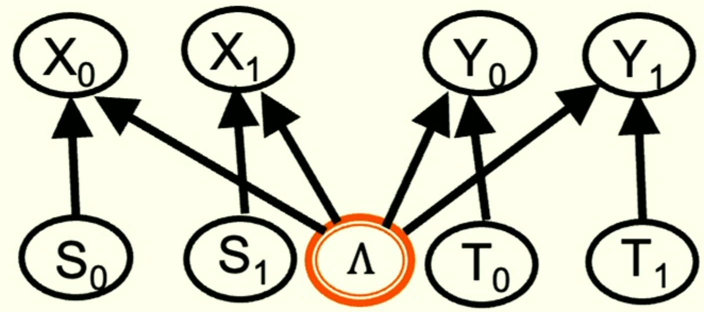
Other inequalities

There are inequalities that distinguish quantum and classical latents

Bell model M



Inflated model M’



fan-out

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

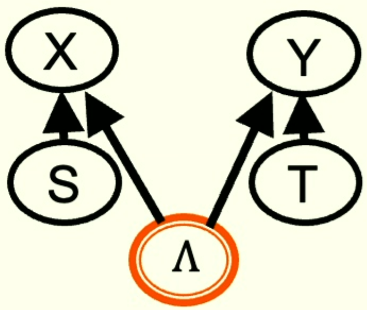
Causal compatibility inequality in M

$$\frac{1}{4}\sum_{x=y} P_{X_0Y_0|S_0T_0}(xy|00) + \frac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0T_1}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{X_1Y_0|S_1T_0}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_1T_1}(xy|11) \leq \frac{3}{4}$$

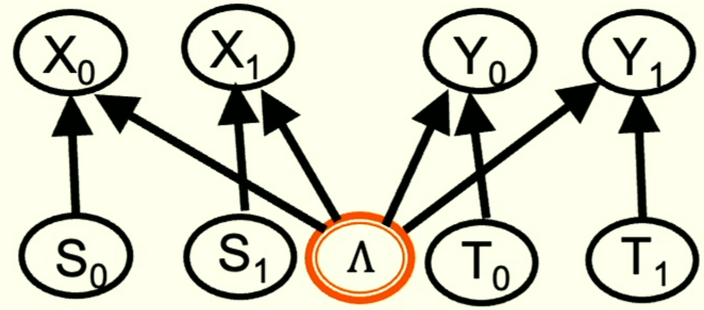
Causal compatibility inequality in M’

not quantum valid

Bell model M



Inflated model M'



fan-out

$I_{st}(X:Y)$ defined by $P_{XY|ST}(\cdot|st)$
$H_s(X)$ defined by $P_{X|S}(\cdot|s)$
$H_t(Y)$ defined by $P_{Y|T}(\cdot|t)$

$I_{st}(X_i:Y_j)$ defined by $P_{X_iY_j|S_iT_j}(\cdot|st)$
$H_s(X_i)$ defined by $P_{X_i|S_i}(\cdot|s)$
$H_t(Y_i)$ defined by $P_{Y_i|T_i}(\cdot|t)$

$$I_{00}(X:Y)+I_{01}(X:Y)+I_{10}(X:Y)-I_{11}(X:Y) \leq H_0(X)+H_0(Y)$$

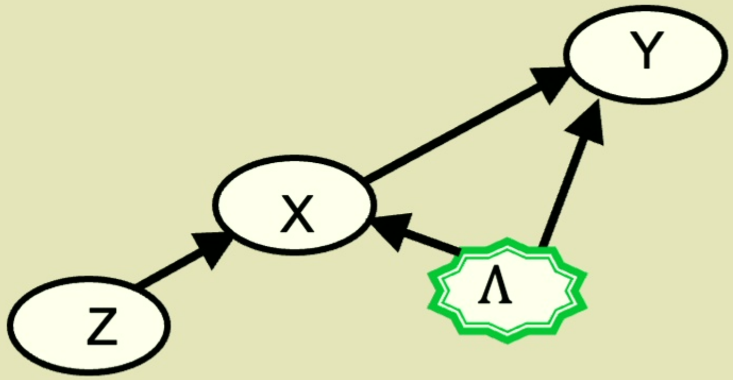
Causal compatibility inequality in M

$$I_{00}(X_0:Y_0)+I_{01}(X_0:Y_1)+I_{10}(X_1:Y_0)-I_{11}(X_1:Y_1) \leq H_0(X_0)+H_0(Y_0)$$

Causal compatibility inequality in M'

not quantum valid

Quantum Instrumental model



$\rho_{X|\Lambda Z}$

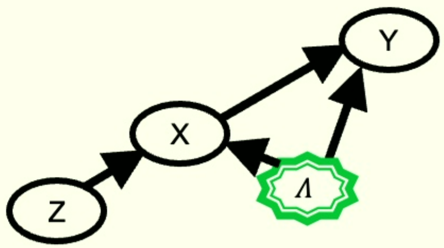
$\rho_{Y|\Lambda X}$

ρ_{Λ}

$$[\rho_{X|\Lambda Z}, \rho_{Y|\Lambda X}] = 0$$

$$P_{XY|Z} = \text{Tr}_{\Lambda}(\rho_{Y|X\Lambda}\rho_{X|Z\Lambda}\rho_{\Lambda})$$

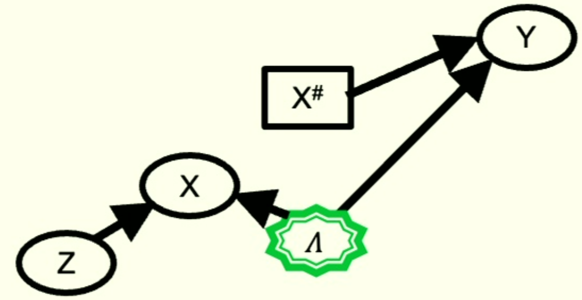
Quantum Instrumental model M



$\rho_{X|\Lambda Z}$
$\rho_{Y|\Lambda X}$
ρ_{Λ}

$$P_{XY|Z}(xy|z)$$
$$= \mathrm{Tr}_{\Lambda}(\langle yx|\rho_{Y|X\Lambda}|yx\rangle\langle xz|\rho_{X|Z\Lambda}|xz\rangle\rho_{\Lambda})$$

Interrupted version M'



$\rho_{X|\Lambda Z}$
$X^{\#} = x$
$\rho_{Y|\Lambda X^{\#}}$
ρ_{Λ}

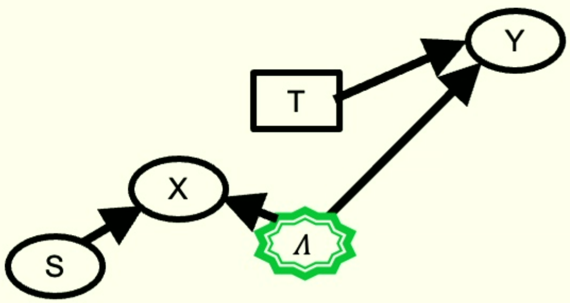
$$P_{XY|ZX^{\#}}(xy|zx)$$
$$= \mathrm{Tr}_{\Lambda}(\langle yx|\rho_{Y|X^{\#}\Lambda}|yx\rangle\langle xz|\rho_{X|Z\Lambda}|xz\rangle\rho_{\Lambda})$$

$P_{XY|Z}$
is compatible with M

$\Rightarrow$

some $P_{XY|ZX^{\#}}$ where
$$P_{XY|ZX^{\#}}(x\cdot|\cdot x) = P_{XY|Z}(x\cdot|\cdot)$$
is compatible with M'

Quantum Bell scenario



$$Y \perp_d S|T \implies P_{Y|ST} = P_{Y|T}$$ Quantum valid

$$P_{XY|ST}(00|00) + P_{XY|ST}(10|00) = P_{Y|T}(0|0)$$
$$P_{XY|ST}(01|00) + P_{XY|ST}(11|00) = P_{Y|T}(1|0)$$
$$P_{XY|ST}(00|10) + P_{XY|ST}(10|10) = P_{Y|T}(0|0)$$
$$P_{XY|ST}(01|10) + P_{XY|ST}(11|10) = P_{Y|T}(1|0)$$

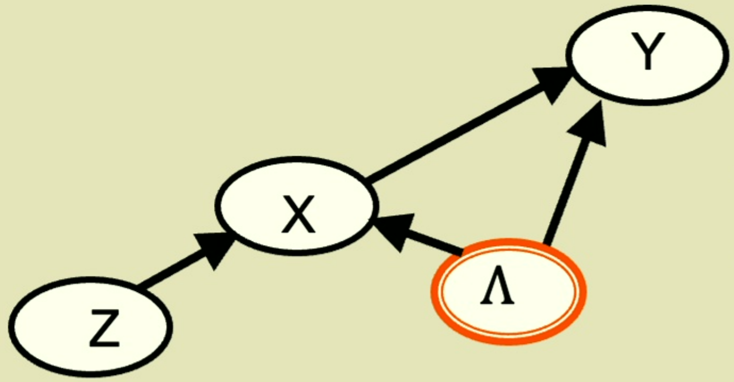
$$P_{XY|ST}(11|10) \geq 0 \implies P_{Y|T}(1|0) - P_{XY|ST}(01|10) \geq 0$$
$$P_{XY|ST}(10|00) \geq 0 \implies P_{Y|T}(0|0) - P_{XY|ST}(00|00) \geq 0$$

$$\implies P_{Y|T}(0|0) - P_{XY|ST}(00|00) + P_{Y|T}(1|0) - P_{XY|ST}(01|10) \geq 0$$

$$\implies P_{XY|ST}(00|00) + P_{XY|ST}(01|10) \leq 1$$

is an implication of equality constraints in the Bell scenario referring only to $\{P_{XY|ZX^{\#}}(xy|zx)\}_{x,y,z}$

Instrumental model



$P_{X|\Lambda Z}$

$P_{Y|\Lambda X}$

P_{Λ}

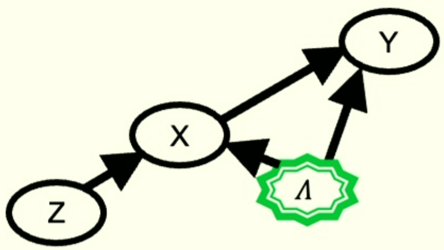
$$P_{XY|Z} = \sum_{\Lambda} P_{Y|X\Lambda} P_{X|Z\Lambda} P_{\Lambda}$$

|Z|=3
|X|=|Y|=2

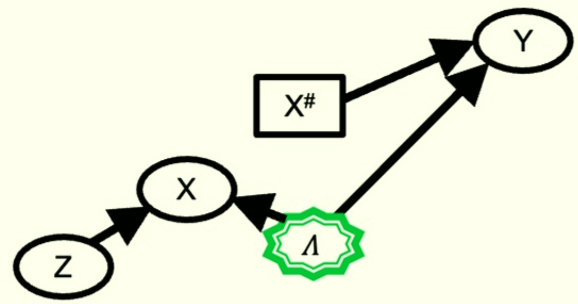
$$P_{XY|Z}(00|0) + P_{XY|Z}(11|0) + P_{XY|Z}(00|1) + P_{XY|Z}(10|1) + P_{XY|Z}(01|2) \leq 2$$

Bonet, 2001

Instrumental model M



Interrupted version M'



$$P_{XY|Z}(00|0) + P_{XY|Z}(01|1) \leq 1$$

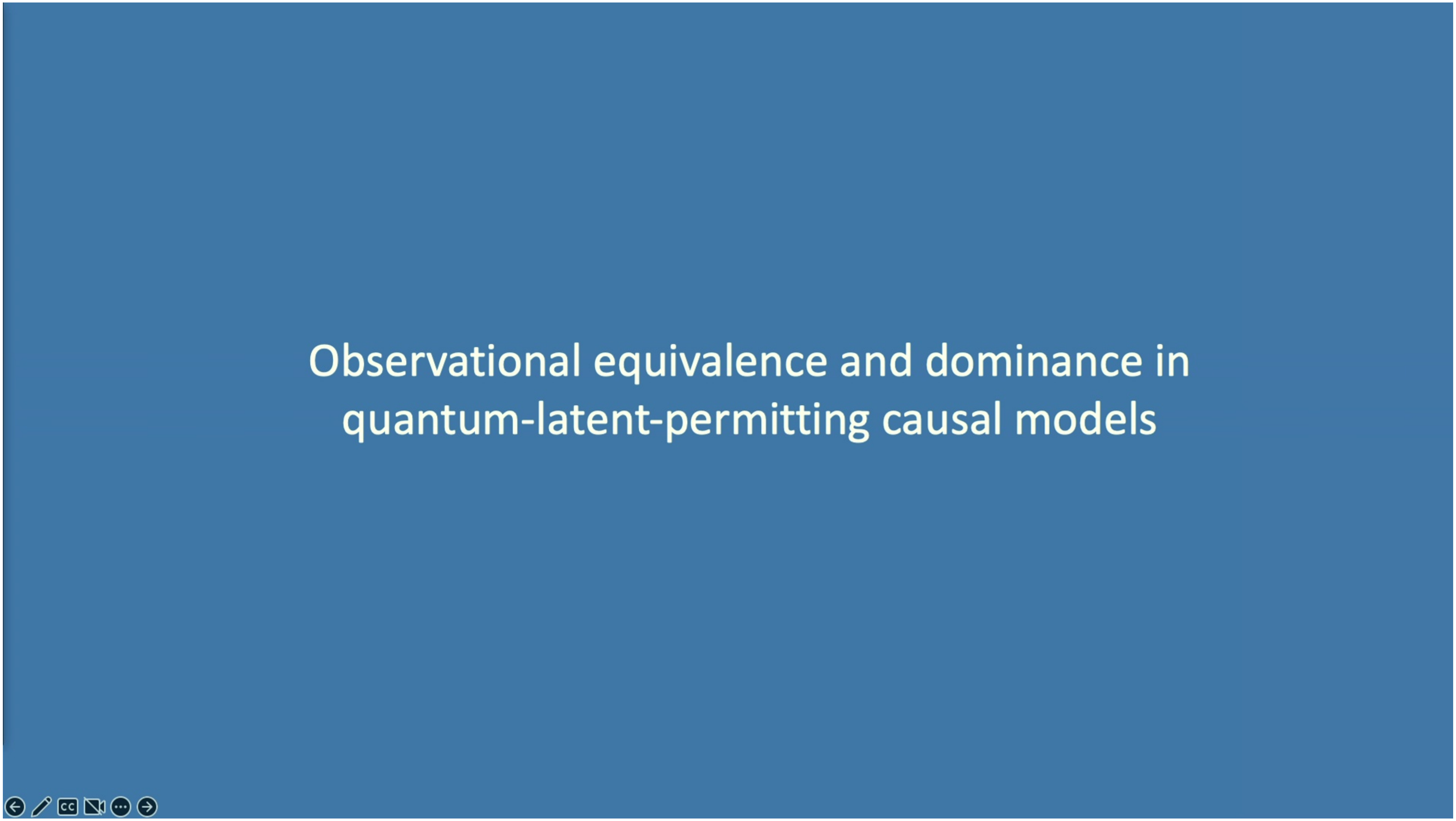
is a causal compatibility inequality in model M

⇐

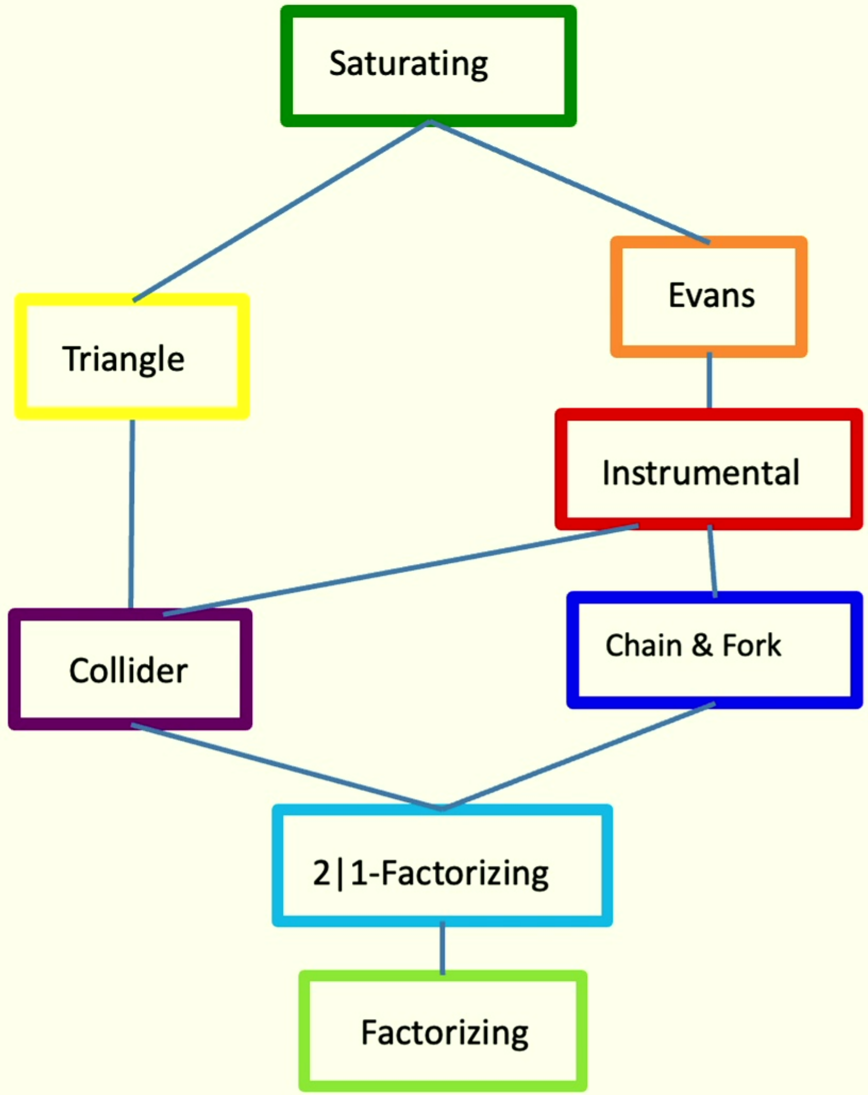
$$P_{XY|ZX^{\#}}(00|00) + P_{XY|ZX^{\#}}(01|10) \leq 1$$

is a causal compatibility inequality in model M'

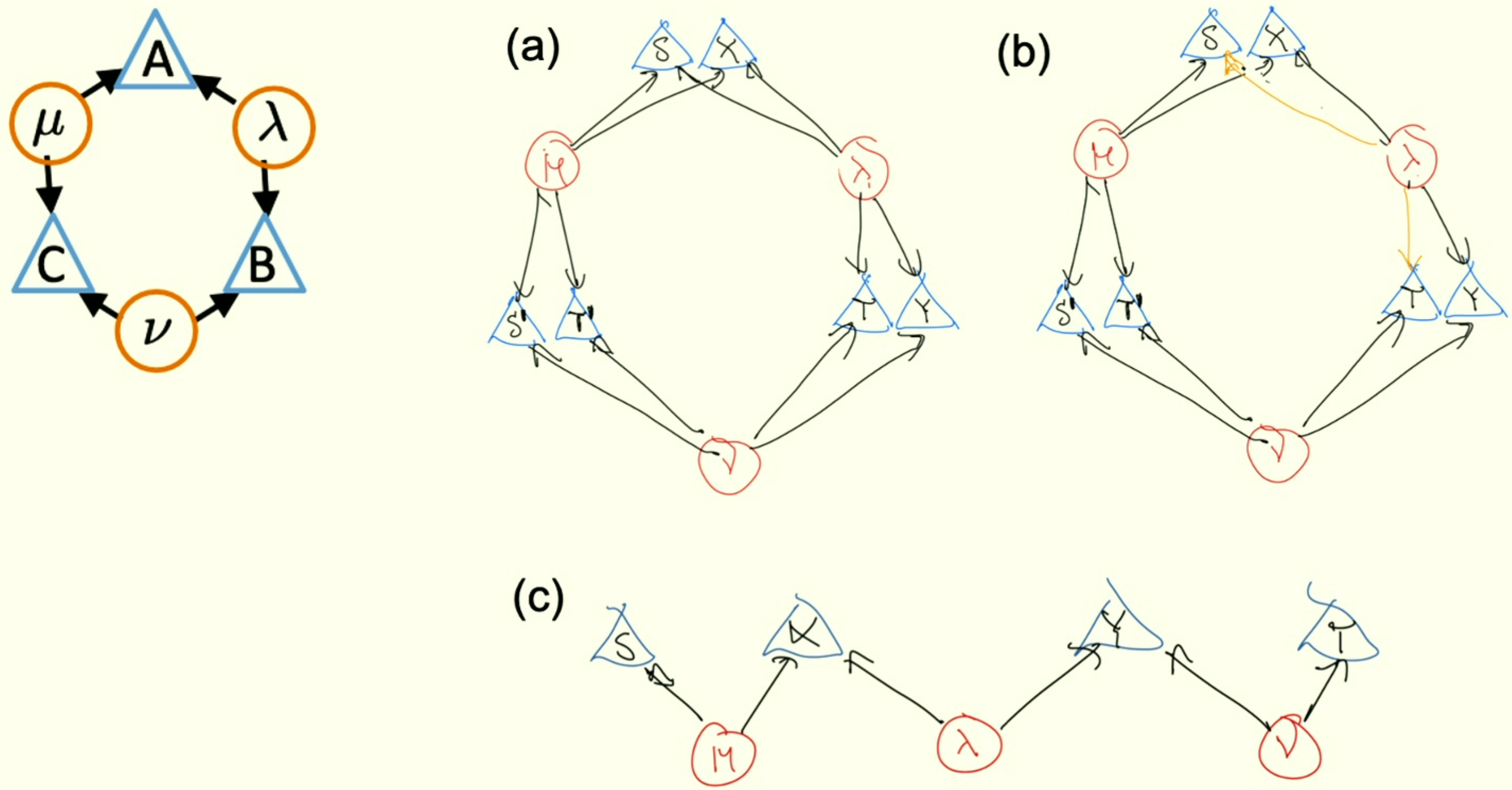
quantum valid


Observational equivalence and dominance in
quantum-latent-permitting causal models

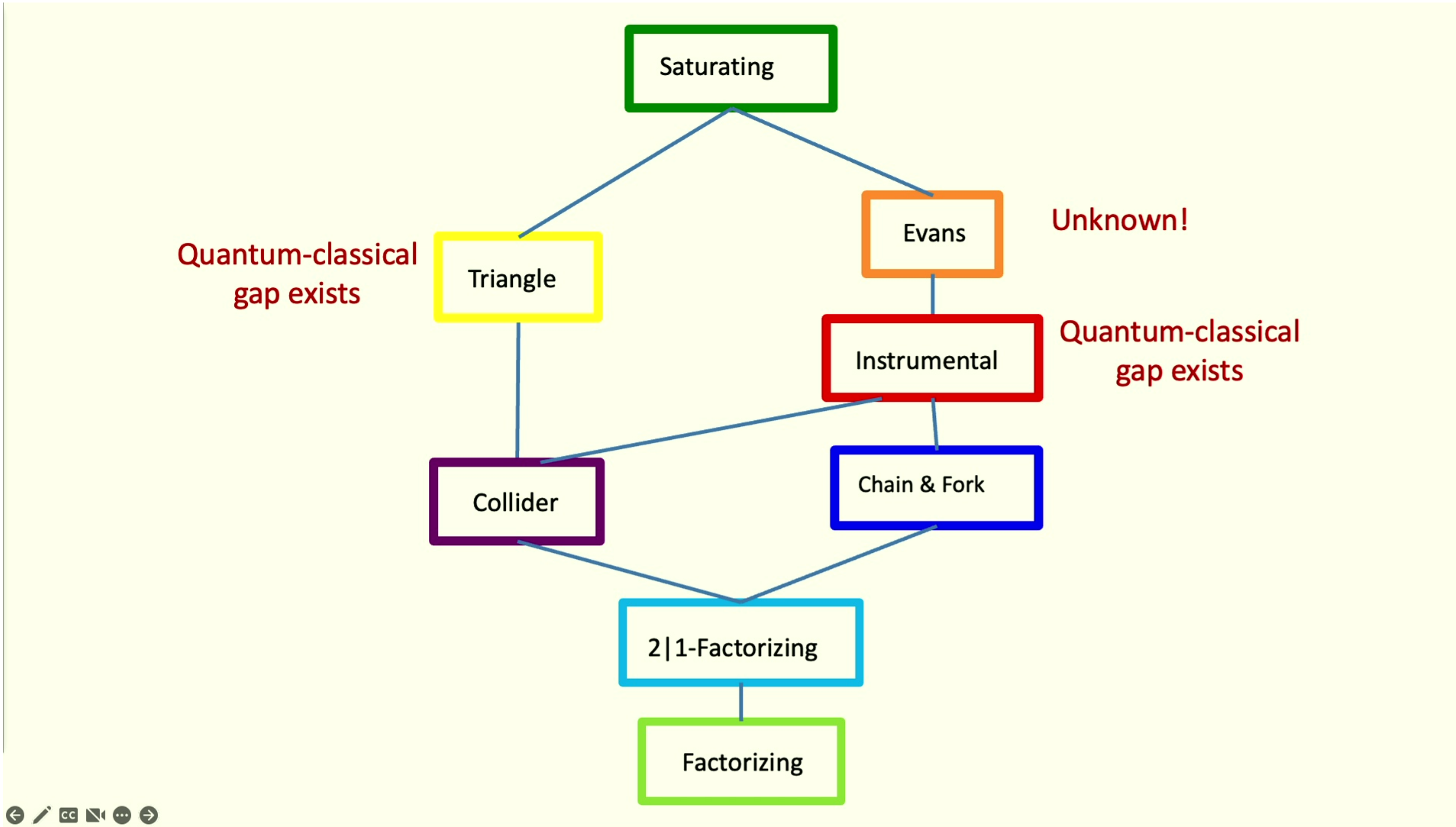
Observational equivalence and dominance in quantum-latent-permitting causal models


Saturating
Triangle
Evans
Instrumental
Collider
Chain & Fork
2|1-Factorizing
Factorizing

Therefore, the only possibility for a quantum-classical gap is in the Triangle, Evans, and Instrumental scenarios



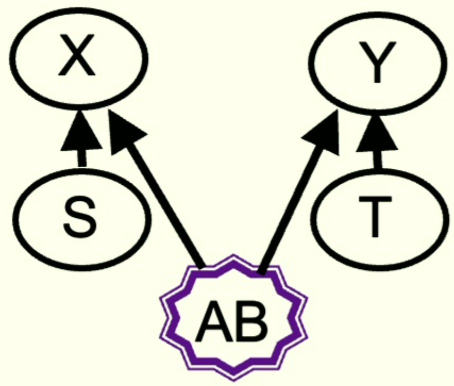
Fritz, New Journal of Physics 14, 103001 (2012).


Saturating
Triangle
Quantum-classical gap exists
Evans
Unknown!
Instrumental
Quantum-classical gap exists
Collider
Chain & Fork
2|1-Factorizing
Factorizing

GPT-latent-permitting causal models (Generalized Probabilistic Theories)

Henson, Lal and Pusey, New J. Phys. 16, 113043 (2014)
Fritz, Comm. Math. Phys. 341, 391 (2016)

GPT-latent Bell model



$$\mathbf{r}^A_{x|s} \in \mathbb{R}^{d_A}$$

$$\mathbf{r}^B_{y|t} \in \mathbb{R}^{d_B}$$

$$\mathbf{s}^{AB} \in \mathbb{R}^{d_A} \otimes \mathbb{R}^{d_B}$$

$$P_{XY|ST}(xy|st) = (\mathbf{r}^A_{x|s} \otimes \mathbf{r}^B_{y|t}) \cdot \mathbf{s}^{AB}$$

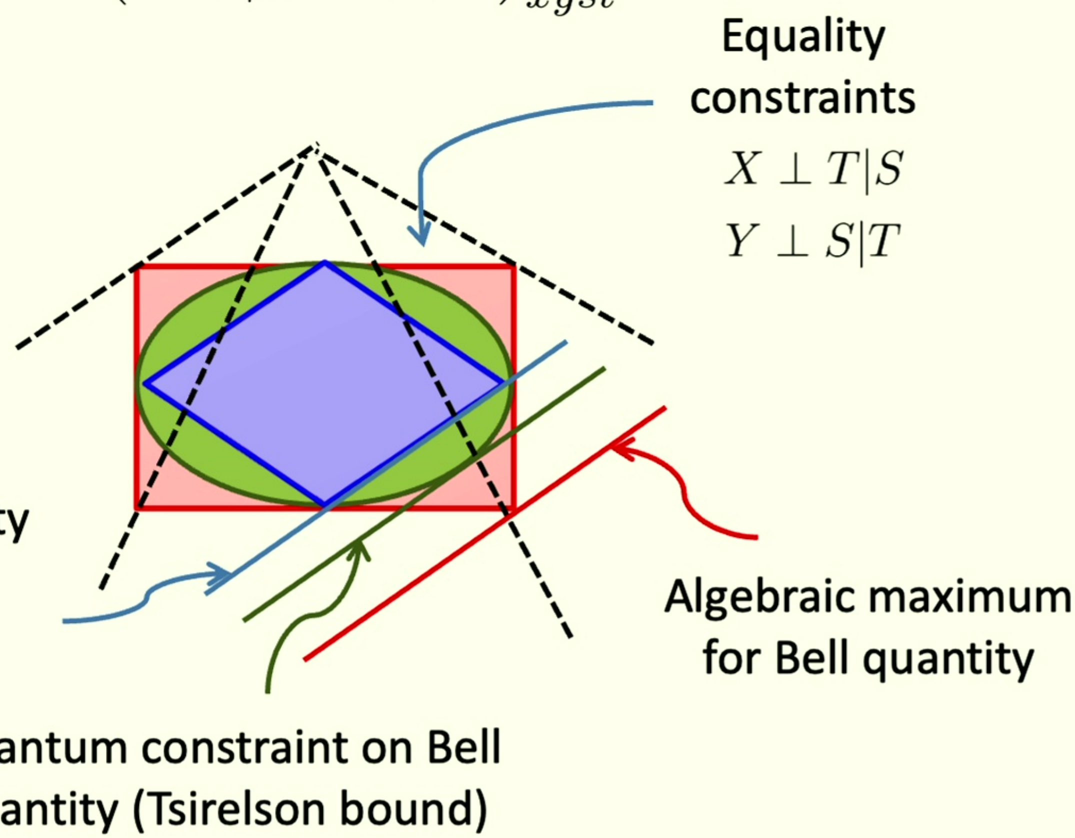
Popescu-Rohrlich box

$$P^{\mathrm{PR}}_{XY|ST}(xy|st) = \begin{cases} \frac{1}{2}[00] + \frac{1}{2}[11] \text{ if } (s,t) \in \{(0,0),(0,1),(1,0)\} \\ \frac{1}{2}[01] + \frac{1}{2}[10] \text{ if } (s,t) = (1,1) \end{cases}$$

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) = 1$$

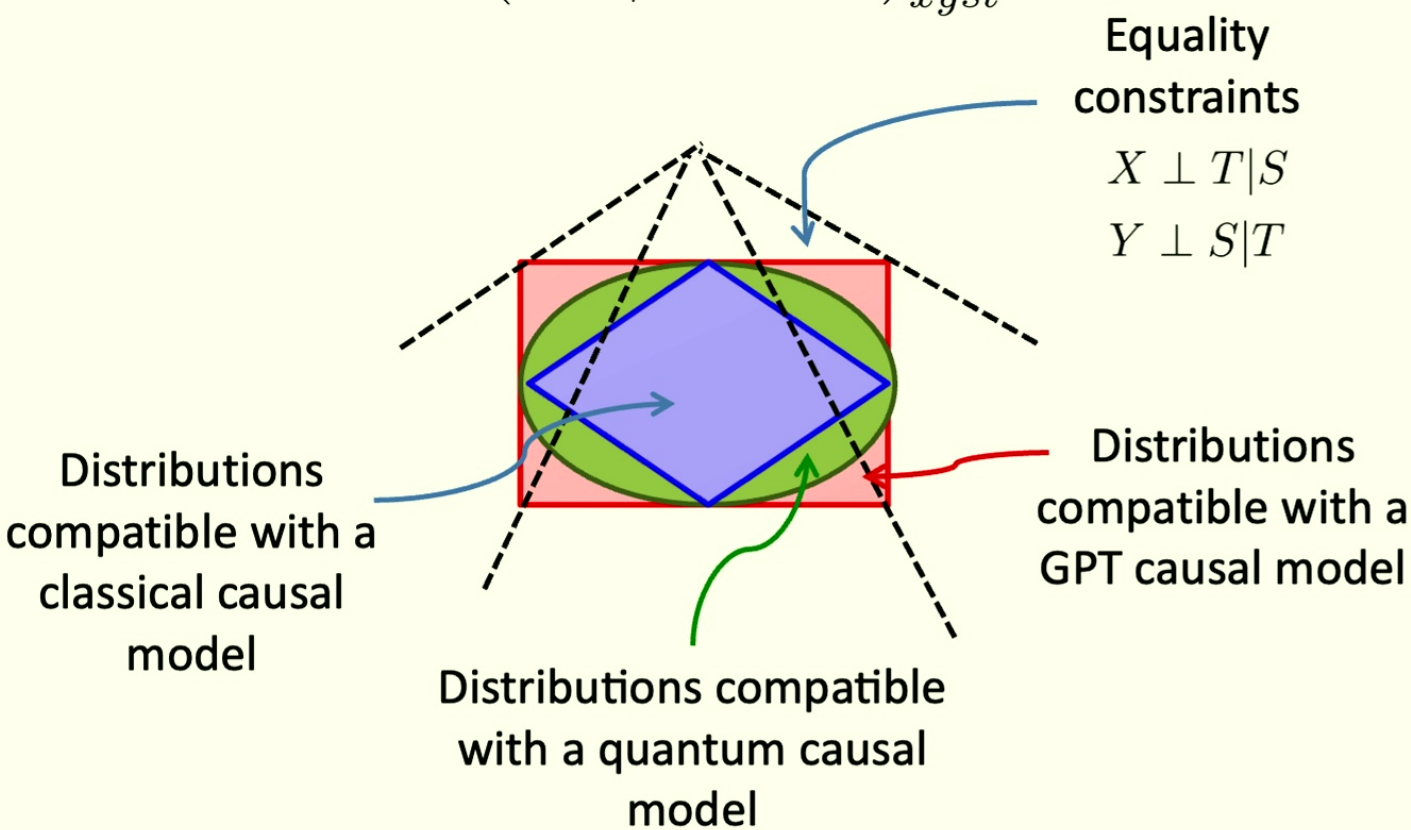
Space of compatible probability distributions

$$\vec{R} = \left(P_{XY|ST}(xy|st)\right)_{xyst}$$

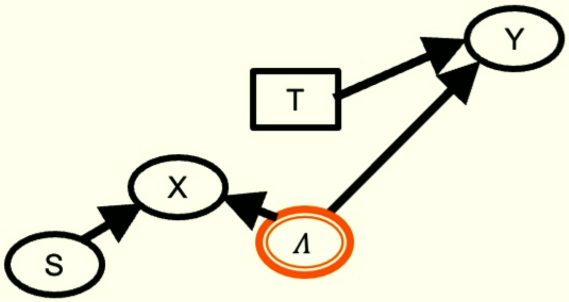


Space of compatible probability distributions

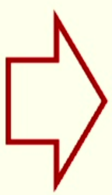
$$\vec{R} = \left(P_{XY|ST}(xy|st)\right)_{xyst}$$



Bell scenario $S \in \{0,1,2\}$



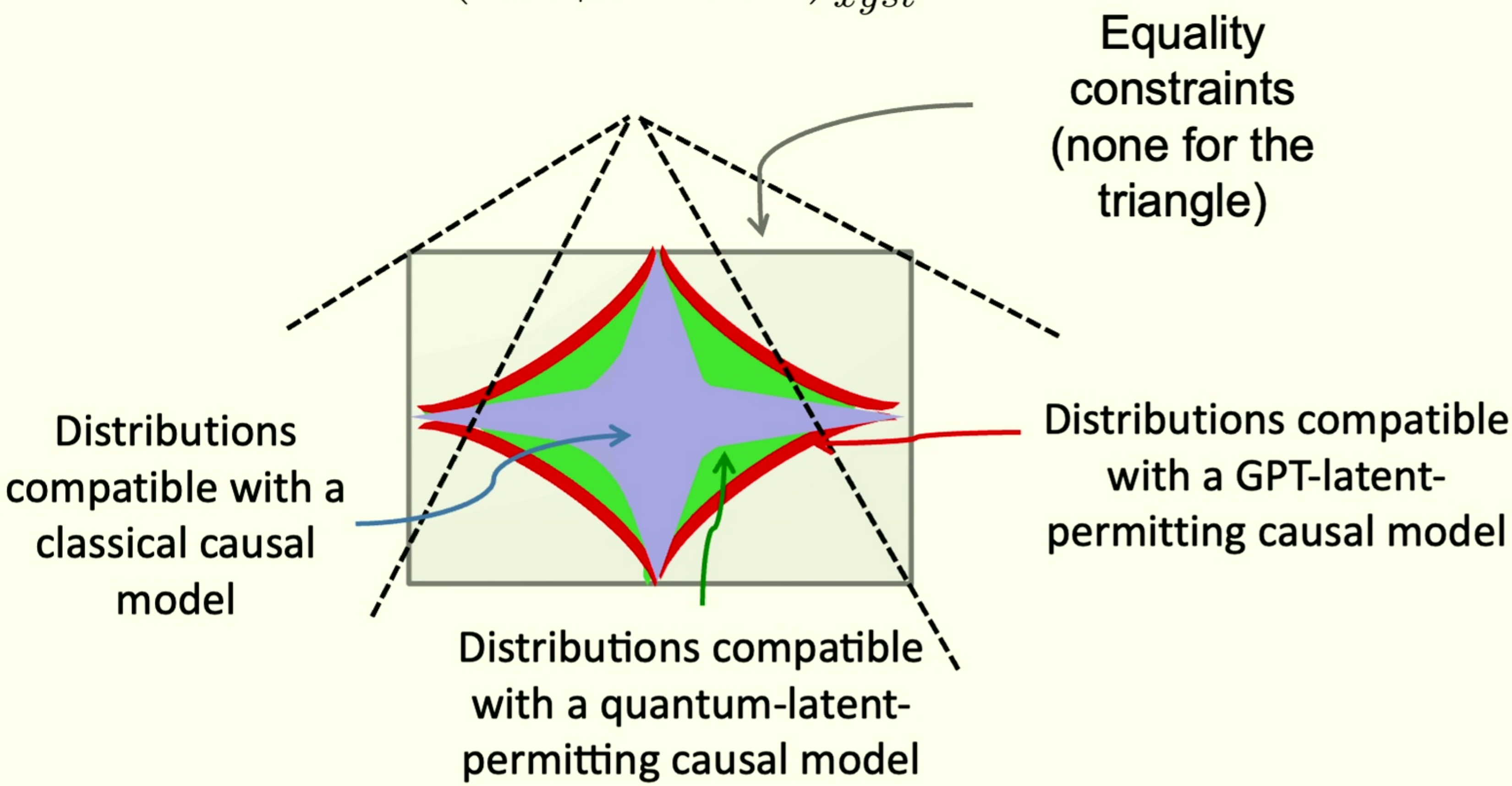
$$\tfrac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \tfrac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\tfrac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \tfrac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) - \tfrac{1}{2}P_{XY|ST}(11|20) \leq \tfrac{3}{4}$$ Classically



$$P_{XY|ST}(00|00) + P_{XY|ST}(11|01)$$
$$+P_{XY|ST}(00|10) + P_{XY|ST}(10|11)$$
$$+P_{XY|ST}(01|20) \leq 2$$ Classically

Space of compatible probability distributions

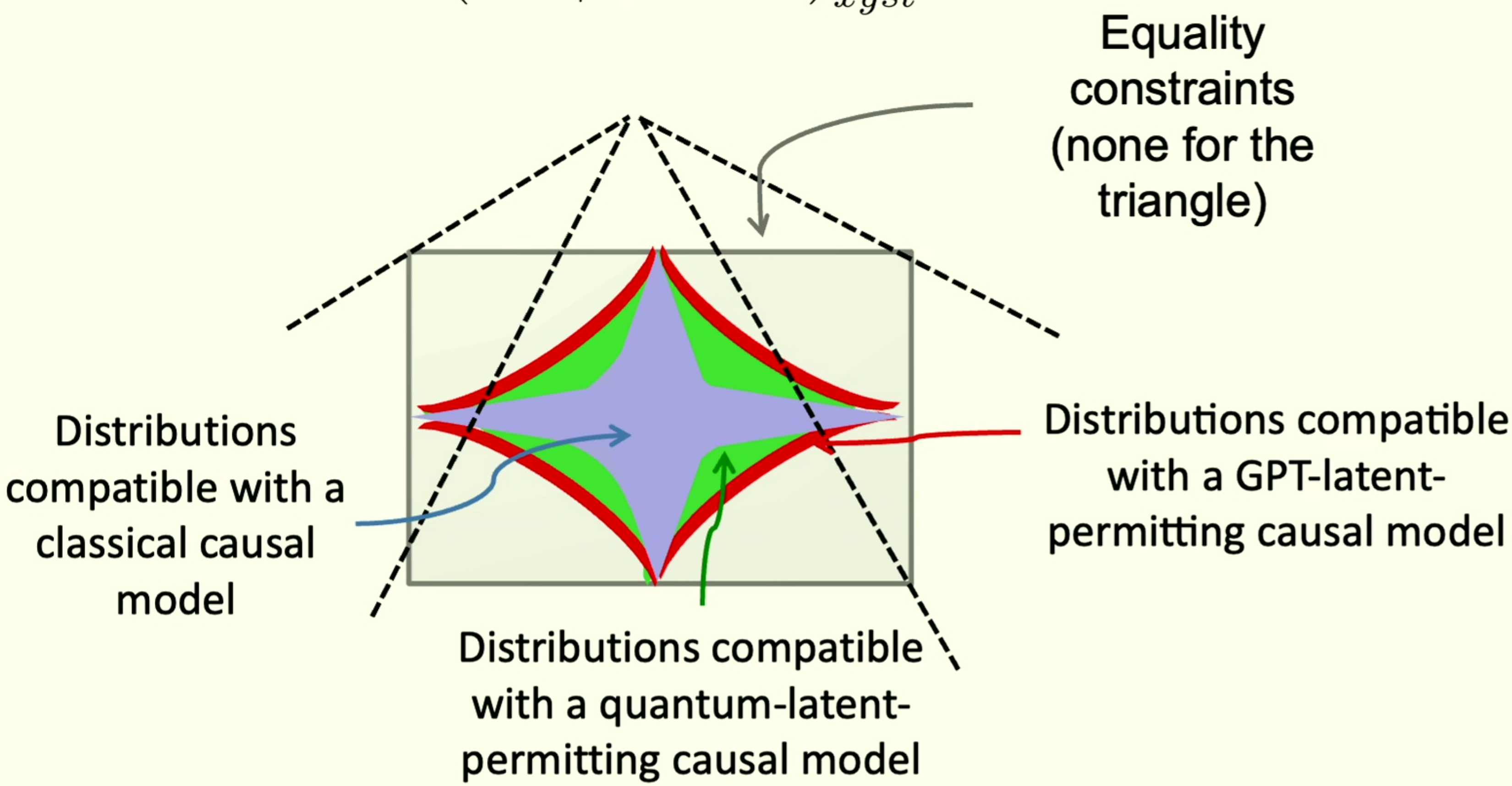
$$\vec{R} = \left(P_{XY|ST}(xy|st)\right)_{xyst}$$



Constraints from non-fan-out inflations are GPT valid

Space of compatible probability distributions

$$\vec{R} = \left(P_{XY|ST}(xy|st)\right)_{xyst}$$



How to characterize the **set** of distributions that are compatible with a quantum-latent-permitting causal model?

See “Quantum inflation technique”
Wolfe, Pozas-Kerstjens, Grinberg, Rosset, Acín, Navascués, Phys. Rev. X **11**, 021043 (2021)

Building on NPA hierarchy for Bell scenario

For convergence result, see:
Ligthart, Gachechiladze, and Gross, arXiv:2110.14659 (2021)

Left outcome and Right outcome

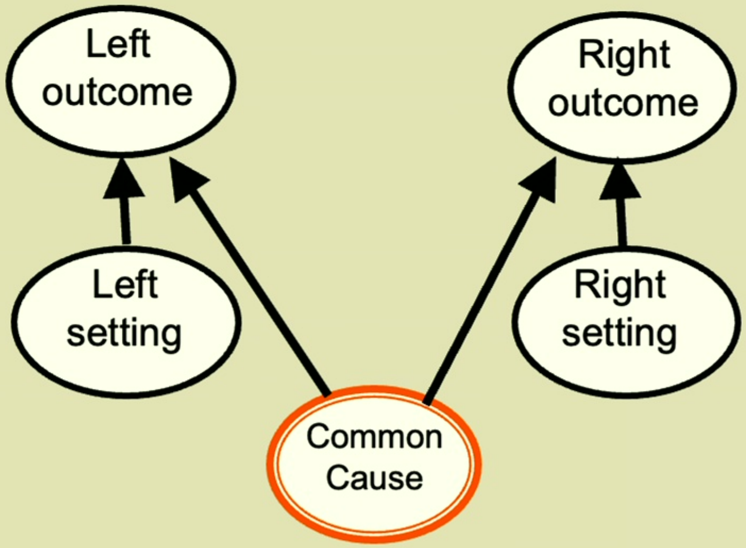
Left setting and Right setting		0 and 0	0 and 1	1 and 0	1 and 1
	0 and 0	43%	7%	7%	43%
	0 and 1	43%	7%	7%	43%
	1 and 0	43%	7%	7%	43%
	1 and 1	7%	43%	43%	7%

The evidence

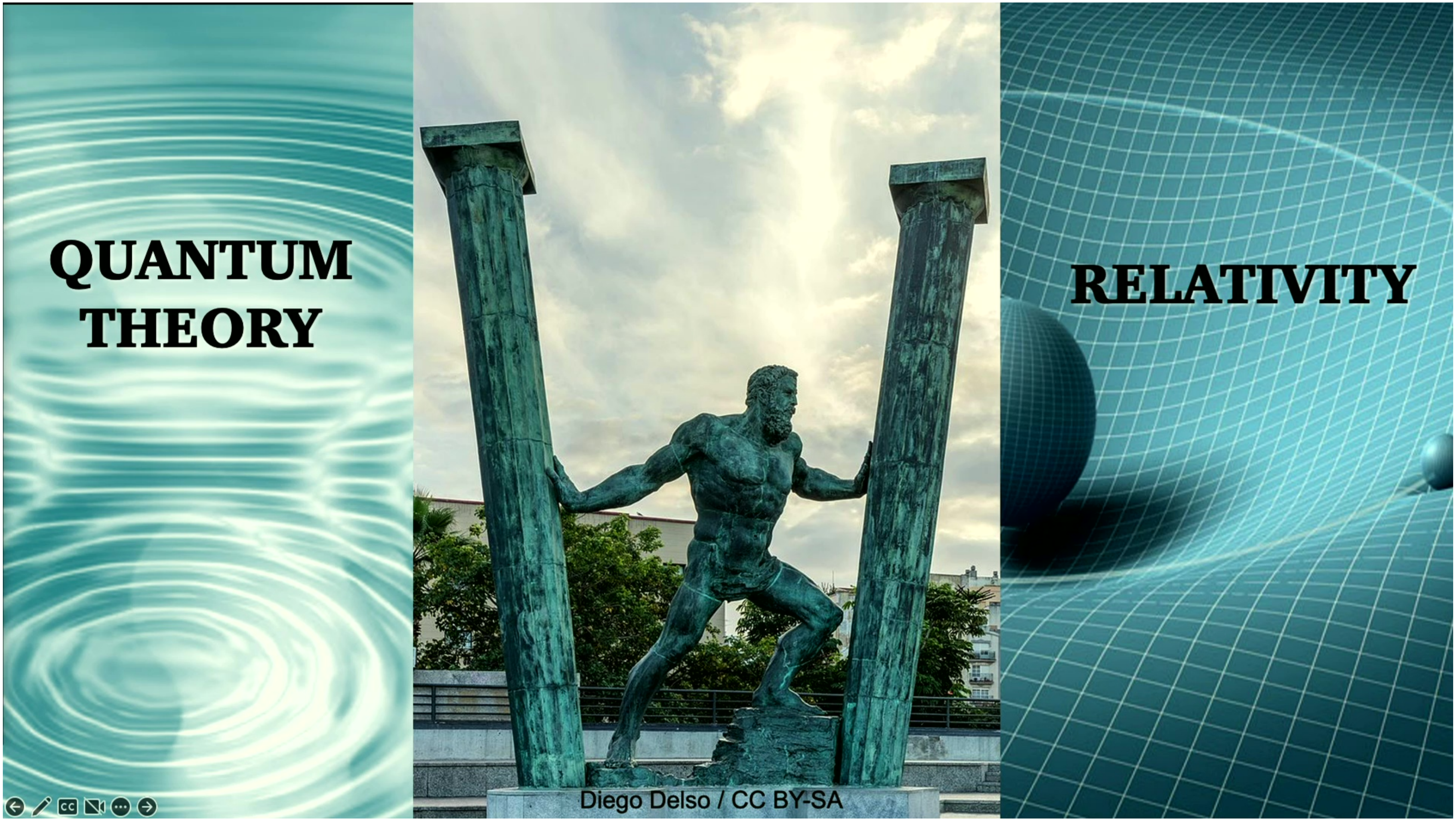
	Left outcome and Right outcome			
Left setting and Right setting	0 and 0	0 and 1	1 and 0	1 and 1
0 and 0	43%	7%	7%	43%
0 and 1	43%	7%	7%	43%
1 and 0	43%	7%	7%	43%
1 and 1	7%	43%	43%	7%

Violates Bell inequalities
(up to Tsirelson bound)

The natural hypothesis



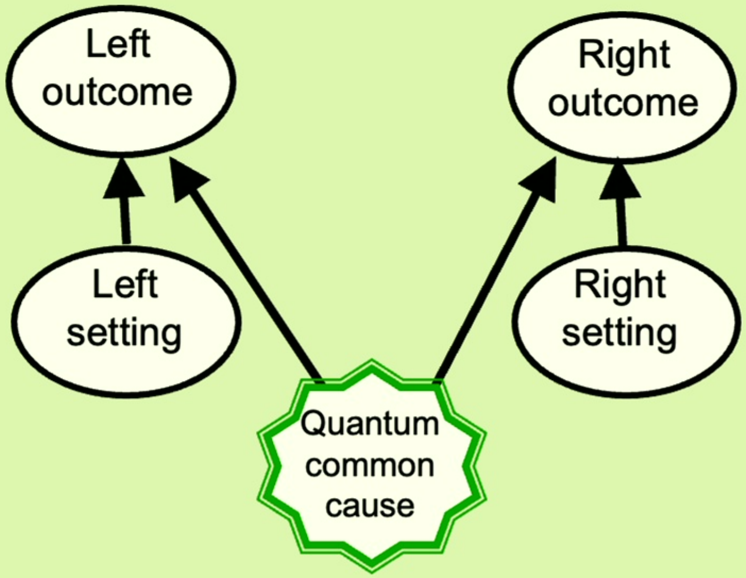
Implies Bell inequalities

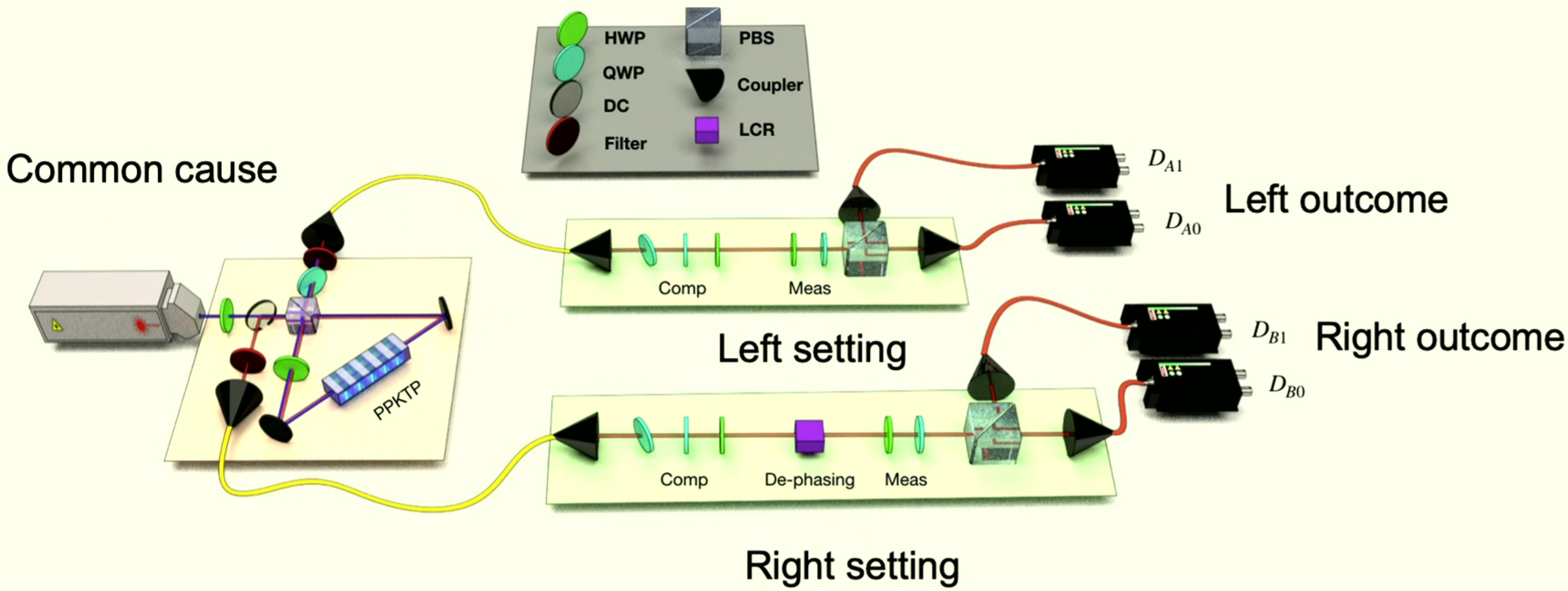

QUANTUM
THEORY
Diego Delso / CC BY-SA
RELATIVITY

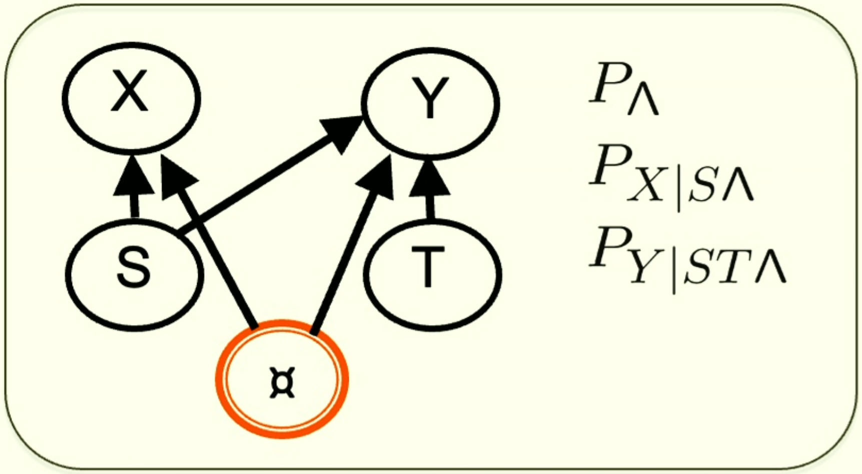
The evidence

		Left outcome and Right outcome			
		0 and 0	0 and 1	1 and 0	1 and 1
Left setting and Right setting	0 and 0	43%	7%	7%	43%
	0 and 1	43%	7%	7%	43%
	1 and 0	43%	7%	7%	43%
	1 and 1	7%	43%	43%	7%

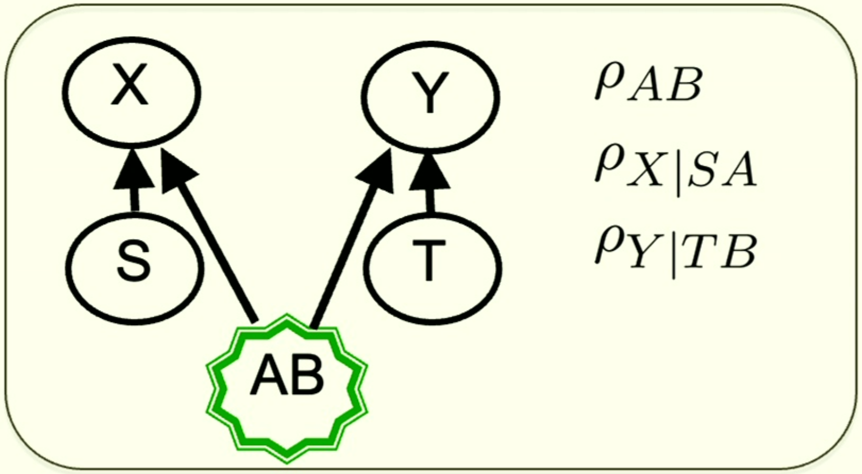
A new possibility

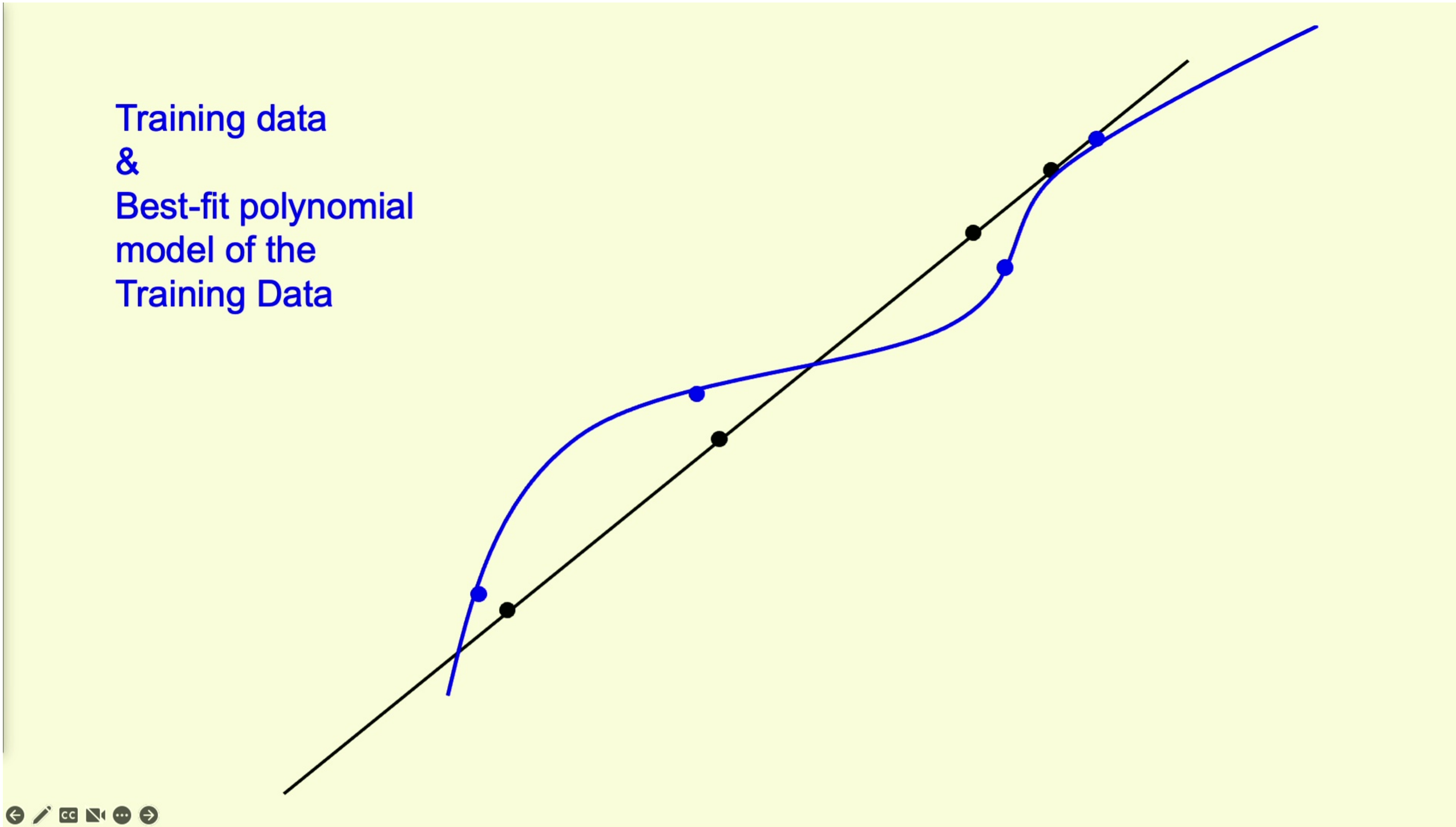



HWP
QWP
DC
Filter
PBS
Coupler
LCR
Common cause
PPKTP
Comp
Meas
Left setting
Comp
De-phasing
Meas
Right setting
D_{A1}
D_{A0}
Left outcome
D_{B1}
D_{B0}
Right outcome

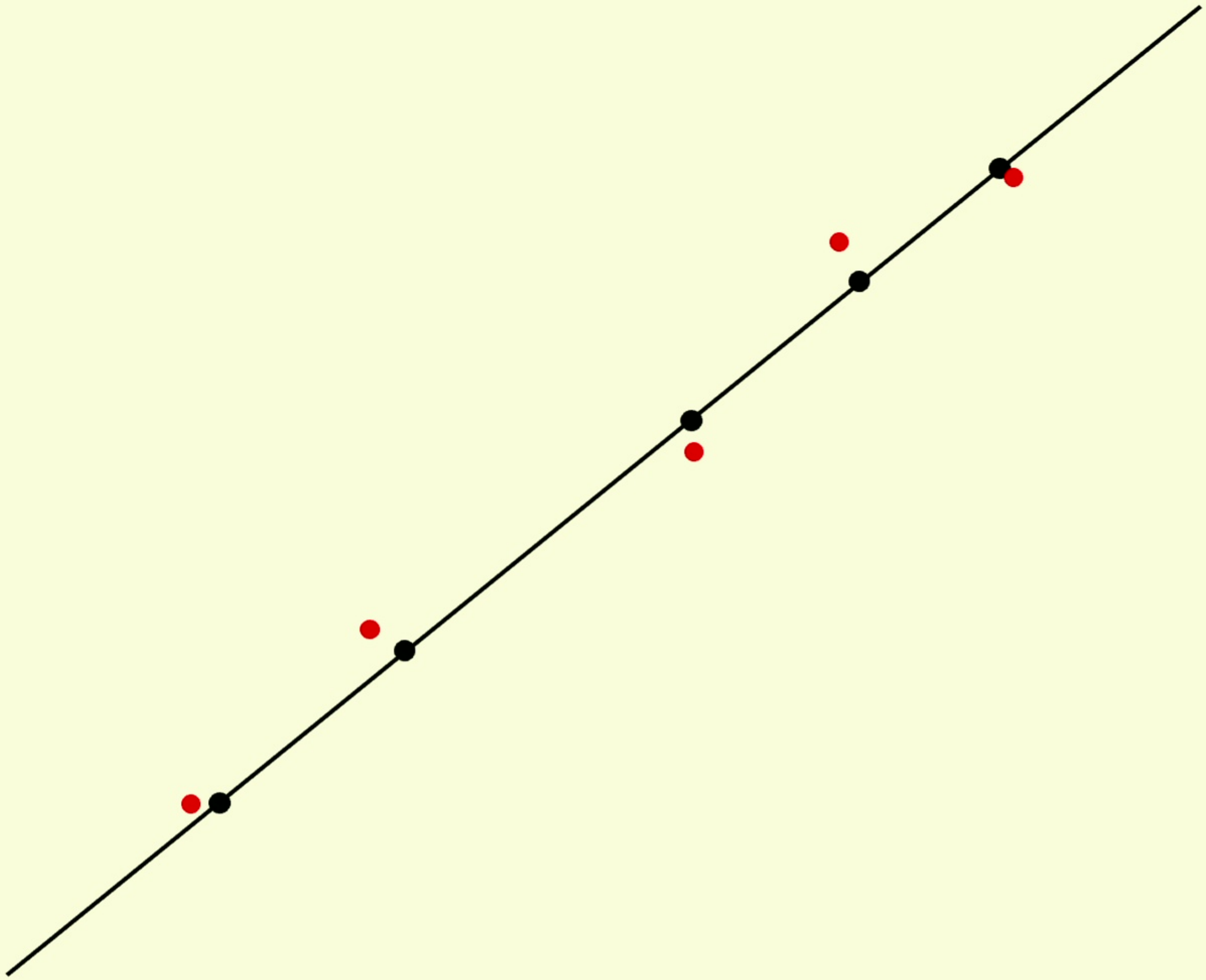

X
Y
S
T
¤
P_Λ
$P_{X|S\Lambda}$
$P_{Y|ST\Lambda}$

vs.

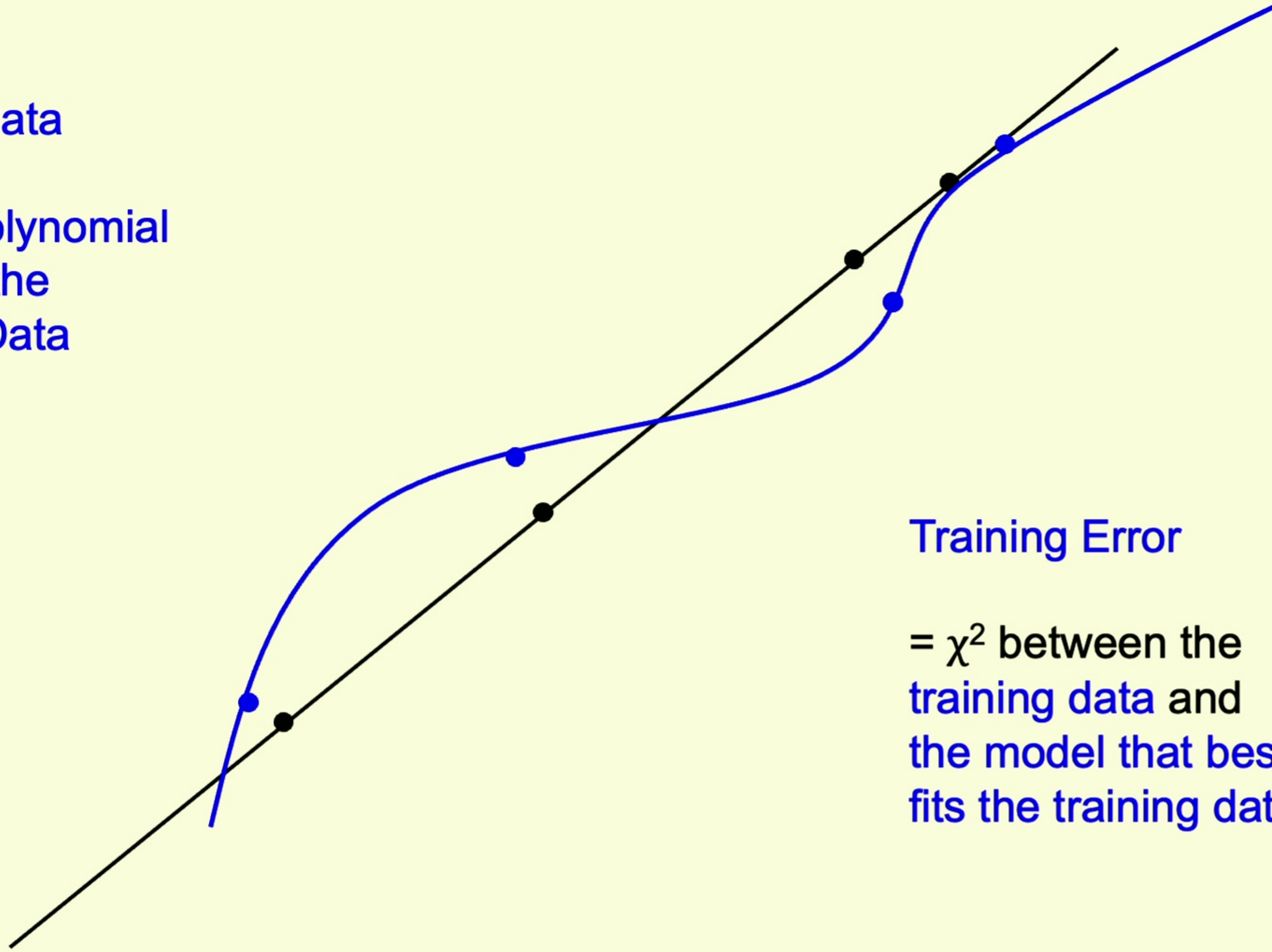

X
Y
S
T
AB
ρ_{AB}
$\rho_{X|SA}$
$\rho_{Y|TB}$


Training data
&
Best-fit polynomial
model of the
Training Data

Test Data

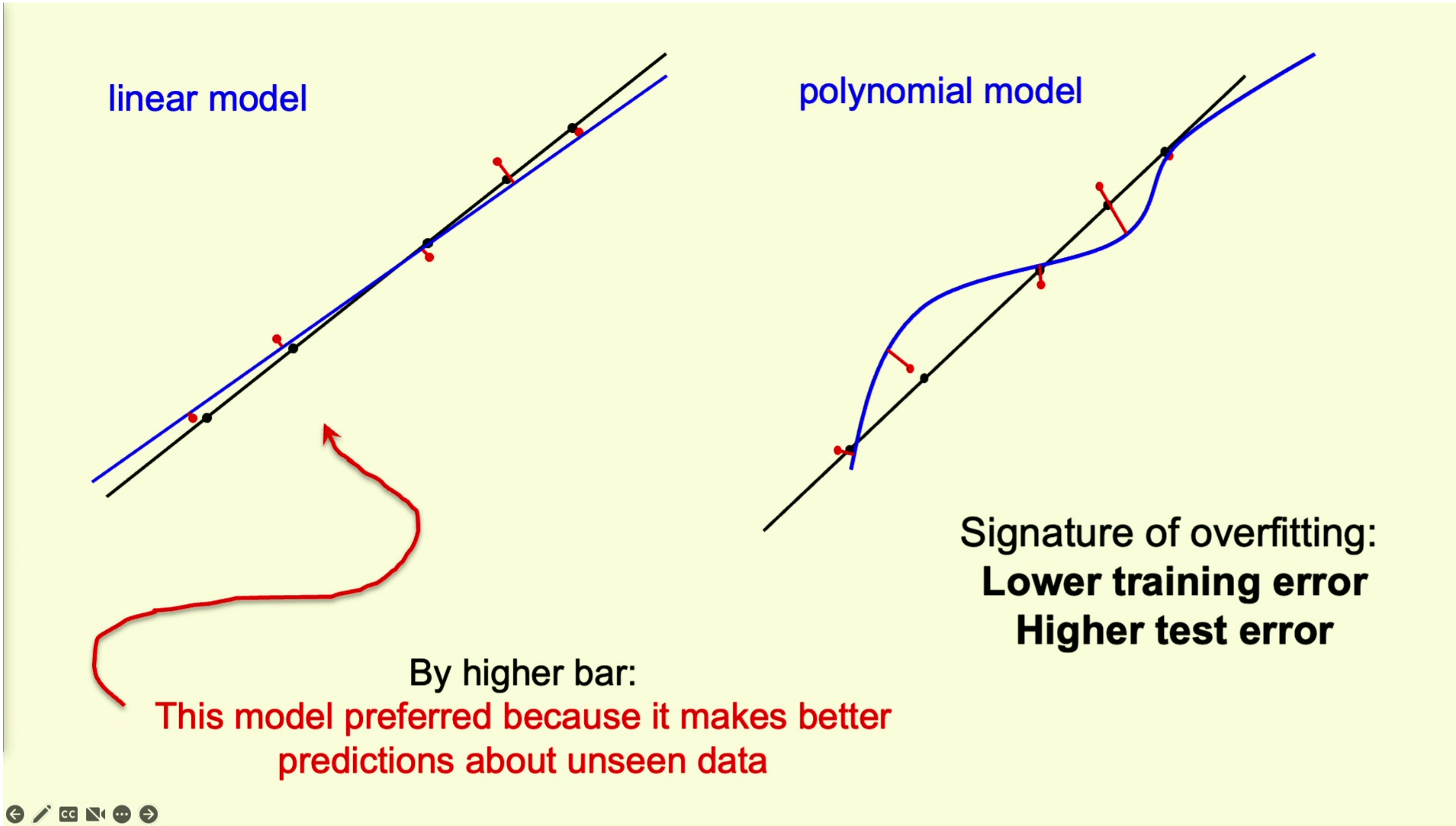


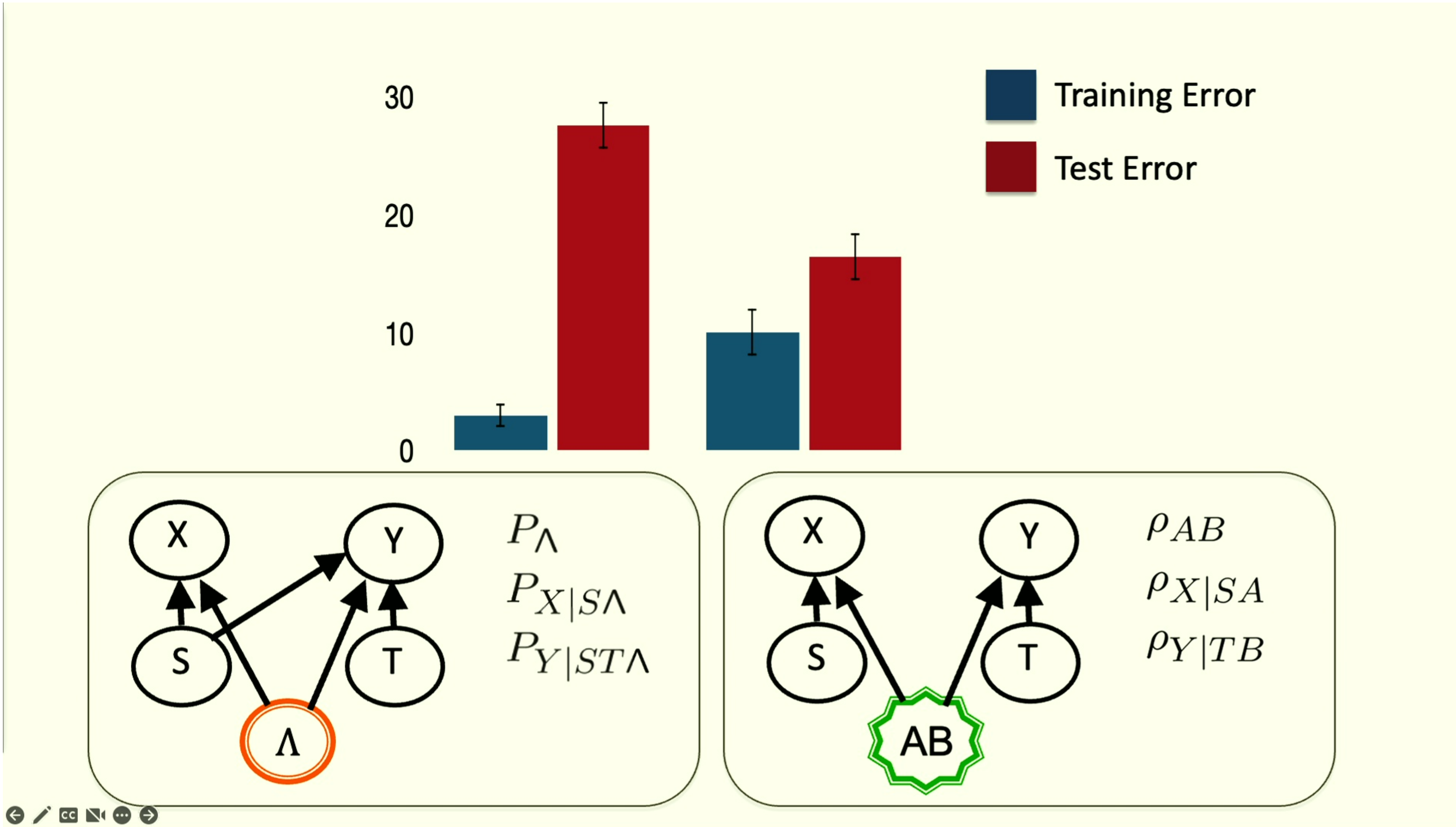
Training data
&
Best-fit polynomial
model of the
Training Data



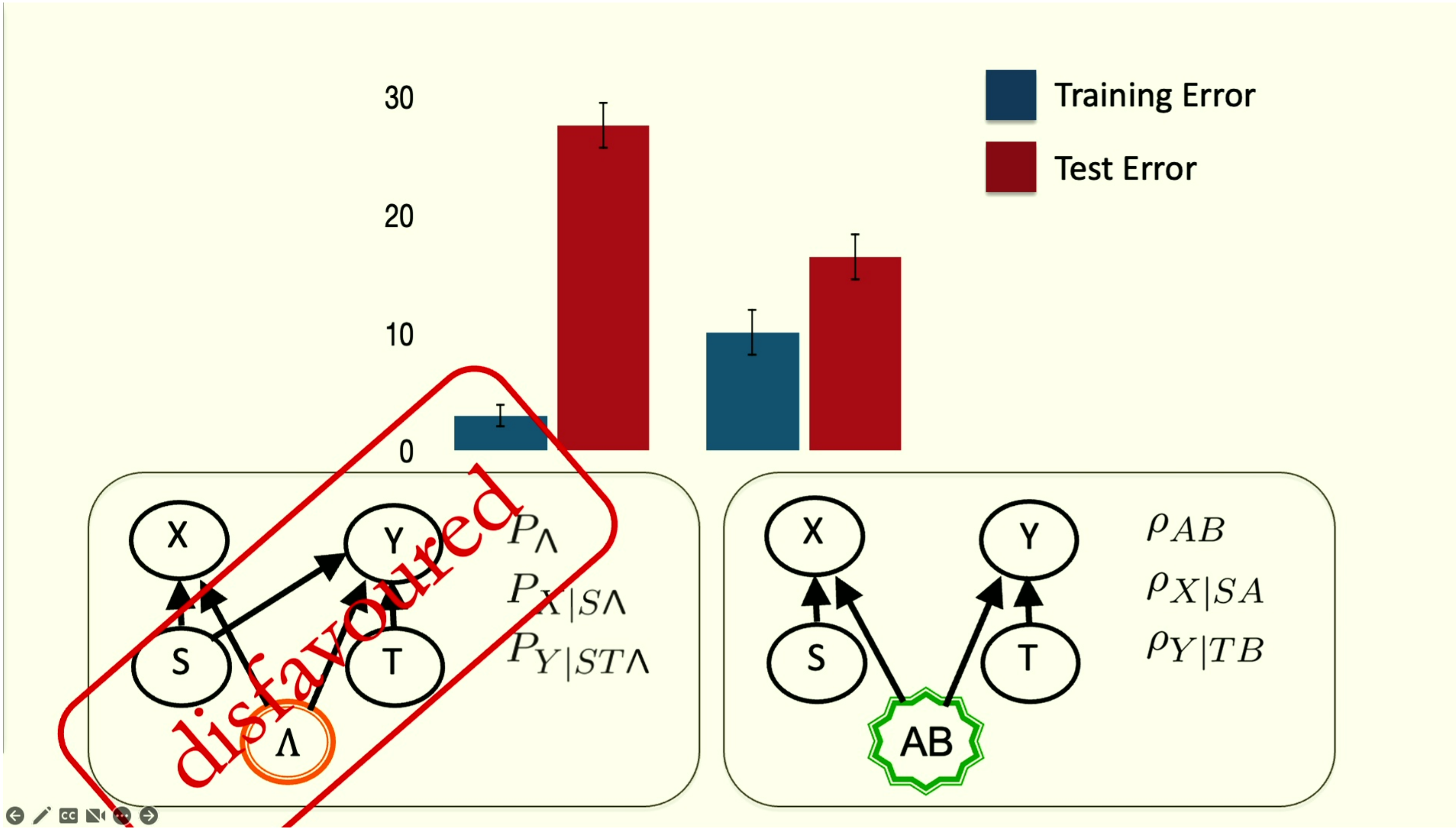
Training Error

= χ^2 between the
training data and
the model that best
fits the training data


linear model
polynomial model
Signature of overfitting:
Lower training error
Higher test error
By higher bar:
This model preferred because it makes better
predictions about unseen data



30
20
10
0
Training Error
Test Error
X
Y
S
T
Λ
P_Λ
$P_{X|S\Lambda}$
$P_{Y|ST\Lambda}$
X
Y
S
T
AB
ρ_{AB}
$\rho_{X|SA}$
$\rho_{Y|TB}$


30
20
10
0
Training Error
Test Error
X
Y
S
T
Λ
P_Λ
$P_{X|S\Lambda}$
$P_{Y|ST\Lambda}$
disfavoured
X
Y
S
T
AB
ρ_{AB}
$\rho_{X|SA}$
$\rho_{Y|TB}$

Quantum causal models with finite dimension of latent systems and various restrictions on parameters.
(Stronger assumptions imply stronger conclusions.)

Identifying and estimating causal effects in quantum causal models

Can we unscramble the omelette of causation and inference in quantum theory?

If so, then can the intrinsically quantum theories of causation and inference be derived from simple axioms that capture the innovation relative to classical theories?


Thanks!