Title: Causal Inference Lecture - 230405

Speakers: Robert Spekkens

Collection: Causal Inference: Classical and Quantum

Date: April 05, 2023 - 10:00 AM

URL: https://pirsa.org/23040001

Abstract: zoom link: https://pitp.zoom.us/j/94143784665?pwd=VFJpajVIMEtvYmRabFYzYnNRSVAvZz09

Quantum causal models

"[...] our present Quantum Mechanical formalism [...] is a peculiar mixture describing in part realities of Nature, in part incomplete human information about Nature all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen how to unscramble."

E.T. Jaynes, 1989

We must unscramble the omelette of causation and inference in quantum theory

Statistical paradigm

↓

Causal paradigm

Operational paradigm

↓

Realist paradigm

Perhaps the notion of realism we should seek
in order to salvage in quantum theory is just this:

Statistical correlations have causal explanations

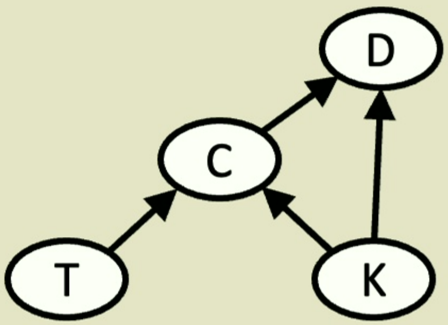
Quantum causal theory

Proposal for how to define A causes B classically

$$P_{B|\mathrm{do}A} \neq P_B$$

Reason to reject it:

Vernam cypher



$D = (C + K)\mathrm{mod}2$

$C = (T + K)\mathrm{mod}2$

$P_K = ½[0]_K + ½[1]_K$

$P_{C|\mathrm{do}\,T} = ½[0]_C + ½[1]_C = P_C$

Yet T is a cause of C

Proposal for how to define A causes B quantumly

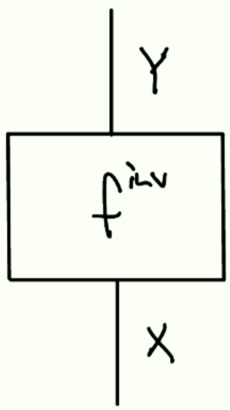
$$\mathcal{E}_{B|A}(\cdot) \neq \rho_B \, \mathrm{Tr}_A(\cdot)$$

Reason to reject it:

Classical is a special case of quantum

Focus on deterministic evolution in closed systems: Unitary evolution

invertible function



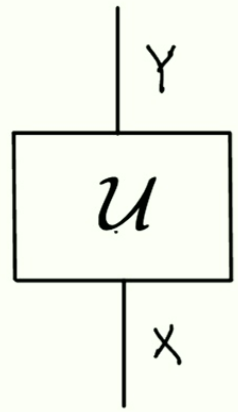
For $|X| = |Y|$

$f^{\mathrm{inv}} : X \to Y$

$:: x \mapsto f^{\mathrm{inv}}(x)$

where f^{inv} is an invertible function

Unitary channel



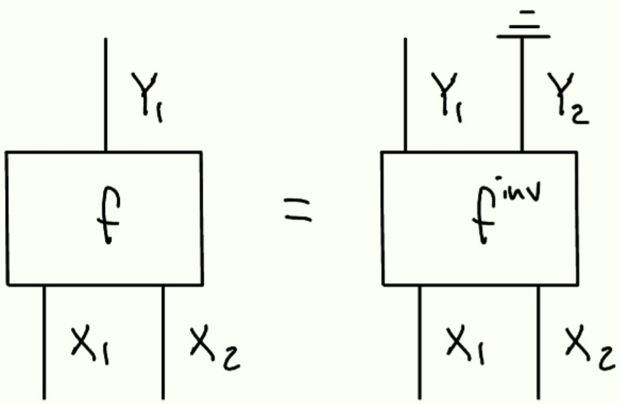
For $\dim(\mathcal{H}_X) = \dim(\mathcal{H}_Y)$

$\mathcal{U} : \mathcal{L}(\mathcal{H}_X) \to \mathcal{L}(\mathcal{H}_Y)$

$:: A \mapsto \mathcal{U}(A) := UAU^\dagger$

where U is a unitary operator

general function



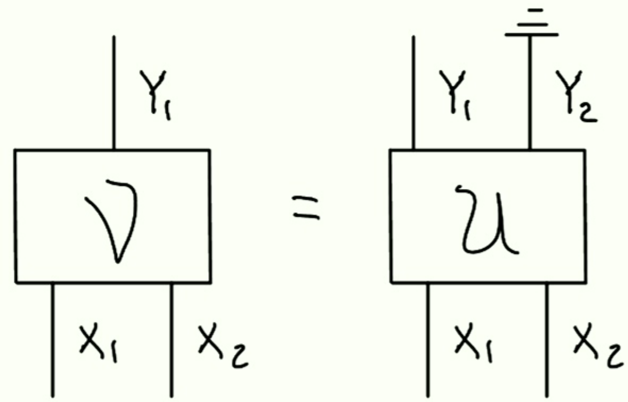
For $|X_1||X_2| = |Y_1||Y_2|$

$$f : X_1 \times X_2 \to Y_1$$

$$f := f^{\text{inv}}|_{Y_1}$$

where f^{inv} is an invertible function

Reduced unitary



For $\dim(\mathcal{H}_{X_1} \otimes \mathcal{H}_{X_2}) = \dim(\mathcal{H}_{Y_1} \otimes \mathcal{H}_{Y_2})$

$$\mathcal{V} : \mathcal{L}(\mathcal{H}_{X_1} \otimes \mathcal{H}_{X_2}) \to \mathcal{L}(\mathcal{H}_{Y_1})$$

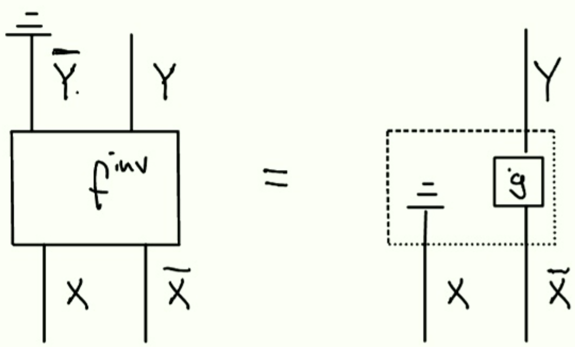
$$\mathcal{V} := \mathrm{Tr}_{Y_2} \circ \mathcal{U}$$

where $\mathcal{U}$ is a unitary channel

Classical

variable X has no influence on variable Y if Y has a trivial dependence on X

for an invertible function f^{inv}

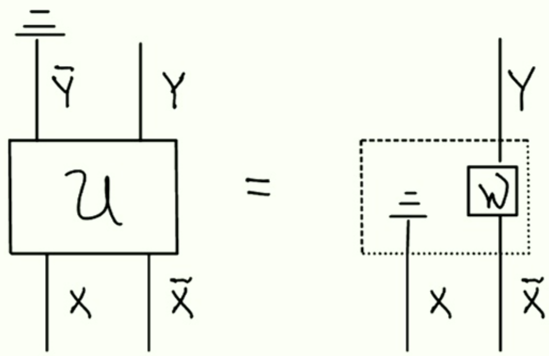


$$f^{\text{inv}}|_Y(X,\bar{X}) = g(\bar{X})$$

Quantum

system X has no influence on system Y if Y has a trivial dependence on X

for a unitary channel U

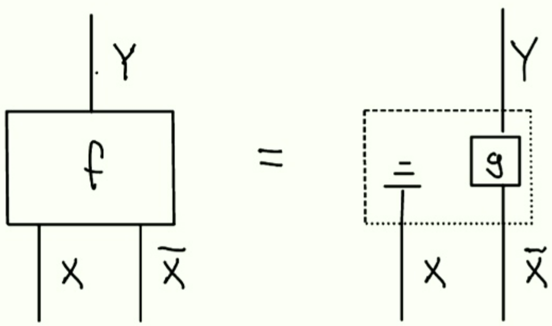


$$\mathrm{Tr}_{\bar{Y}} \circ \mathcal{U}_{\bar{Y}Y|X\bar{X}} = \mathcal{W}_{Y|\bar{X}} \otimes \mathrm{Tr}_X$$

Classical

variable X has no influence on variable Y if Y has a trivial dependence on X

for a general function f

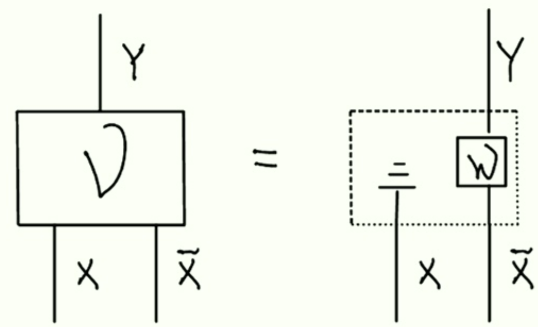


$$f(X,\bar{X}) = g(\bar{X})$$

Quantum

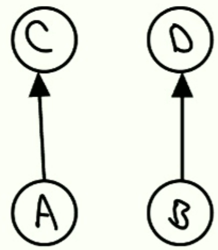
system X has no influence on system Y if Y has a trivial dependence on X

for a reduced unitary channel V

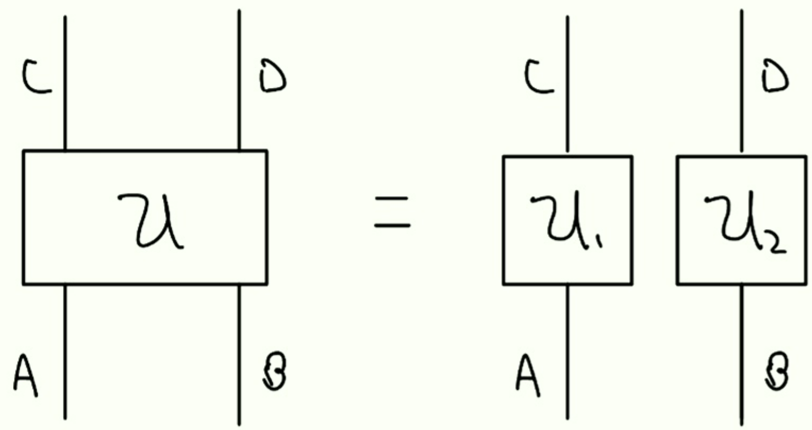


$$\mathcal{V}_{Y|X\bar{X}} = \mathrm{Tr}_X \otimes \mathcal{W}_{Y|\bar{X}}$$

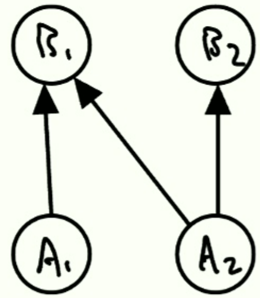
A only influences C
B only influences D



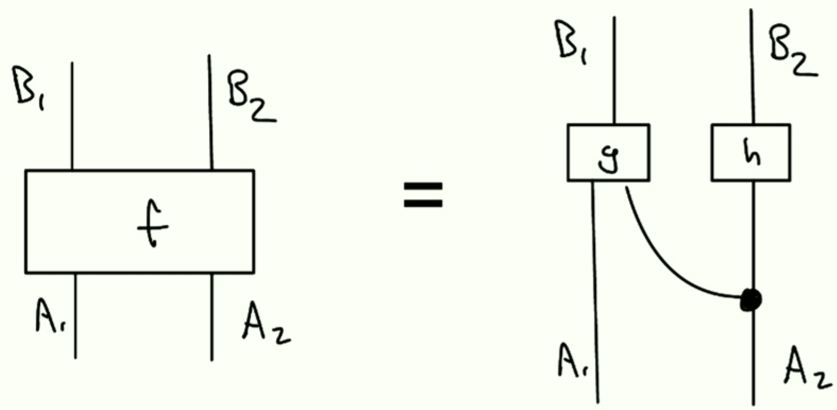
Quantumly:



A_1 only influences B_1
A_2 influences B_1 and B_2



Classically



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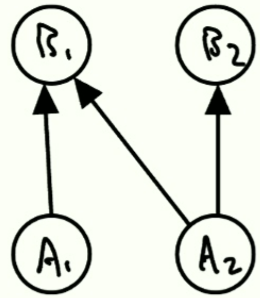
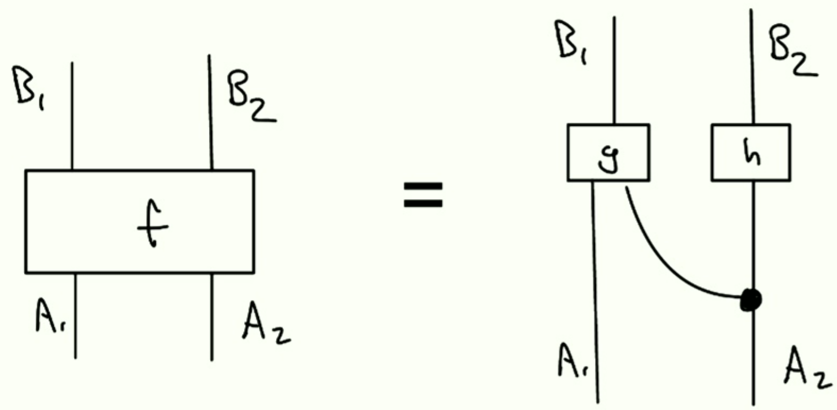
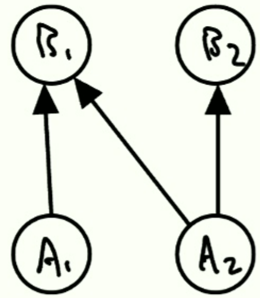


Classically

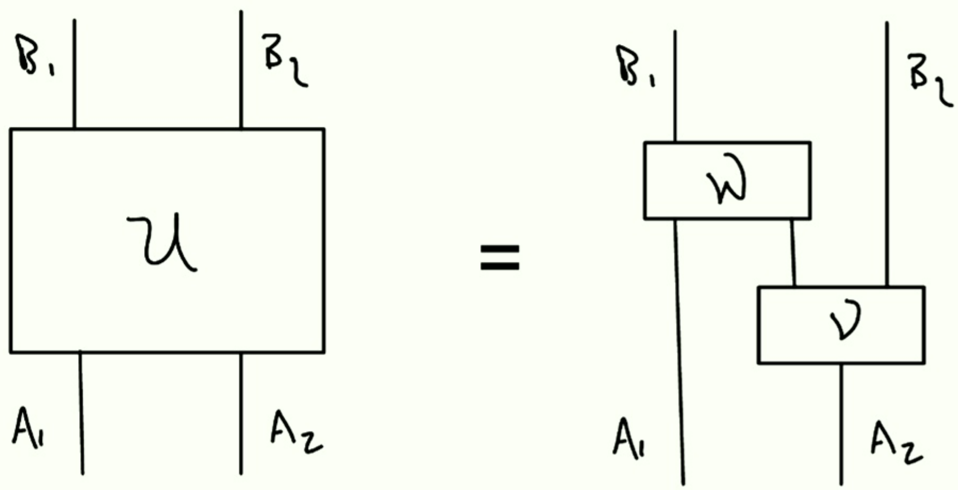


But there is no cloning in quantum theory

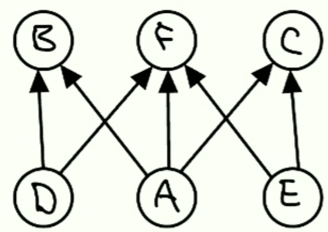
A_1 only influences B_1
A_2 influences B_1 and B_2



Quantumly:

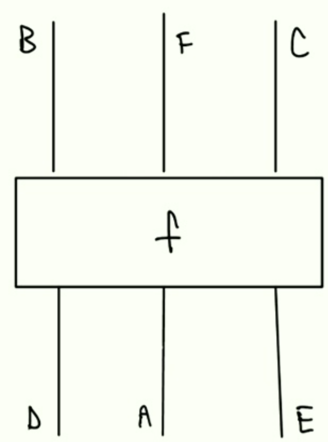


A is the complete common cause of B and C



Classically

one can express the function as



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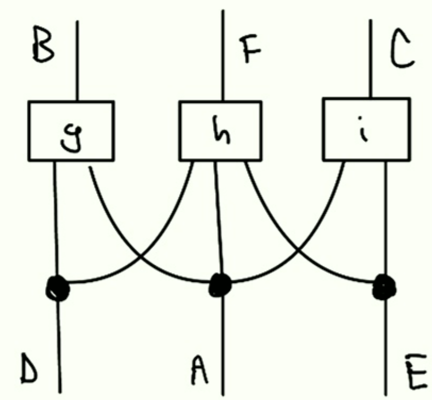

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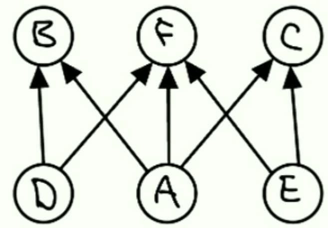
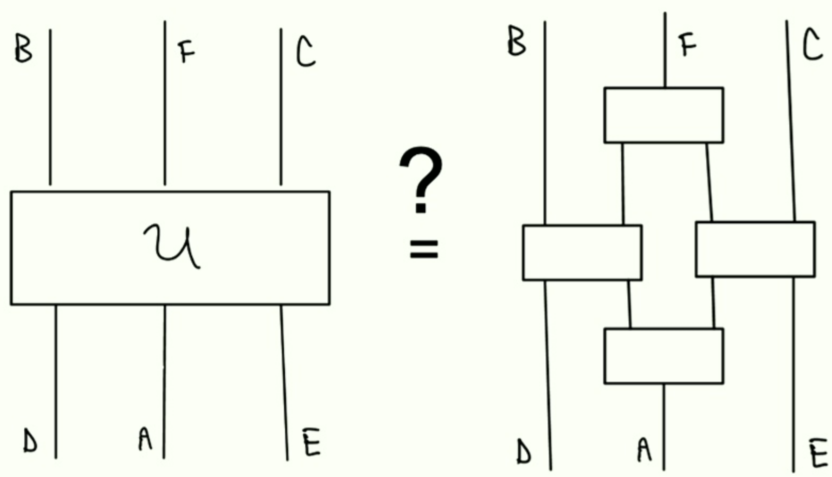


Quantumly

Is it always the case that we can find a decomposition



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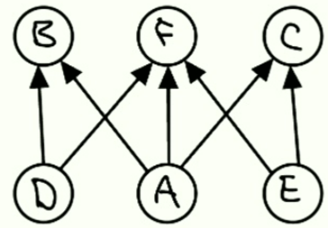
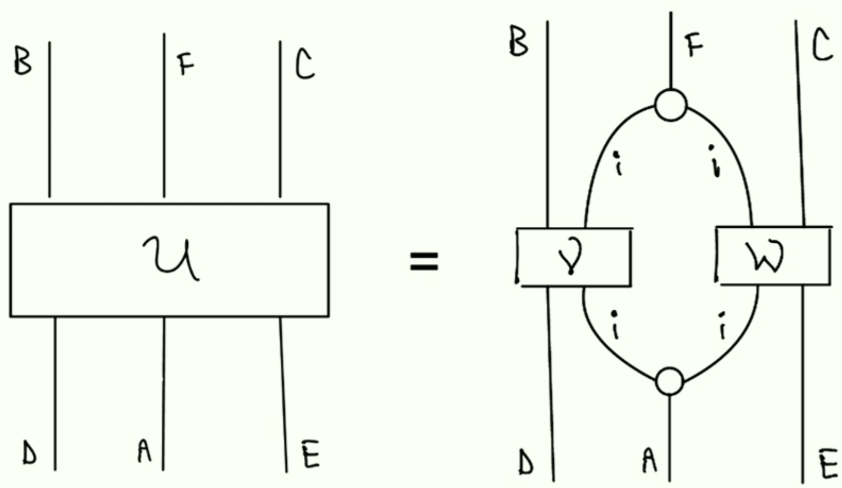
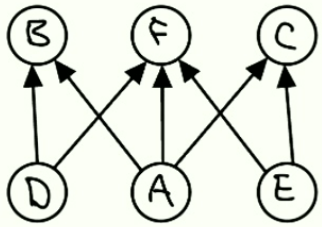
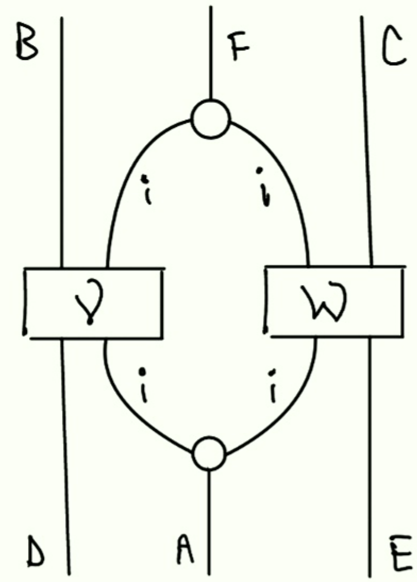


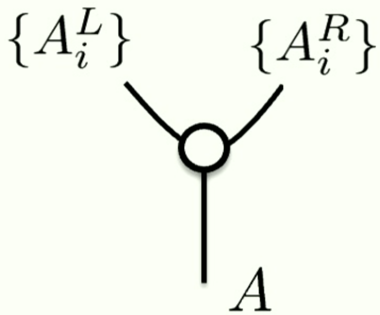
But we *can* always find a decomposition



See: Allen et al., Phys. Rev. X 7, 031021 (2017)







The dot describes a
"factorization within subspaces" for A

$$\mathcal{H}_A = \bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R}$$

See: Hayden et al., Comm. Math. Phys. 246, 359 (2004)

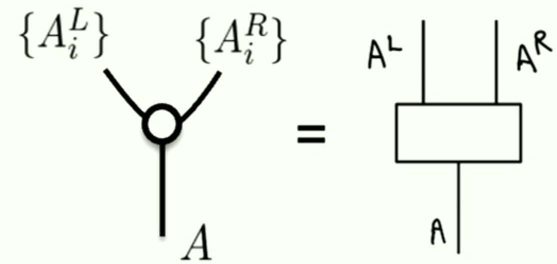
Special cases of "factorization within subspaces"

$$\mathcal{H}_A = \bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R}$$

Pure factorization

$$\mathcal{H}_A = \mathcal{H}_{A^L} \otimes \mathcal{H}_{A^R}$$

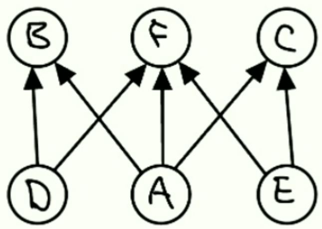
$$\dim(\mathcal{H}_{A^L}) \times \dim(\mathcal{H}_{A^R}) = \dim(\mathcal{H}_A)$$

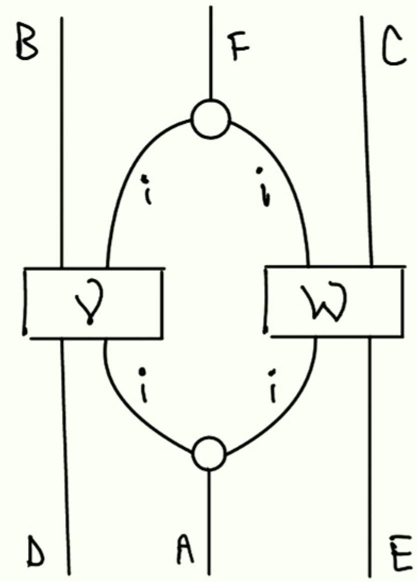


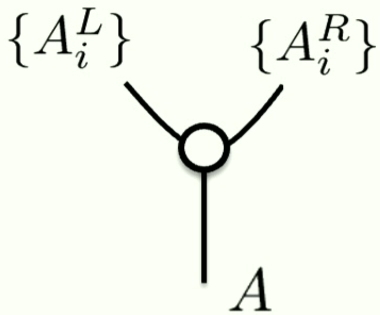
Coherent copy

$$\mathcal{H}_A = \bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R}$$

$$\dim(\mathcal{H}_{A_i^L}) = \dim(\mathcal{H}_{A_i^R}) = 1 \ \ \forall i$$







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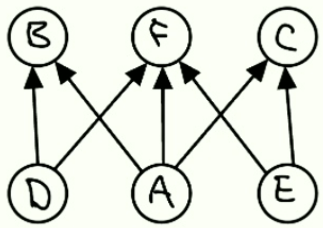
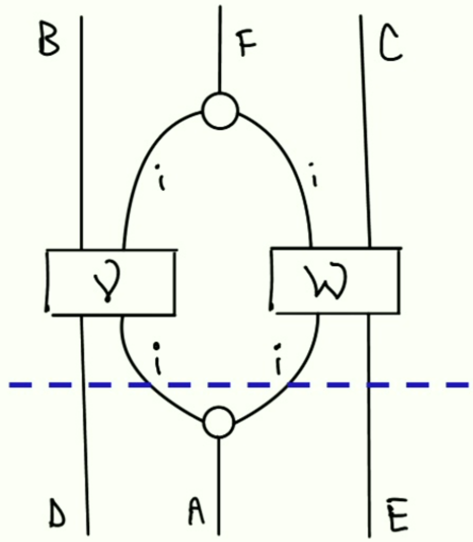
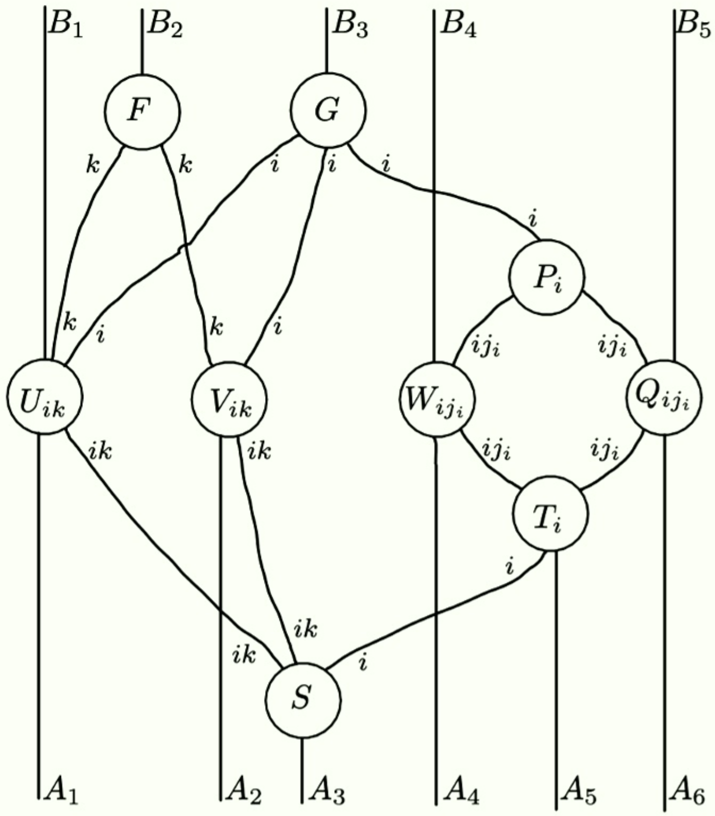


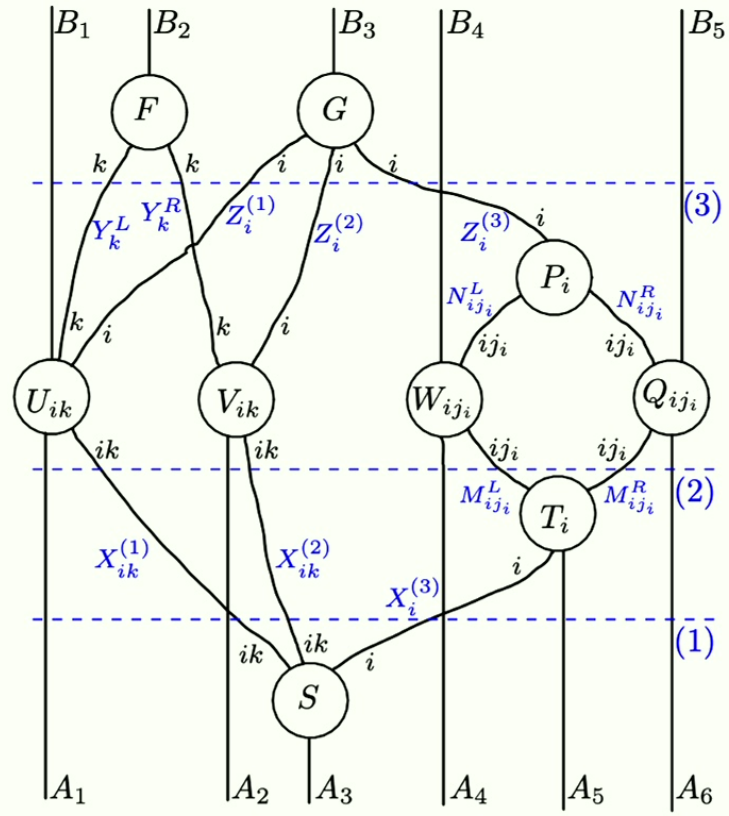


$$\mathcal{H}_D \otimes \left(\bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R} \right) \otimes \mathcal{H}_E$$

$\mathcal{V}$ is block-diagonal across the *i* sectors and is nontrivial only on $\mathcal{H}_{A_i^L}$

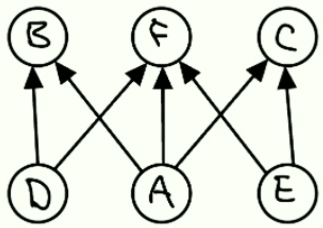
More complicated case

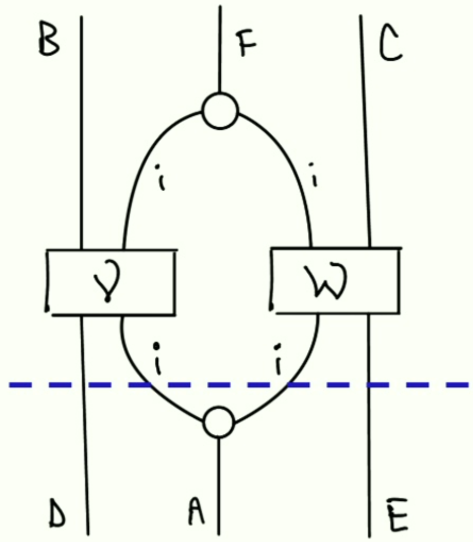




Lorenz and Barrett, Quantum 5, 511 (2021)

A circuit decomposition is **causally faithful** if influences are given by structure of diagram





$$\mathcal{H}_D \otimes \left(\bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R} \right) \otimes \mathcal{H}_E$$

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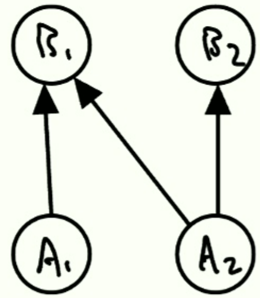
A circuit decomposition is **causally faithful** if influences are given by structure of diagram

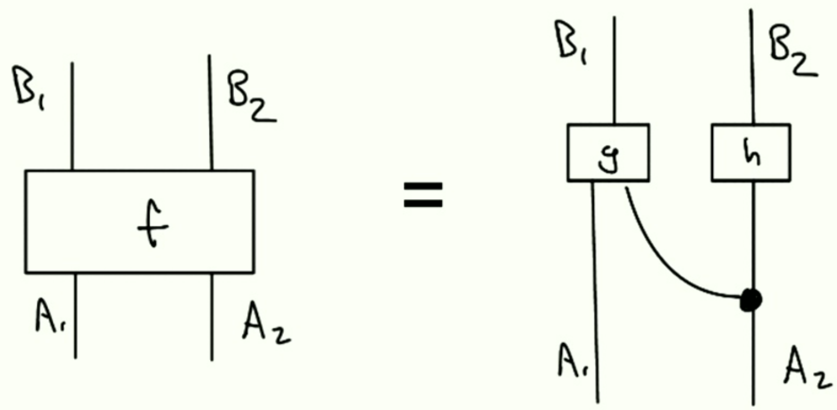
It is an open question whether every multipartite unitary channel admits a causally faithful decomposition

Quantum inferential theory

A_1 only influences B_1
A_2 influences B_1 and B_2

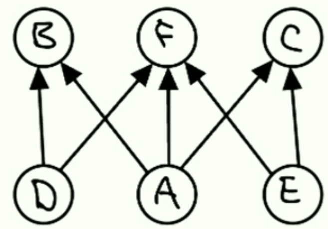


Classically



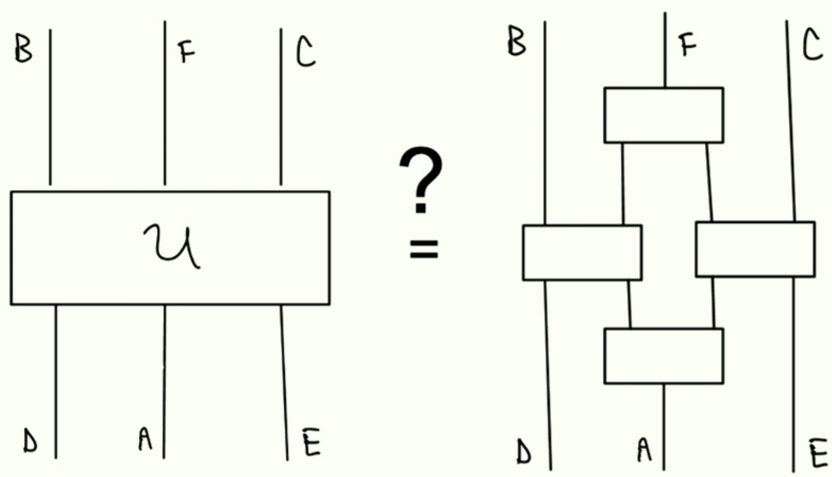
But there is no cloning in quantum theory

A is the complete common cause of B and C



Quantumly

Is it always the case that we can find a decomposition



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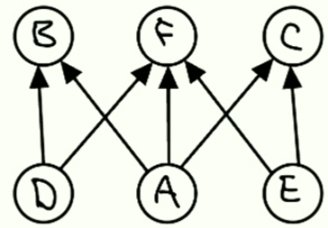

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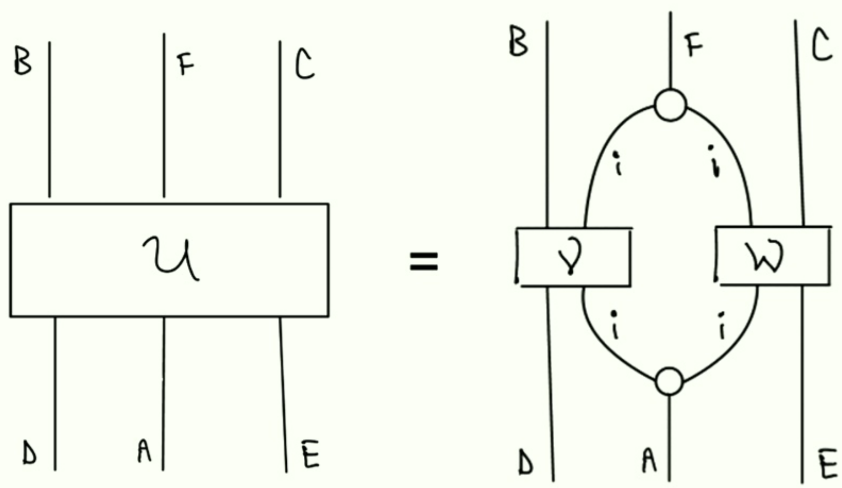
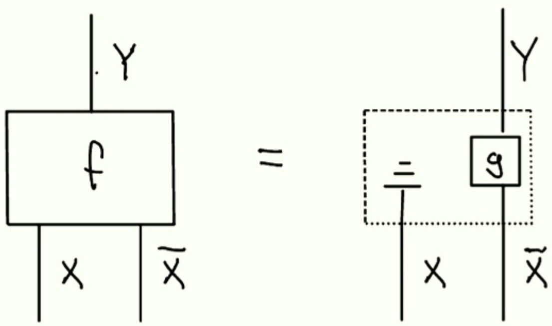


See: Allen et al., Phys. Rev. X 7, 031021 (2017)

Classical

variable X has no influence on variable Y if Y has a trivial dependence on X

for a general function f

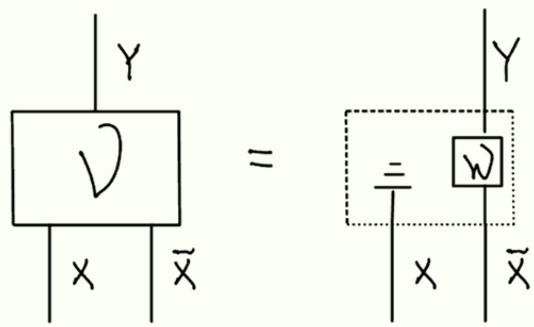


$$f(X, \bar{X}) = g(\bar{X})$$

Quantum

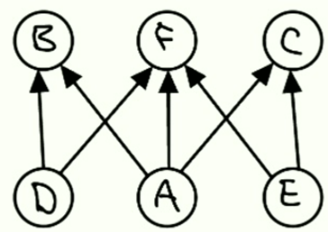
system X has no influence on system Y if Y has a trivial dependence on X

for a reduced unitary channel V



$$\mathcal{V}_{Y|X\bar{X}} = \mathrm{Tr}_X \otimes \mathcal{W}_{Y|\bar{X}}$$

A is the complete common cause of B and C



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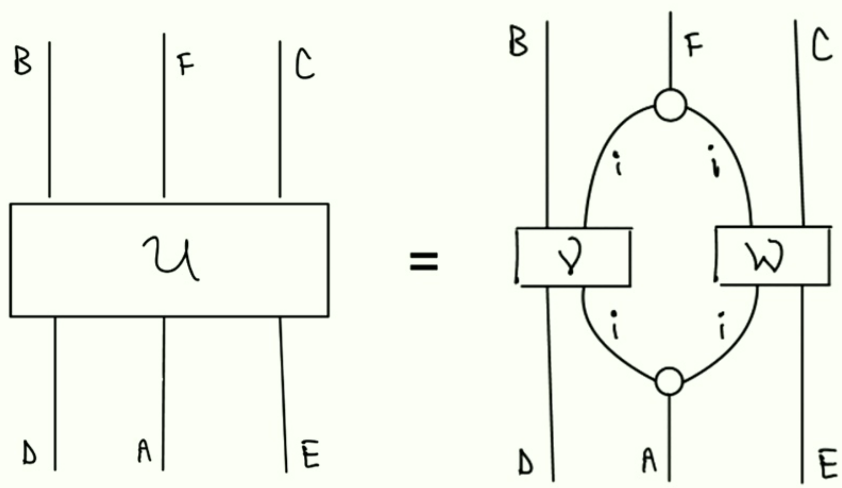

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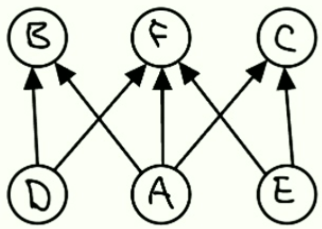
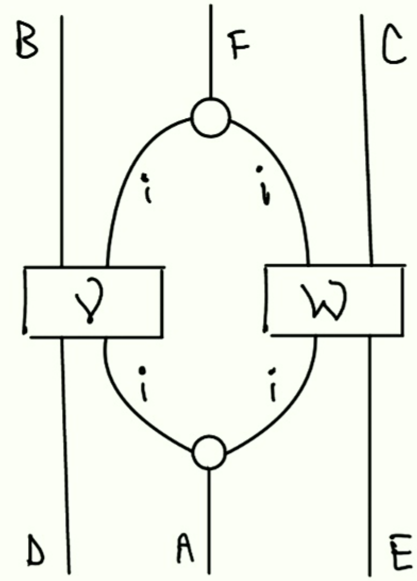
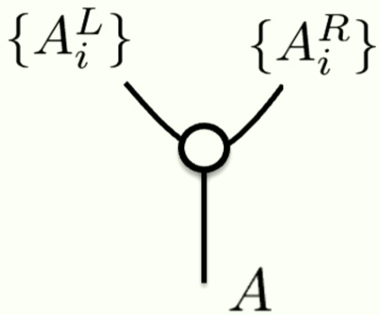






The dot describes a "factorization within subspaces" for A

$$\mathcal{H}_A = \bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R}$$

See: Hayden et al., Comm. Math. Phys. 246, 359 (2004)

	Classical	Quantum
State of knowledge	P_A	ρ_A
Normalization	$\sum_A P_A = 1$	$\mathrm{Tr}_A(\rho_A) = 1$
Joint state	P_{AB}	ρ_{AB}
Marginalization	$P_B = \sum_A P_{AB}$	$\rho_B = \mathrm{Tr}_A(\rho_{AB})$

Joint states for systems that are common-cause connected

Quantum marginal independence

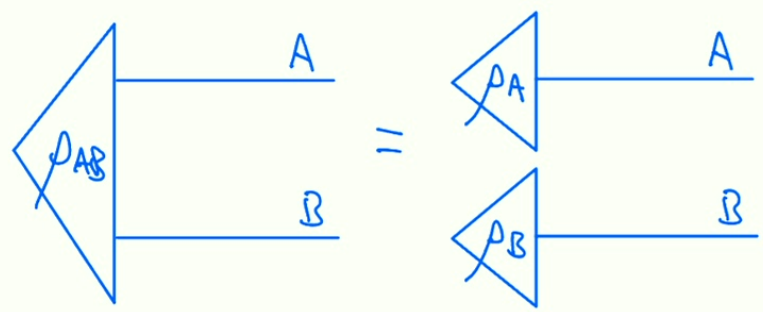
$$\rho_{AB} = \rho_A \otimes \rho_B$$

Denote this
$(A \perp B)$

$$I(A:B) = 0$$

where

$$I(A:B) = S(A) + S(B) - S(AB)$$
$$S(X) = -\mathrm{Tr}(\rho_X \log \rho_X)$$



Is there a quantum state on A, B, C that has the following marginals?

$$\rho_{AB} = \frac{1}{2}|0\rangle_A\langle 0| \otimes |0\rangle_B\langle 0| + \frac{1}{2}|1\rangle_A\langle 1| \otimes |1\rangle_B\langle 1|$$
$$\rho_{AC} = \frac{1}{2}|0\rangle_A\langle 0| \otimes |0\rangle_C\langle 0| + \frac{1}{2}|1\rangle_A\langle 1| \otimes |1\rangle_C\langle 1|$$
$$\rho_{BC} = \frac{1}{2}|0\rangle_B\langle 0| \otimes |0\rangle_C\langle 0| + \frac{1}{2}|1\rangle_B\langle 1| \otimes |1\rangle_C\langle 1|$$

Quantum marginal problem

Is there a quantum state on A, B, C that has the following marginals?

$$\rho_{AB} = |\Phi^+\rangle_{AB}\langle\Phi^+|$$

$$|\Phi^+\rangle_{AB} = \tfrac{1}{2}|00\rangle_{AB} + \tfrac{1}{2}|11\rangle_{AB}$$

$$\rho_{AC} = \tfrac{1}{2}I_A \otimes \tfrac{1}{2}I_C$$

$$\rho_{BC} = \tfrac{1}{2}I_B \otimes \tfrac{1}{2}I_C$$

Is there a quantum state on A, B, C that has the following marginals?

$$\rho_{AB} = |\Phi^+\rangle_{AB}\langle\Phi^+|$$

$$|\Phi^+\rangle_{AB} = \frac{1}{2}|00\rangle_{AB} + \frac{1}{2}|11\rangle_{AB}$$

$$\rho_{AC} = \frac{1}{2}I_A \otimes \frac{1}{2}I_C$$

$$\rho_{BC} = \frac{1}{2}I_B \otimes \frac{1}{2}I_C$$

Yes! $\rho_{ABC} = |\Phi^+\rangle_{AB}\langle\Phi^+| \otimes \frac{1}{2}I_C$

Classical marginal inequality

$$0 \leq 1 - P_X - P_Y - P_Z + P_{XY} + P_{XZ} + P_{YZ} \leq 1$$

Is there a quantum state on A, B, C that has the following marginals?

$$\rho_{AB} = |\Phi^+\rangle_{AB}\langle\Phi^+|$$
$$\rho_{AC} = |\Phi^+\rangle_{AC}\langle\Phi^+|$$
$$\rho_{BC} = |\Phi^+\rangle_{BC}\langle\Phi^+|$$

No!

Classical marginal inequality

$$0 \leq 1 - P_X - P_Y - P_Z + P_{XY} + P_{XZ} + P_{YZ} \leq 1$$

Quantum marginal inequality

$$0 \leq I - \rho_A - \rho_B - \rho_C + \rho_{AB} + \rho_{AC} + \rho_{BC} \leq I$$

Butterley, Sudbery, Szulc, Found. Phys. 36, 83-101 (2006)

Classical belief propagation

$$P_B = \Gamma_{B|A}[P_A]$$

Stochastic map preserving positivity and normalization

Quantum belief propagation

$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

Completely positive trace-preserving map

Classical belief propagation	Quantum belief propagation
$P_B = \Gamma_{B\|A}[P_A]$	$\rho_B = \mathcal{E}_{B\|A}(\rho_A)$
Stochastic map preserving positivity and normalization	Completely positive trace-preserving map
In terms of a conditional	In terms of a conditional
$P_B = \sum_A P_{B\|A} P_A$	$\rho_B = \mathrm{Tr}_A(\rho_{B\|A}\rho_A)$
$\sum_B P_{B\|A} = 1$	$\mathrm{Tr}_B(\rho_{B\|A}) = I_A$
$P_{B\|A} \geq 0$	$\rho_{B\|A}^{T_A} \geq 0$

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{B|A}(\cdot) = \mathrm{Tr}_A(\rho_{B|A} \cdot)$$

Proof: $\mathrm{Tr}_A(\rho_{B|A}\rho_A) = \sum_{j,k} \Phi_{B|A'}(|j\rangle\langle k|_{A'})\langle j|\rho_A|k\rangle$

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{B|A}(\cdot) = \mathrm{Tr}_A(\rho_{B|A} \cdot)$$

Proof: $$\mathrm{Tr}_A(\rho_{B|A}\rho_A) = \sum_{j,k} \Phi_{B|A'}(|j\rangle\langle k|_{A'})\langle j|\rho_A|k\rangle$$

$$= \Phi_{B|A}(\sum_{j,k} |j\rangle\langle j|_A \rho_A |k\rangle\langle k|_A)$$

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{B|A}(\cdot) = \mathrm{Tr}_A(\rho_{B|A}\cdot)$$

Proof:
$$\mathrm{Tr}_A(\rho_{B|A}\rho_A) = \sum_{j,k} \Phi_{B|A'}(|j\rangle\langle k|_{A'})\langle j|\rho_A|k\rangle$$
$$= \Phi_{B|A}(\sum_{j,k} |j\rangle\langle j|_A \rho_A |k\rangle\langle k|_A)$$
$$= \Phi_{B|A}(\rho_A) \qquad \text{QED}$$

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{B|A}(\cdot) = \mathrm{Tr}_A(\rho_{B|A} \cdot)$$

$$\Phi_{B|A'} \text{ is trace-preserving} \quad \leftrightarrow \quad \mathrm{Tr}_B(\rho_{B|A}) = I_A$$

Proof: $\mathrm{Tr}_B(\rho_{B|A}) = \mathrm{Tr}_B \left[(\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A) \right]$

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{B|A}(\cdot) = \mathrm{Tr}_A(\rho_{B|A} \cdot)$$

$$\Phi_{B|A'} \text{ is trace-preserving} \quad \leftrightarrow \quad \mathrm{Tr}_B(\rho_{B|A}) = I_A$$

Proof:

$$\begin{aligned} \mathrm{Tr}_B(\rho_{B|A}) &= \mathrm{Tr}_B \left[(\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A) \right] \\ &= \sum_{j,k} \mathrm{Tr}_B \circ \Phi_{B|A'} (|j\rangle\langle k|_{A'}) \, |k\rangle\langle j|_A \\ &= \sum_{j,k} \mathrm{Tr}_{A'} (|j\rangle\langle k|_{A'}) \, |k\rangle\langle j|_A \\ &= \sum_{j,k} \delta_{j,k} |k\rangle\langle j|_A \end{aligned}$$

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{B|A}(\cdot) = \mathrm{Tr}_A(\rho_{B|A} \cdot)$$

$$\Phi_{B|A} \text{ is completely positive } \leftrightarrow \rho_{B|A}^{T_A} \geq 0$$

$$\rho_{B|A} \text{ is PPT}$$

Proof: $\rho_{B|A} = (\Phi_{B|A'} \otimes \mathrm{id}_A)\left[(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |j\rangle\langle k|_A)^{T_A}\right]$

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

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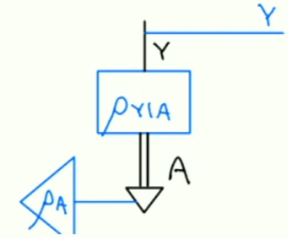
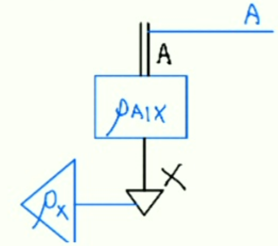
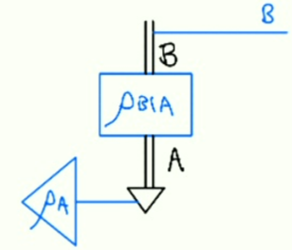
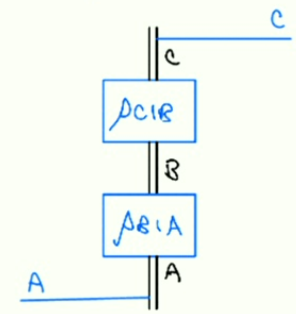
Proof: $\rho_{B|A} = (\Phi_{B|A'} \otimes \mathrm{id}_A)\left[(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |j\rangle\langle k|_A)^{T_A}\right]$

$$= \left[(\Phi_{A' \to B} \otimes \mathrm{id}_A)(d_A |\Psi^+\rangle_{A'A}\langle\Psi^+|)\right]^{T_A}$$

QED

	Conventional expression	In terms of conditional states	
Born's rule	$\forall y : P_Y(y) = \mathrm{Tr}_A(E_y^A \rho_A)$	$\rho_Y = \mathrm{Tr}_A(\rho_{Y\|A}\rho_A)$	
Ensemble averaging	$\rho_A = \sum_x P_X(x)\rho_x^A$	$\rho_A = \mathrm{Tr}_X(\rho_{A\|X}\rho_X)$	
Action of quantum channel	$\rho_B = \mathcal{E}_{B\|A}(\rho_A)$	$\rho_B = \mathrm{Tr}_A(\varrho_{B\|A}\rho_A)$	
Composition of channels	$\mathcal{E}_{C\|A} = \mathcal{E}_{C\|B} \circ \mathcal{E}_{B\|A}$	$\varrho_{C\|A} = \mathrm{Tr}_B(\varrho_{C\|B}\varrho_{B\|A})$	

A and B are conditionally independent given C

$$\rho_{AB|C} = \rho_{A|C}\rho_{B|C}$$

Denote this $(A \perp B|C)$

$$I(A:B|C) = 0$$

where

$$I(A:B|C) := S(AB) + S(AC) - S(ABC) - S(A)$$

for $\rho_{ABC} = \rho_{AB|C}^{T_C} \frac{1}{d_C}$

$$= \mathrm{Tr}_{C'}\left[\rho_{AB|C'}(|\Phi^+\rangle_{CC'}\langle\Phi^+|)\right]$$

A and B are marginally independent

$$\rho_{B|A} = \rho_B$$

$$I(A:B) = 0$$

where

$$I(A:B) = S(A) + S(B) - S(AB)$$

for $\rho_{AB} = (\rho_{B|A})^{T_A} \frac{1}{d_A}$

Puzzles:

What is a joint state over systems that are cause-effect related?

What is a conditional state between systems that are common-cause or common-effect related?

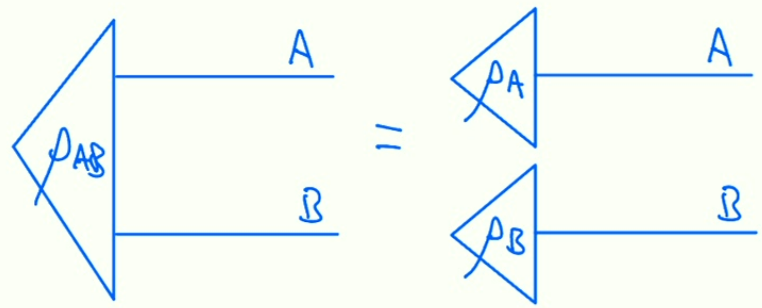
Relation of conditional to joint	$P_{B\|A} = \frac{P_{AB}}{P_A}$ $P_{AB} = P_{B\|A} P_A$	$\rho_{B\|A} = \rho_A^{-1/2} \rho_{AB} \rho_A^{-1/2}$ $\rho_{AB} = \rho_A^{1/2} \rho_{B\|A} \rho_A^{1/2}$
Normalization condition	$\sum_B P_{B\|A} = 1$	$\mathrm{Tr}_B(\rho_{B\|A}) = I_A$
Belief propagation	$P_B = \sum_A P_{B\|A} P_A$	$\rho_B = \mathrm{Tr}_A(\rho_{B\|A} \rho_A)$

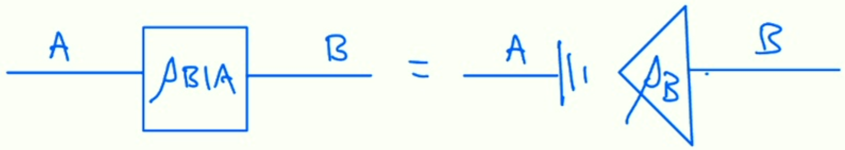
See: Leifer, PRA 74, 042310 (2006)
Leifer & Spekkens, PRA A 88, 052130 (2013)

A and B are marginally independent

$$\rho_{AB} = \rho_A \otimes \rho_B$$

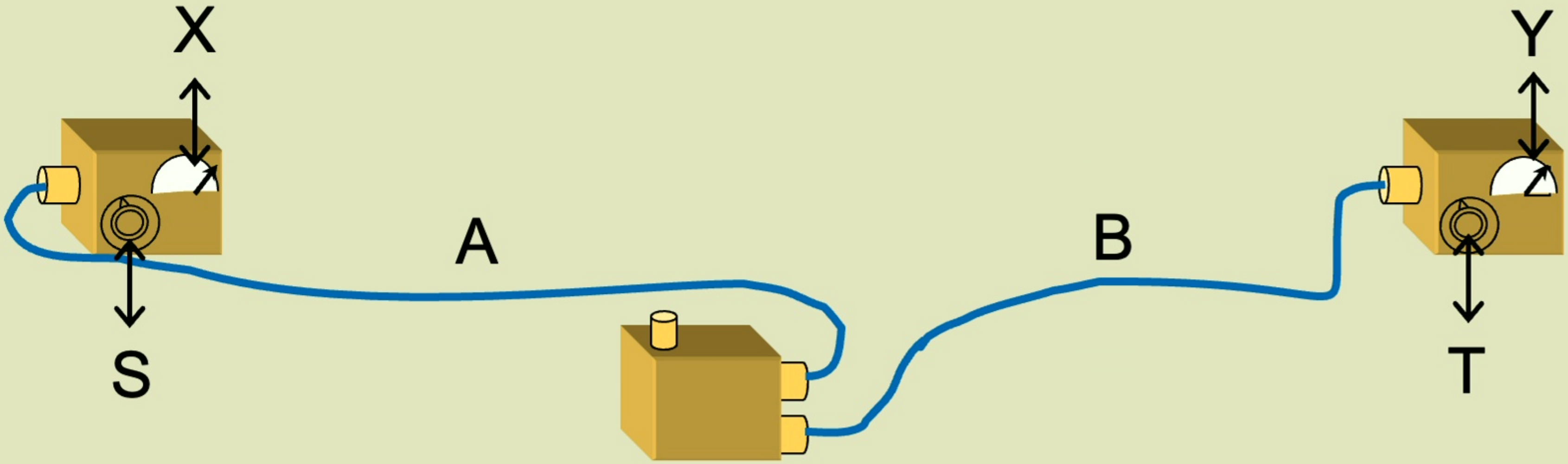
$$\rho_{B|A} = \rho_B$$

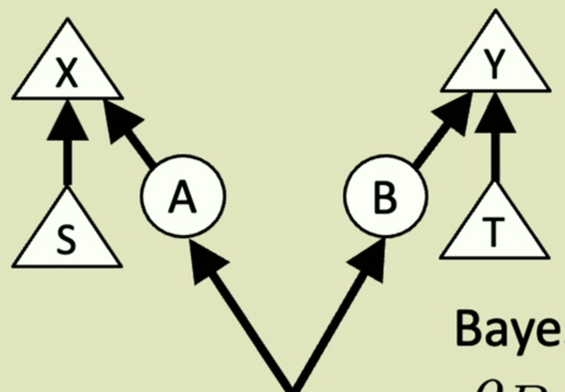




Joint states for systems that are cause-effect connected

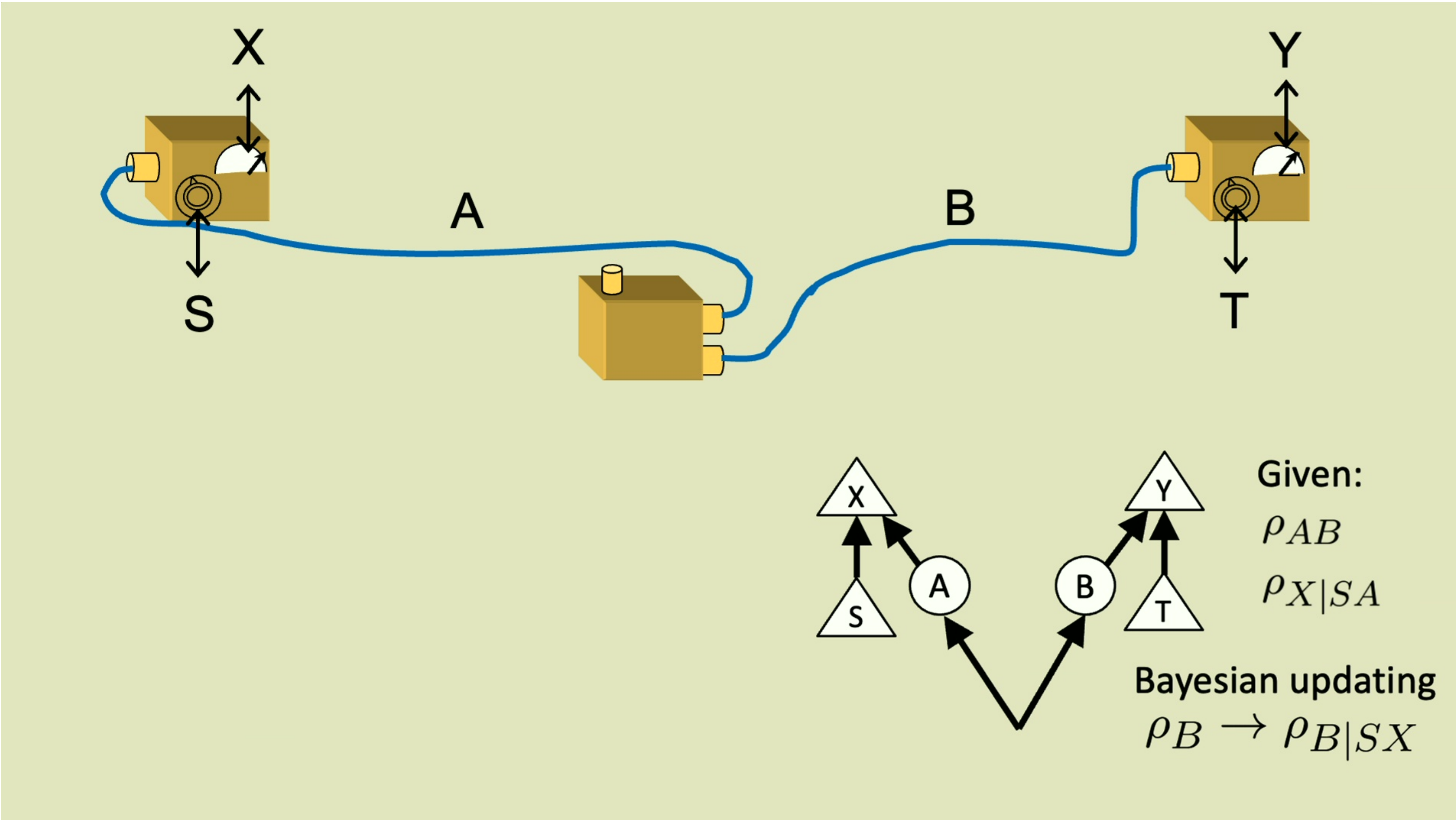
A necessary condition for any claim that the omelette of causation and inference in quantum theory has been unscrambled

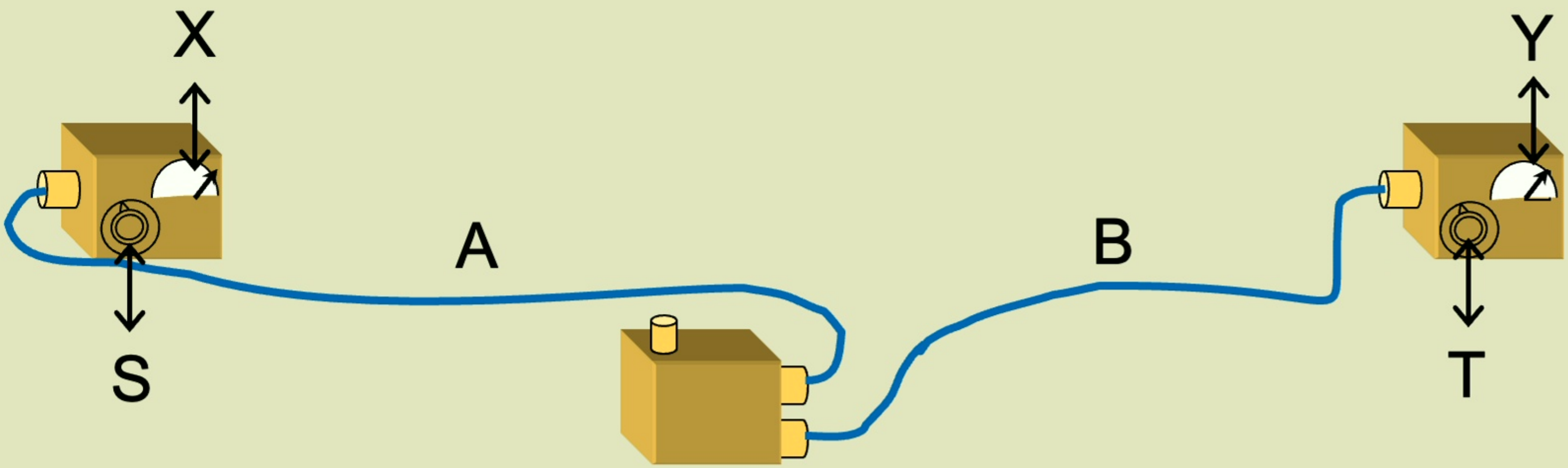




Bayesian updating

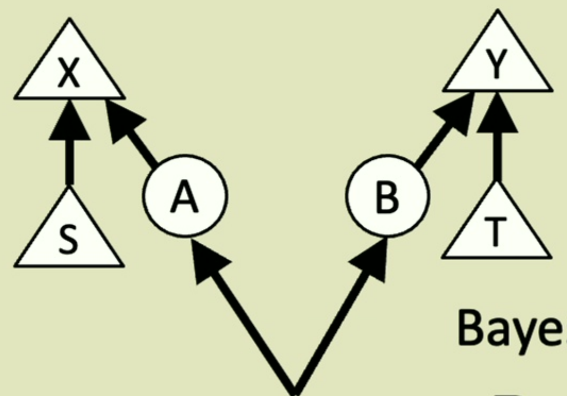
$$\rho_B \rightarrow \rho_{B|SX}$$


X
Y
A
B
S
T
X
Y
A
B
S
T
Given:
ρ_{AB}
$\rho_{X|SA}$
Bayesian updating
$\rho_B \rightarrow \rho_{B|SX}$



Bayesian inversion

$$P_{A|SX} = \frac{P_{X|AS}P_A}{P_{X|S}}$$



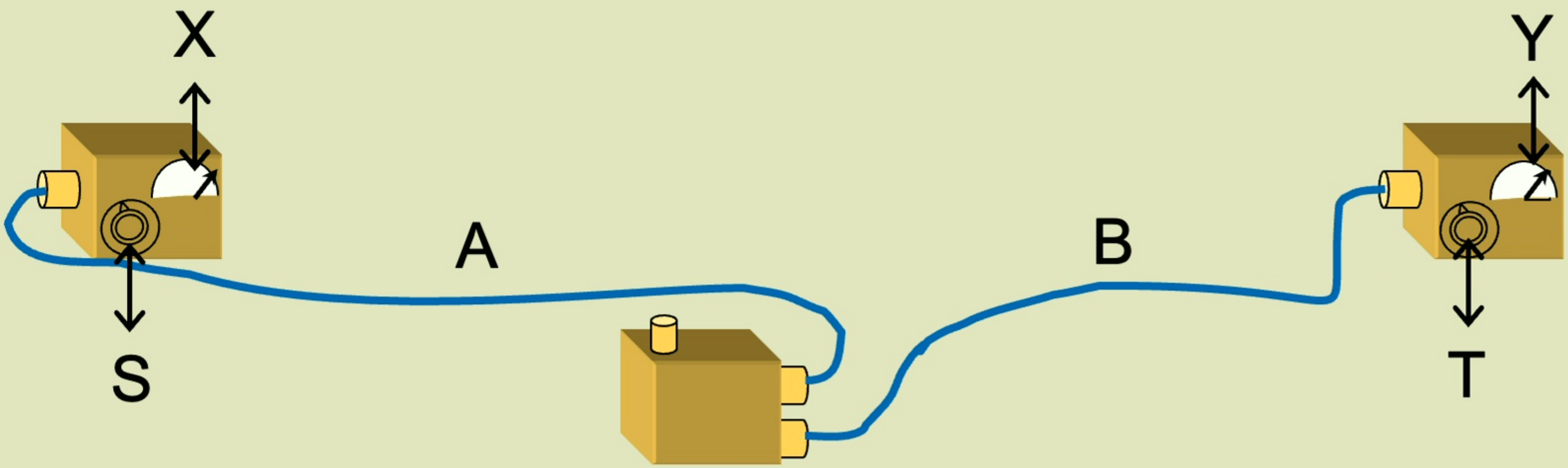
Given:

P_{AB}

$P_{X|AS}$

Bayesian updating

$$P_B \to P_{B|SX}$$

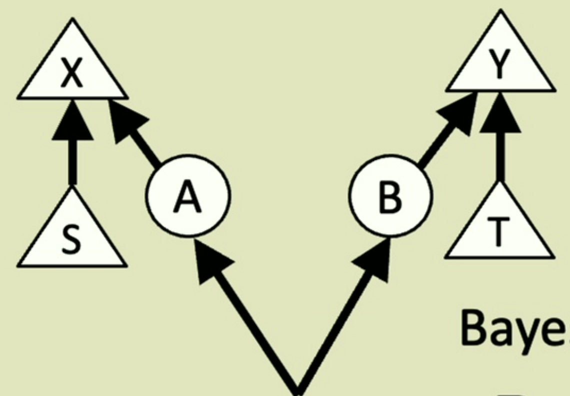


Bayesian inversion

$$P_{A|SX} = \frac{P_{X|AS} P_A}{P_{X|S}}$$

Conditional from joint

$$P_{B|A} = \frac{P_{AB}}{P_A}$$



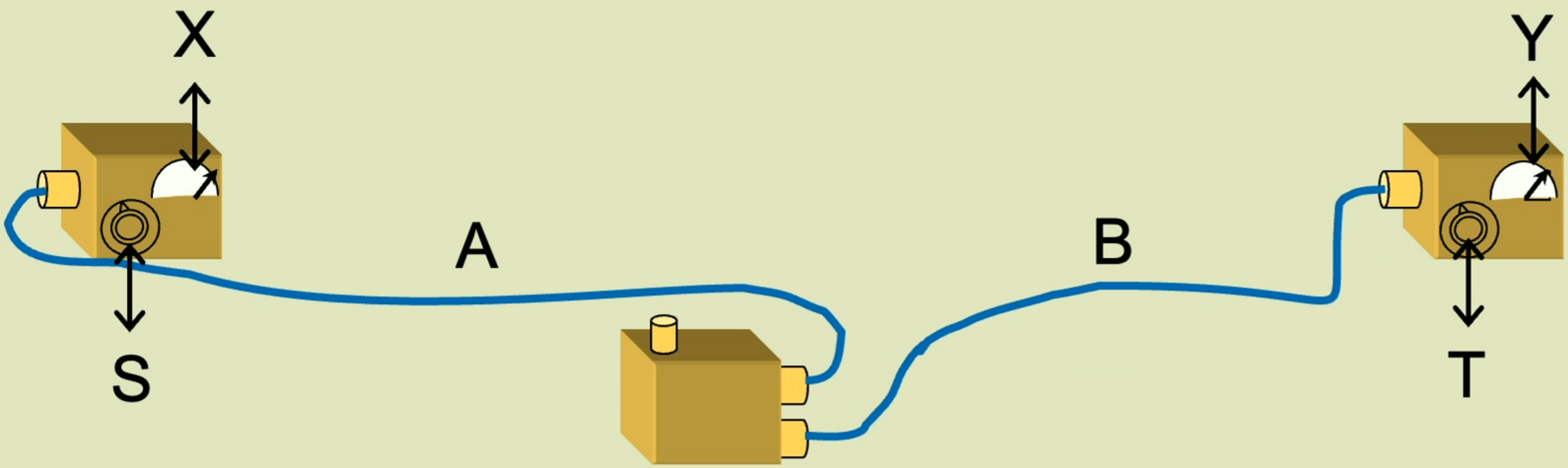
Given:

P_{AB}

$P_{X|AS}$

Bayesian updating

$$P_B \to P_{B|SX}$$



Bayesian inversion

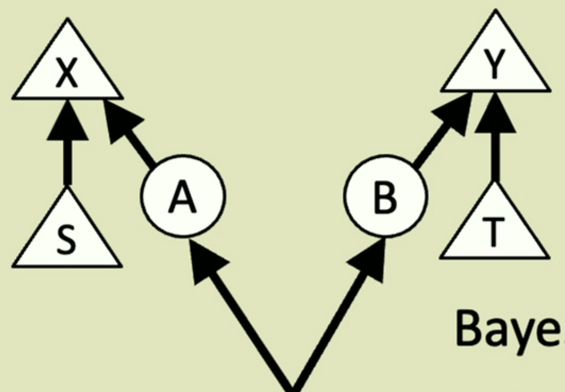
$$P_{A|SX} = \frac{P_{X|AS}P_A}{P_{X|S}}$$

Conditional from joint

$$P_{B|A} = \frac{P_{AB}}{P_A}$$

Belief propagation

$$P_{B|SX} = \sum_A P_{B|A}P_{A|SX}$$



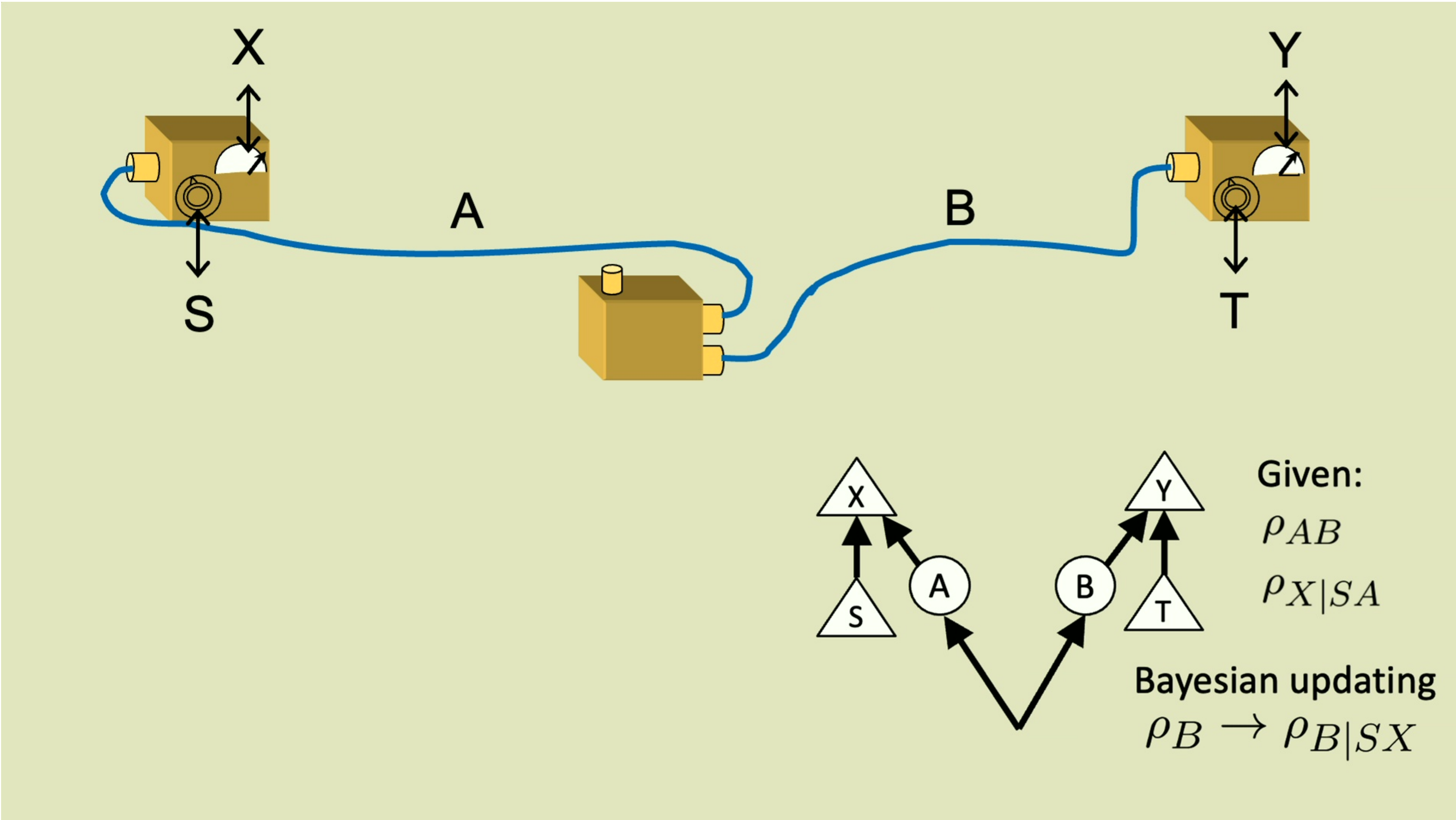
Given:

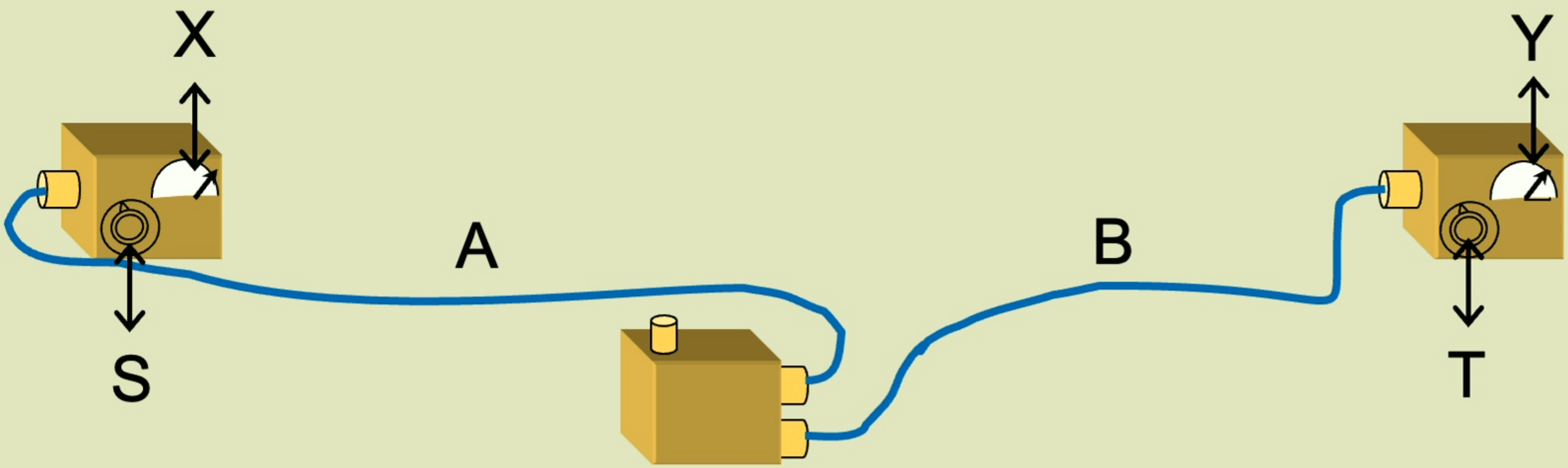
P_{AB}

$P_{X|AS}$

Bayesian updating

$$P_B \to P_{B|SX}$$


X
Y
A
B
S
T
X
Y
A
B
S
T
Given:
ρ_{AB}
$\rho_{X|SA}$
Bayesian updating
$\rho_B \to \rho_{B|SX}$



Bayesian inversion

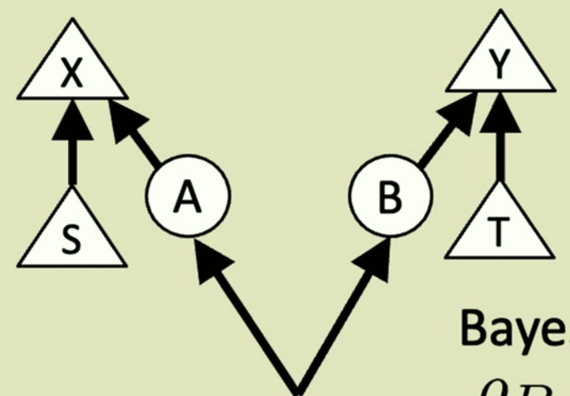
$$\rho_{A|XS} = \rho_{X|AS} \star \rho_A \rho_{X|S}^{-1}$$

Conditional from joint

$$\rho_{B|A} = \rho_{AB} \star \rho_A^{-1}$$

Belief propagation

$$\rho_{B|SX} = \text{tr}_A(\rho_{B|A}\rho_{A|SX})$$



Given:

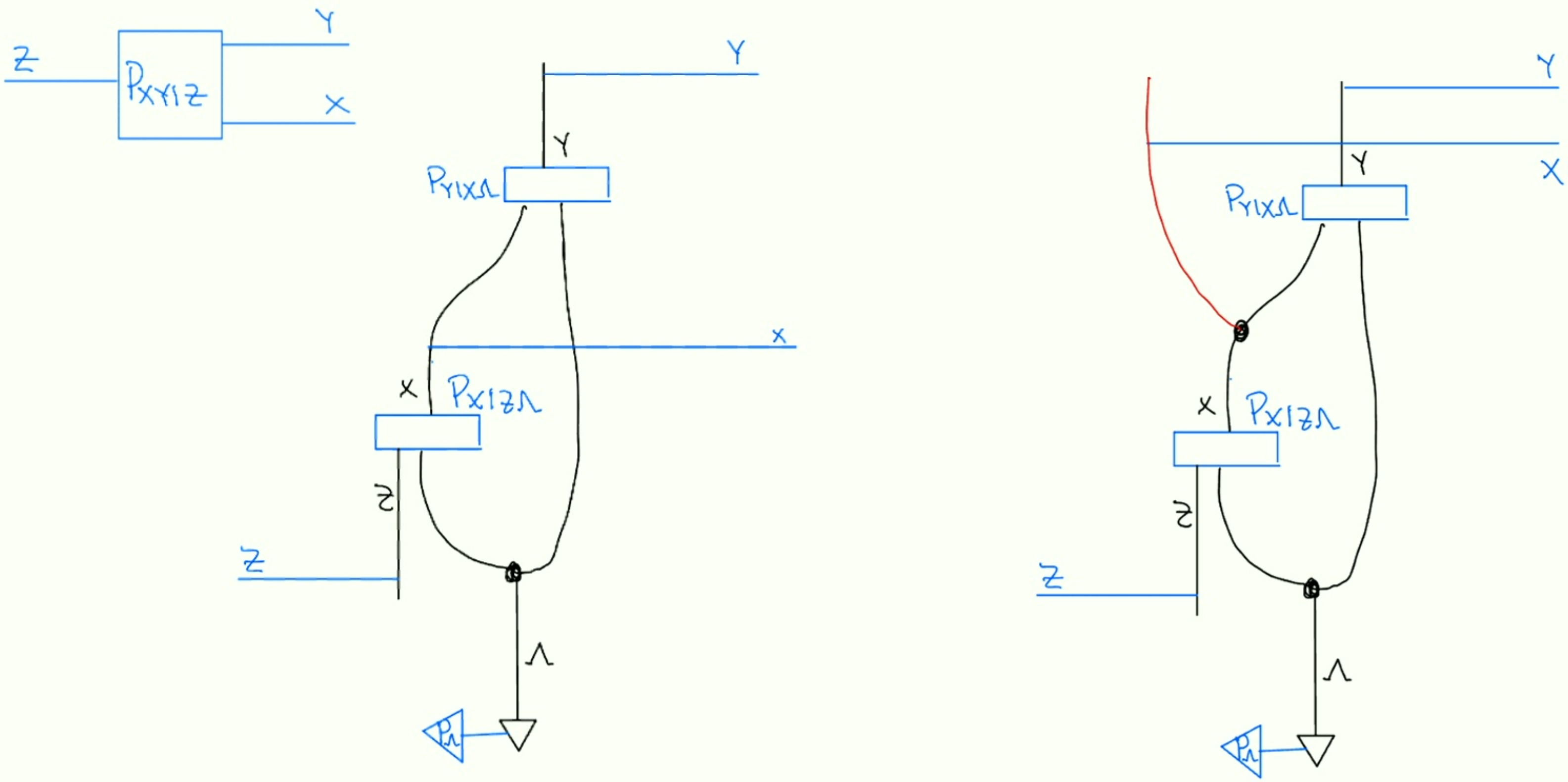
ρ_{AB}

$\rho_{X|SA}$

Bayesian updating

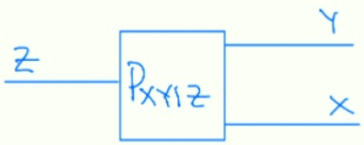
$$\rho_B \to \rho_{B|SX}$$

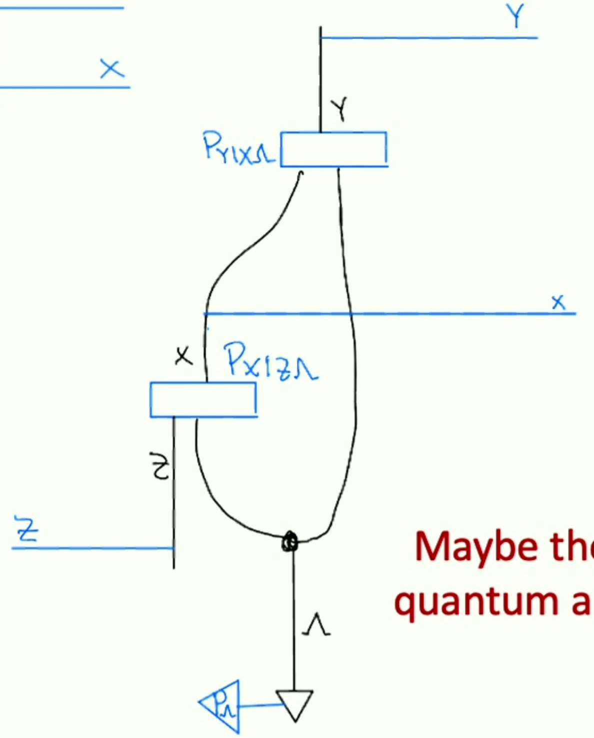
Interplay of causal and inferential theories



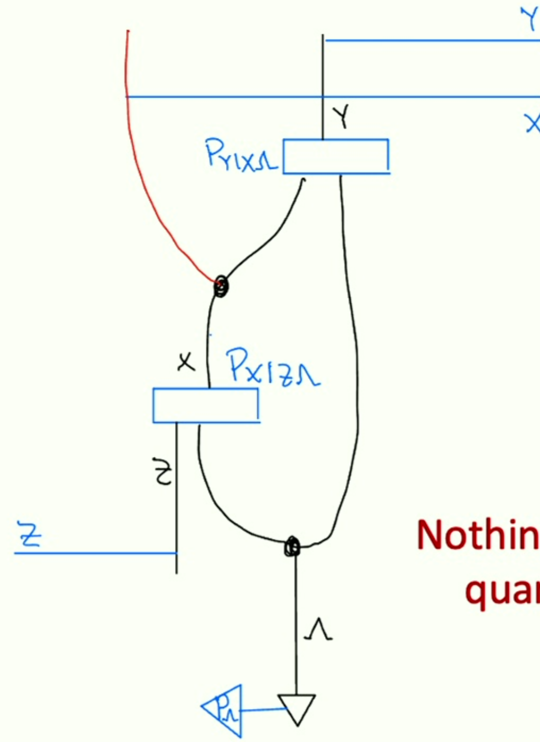
Markov condition

$$P_{XY|Z} = \sum_{\Lambda} P_{Y|X\Lambda} P_{X|Z\Lambda} P_{\Lambda}$$





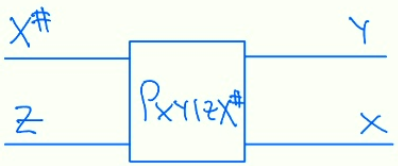
Maybe there is a quantum analogue

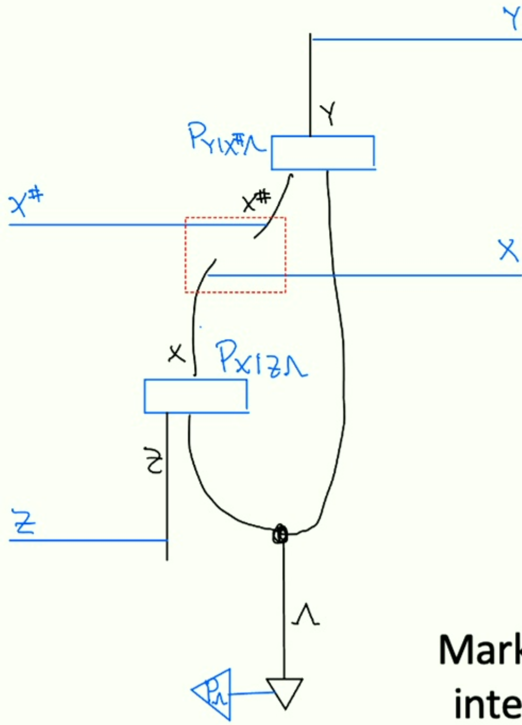


Nothing like this quantumly

Markov condition

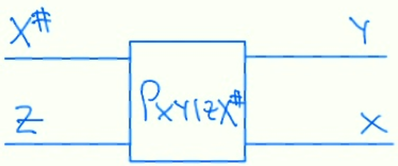
$$P_{XY|Z} = \sum_{\Lambda} P_{Y|X\Lambda} P_{X|Z\Lambda} P_{\Lambda}$$

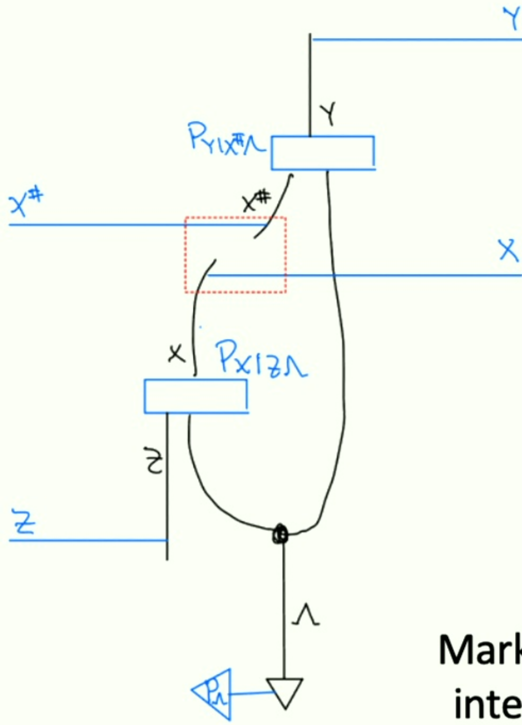




Markov condition for split-node intervention probing schemes

$$P_{XY|ZX^{\#}} = \sum_{\Lambda} P_{Y|X^{\#}\Lambda} P_{X|Z\Lambda} P_{\Lambda}$$



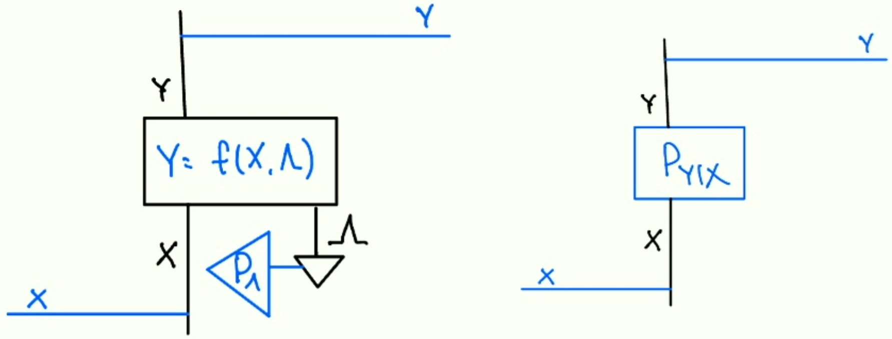


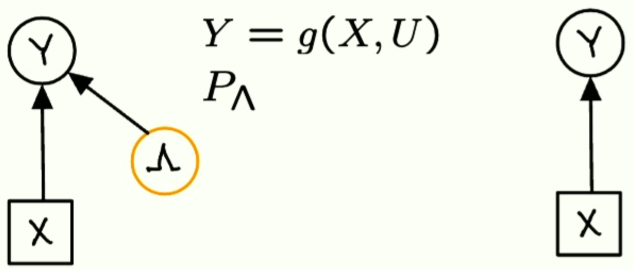
Quantum analogue exists

Markov condition for split-node intervention probing schemes

$$P_{XY|ZX\#} = \sum_{\Lambda} P_{Y|X\#\Lambda} P_{X|Z\Lambda} P_{\Lambda}$$

structural equation model | **Bayesian causal model**





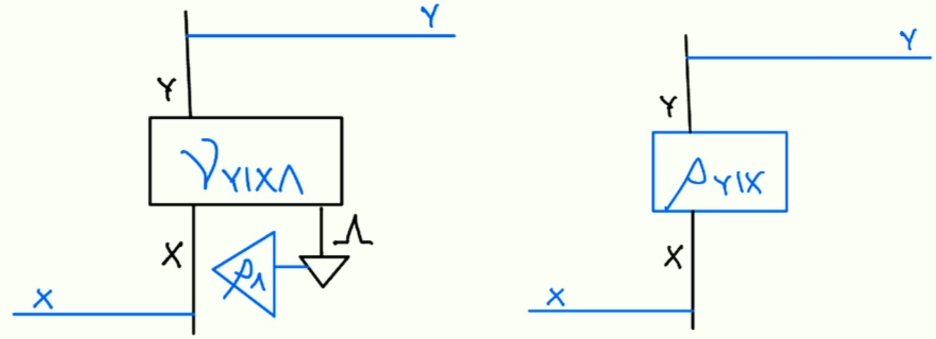
$Y = g(X,U)$
P_Λ

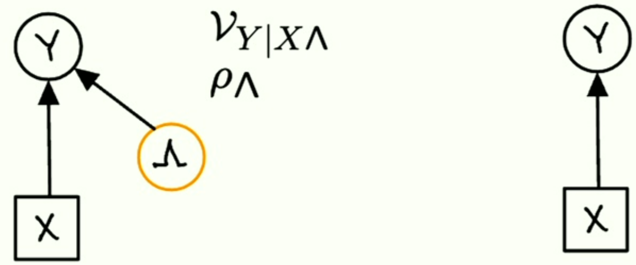
$P_{Y|X}$

$$P_{Y|X} = \sum_\Lambda P^{\text{det}}_{Y|X\Lambda} P_\Lambda$$

where $P^{\text{det}}_{Y|X\Lambda} = \delta_{Y,f(X,\Lambda)}$

structural equation model | **Bayesian causal model**





$\mathcal{V}_{Y|X\Lambda}$
ρ_Λ

$\rho_{Y|X}$

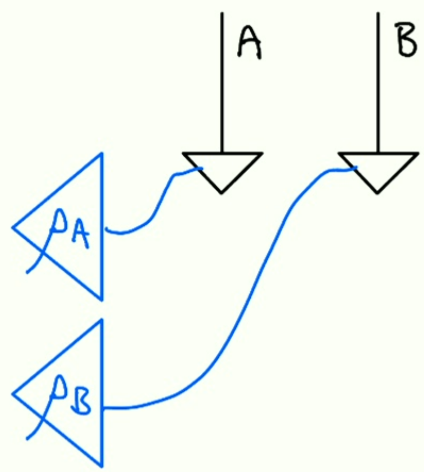
$$\rho_{Y|X} = \text{Tr}_\Lambda \left(\rho^{\text{det}}_{Y|X\Lambda} \rho_\Lambda \right)$$

where $\rho^{\text{det}}_{Y|X\Lambda}$ is CJ-isomorphic to $\mathcal{V}_{Y|X\Lambda}$

If A and B have no common ancestry, then

$$A \perp B$$

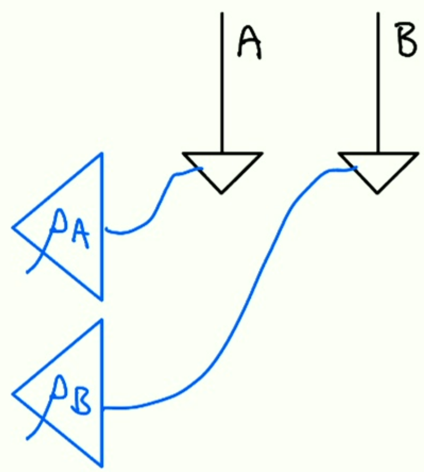
$$\rho_{AB} = \rho_A \otimes \rho_B$$



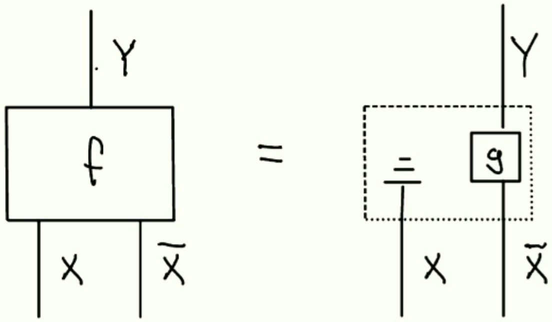
If A and B have no common ancestry, then

$$A \perp B$$

$$\rho_{AB} = \rho_A \otimes \rho_B$$



i.e., need a reason to posit correlated statistical sources variable X has no influence on variable Y

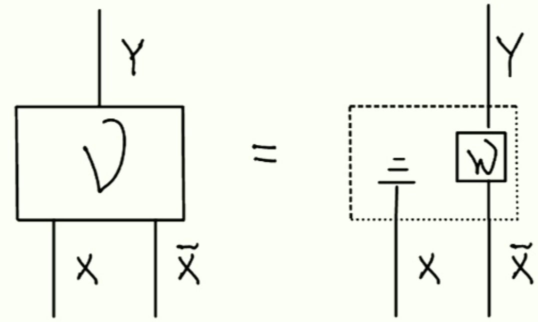


$$f(X,\bar{X}) = g(\bar{X})$$

$$P^{\text{det}}_{Y|X\bar{X}} = P^{\text{det}}_{Y|\bar{X}}$$

$$Y \perp X|\bar{X}$$

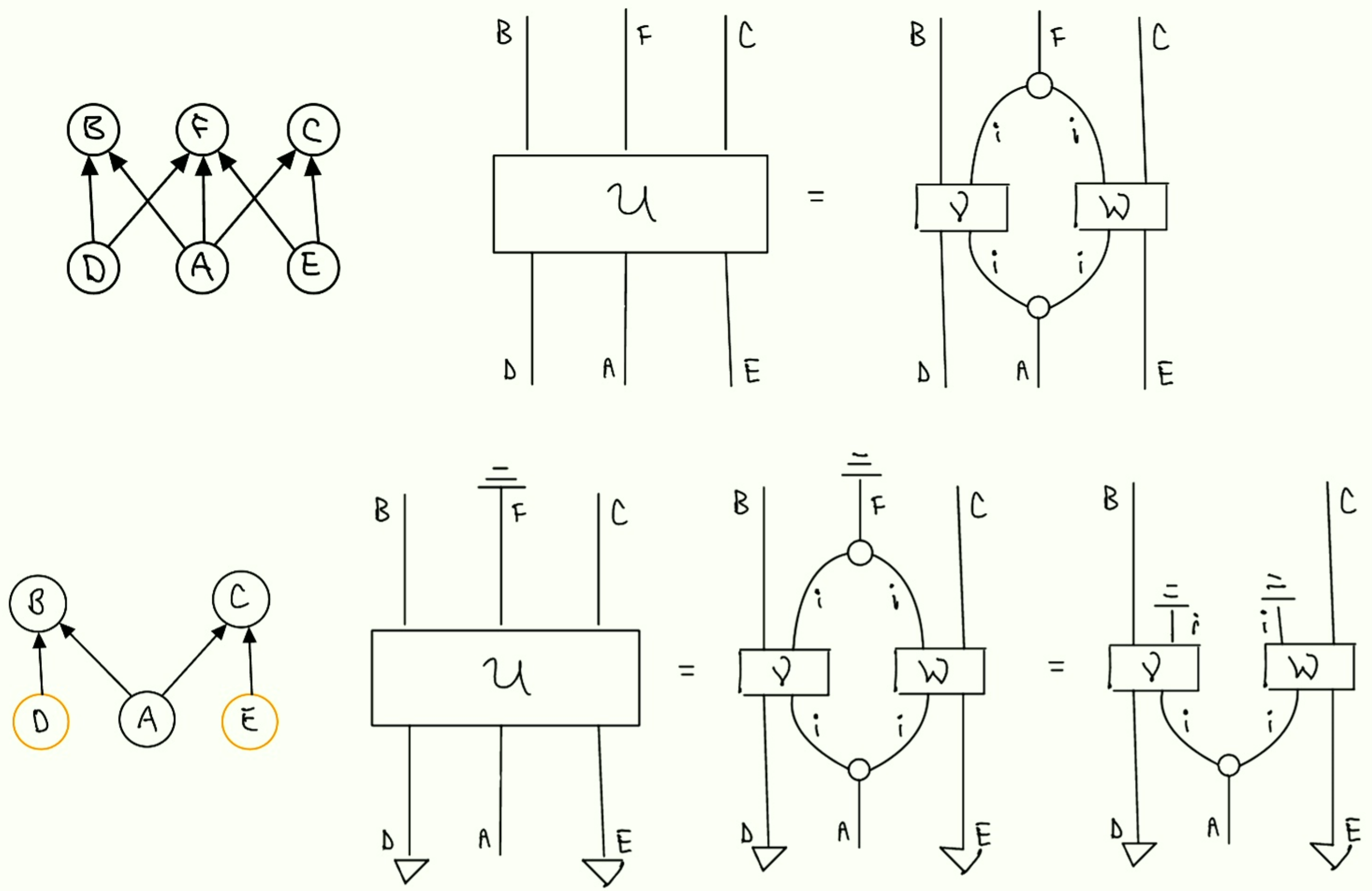
system X has no influence on system Y

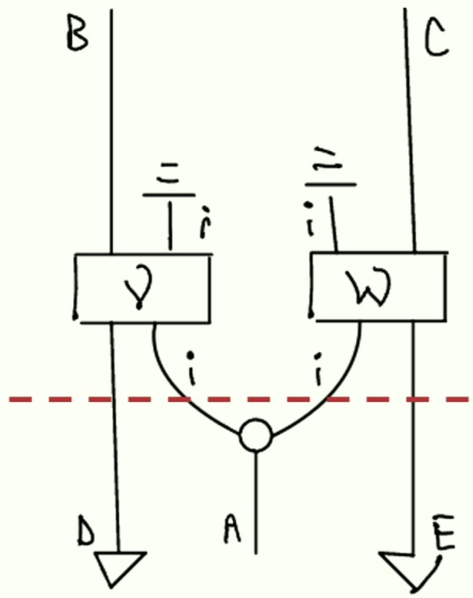


$$\mathcal{V}_{Y|X\bar{X}} = \mathrm{Tr}_X \otimes \mathcal{V}_{Y|\bar{X}}$$

$$\rho^{\text{det}}_{Y|X\bar{X}} = \rho^{\text{det}}_{Y|\bar{X}}$$

$$Y \perp X|\bar{X}$$


B F C
D A E
U
=
V W
i



$\mathcal{V}$ is block-diagonal across the *i* sectors and is nontrivial only on $\mathcal{H}_{A_i^L}$

$$\mathcal{H}_D \otimes \left(\bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R} \right) \otimes \mathcal{H}_E$$

$$\rho^{\text{det}}_{BC|ADE} := (\mathcal{U}_{BC|A'D'E'} \otimes \text{id}_{ADE})(\textstyle\sum_{j,k} |j\rangle\langle k|_{A'D'E'} \otimes |k\rangle\langle j|_{ADE})$$

$$\rho^{\text{det}}_{BC|ADE} = \rho^{\text{det}}_{B|AD} \rho^{\text{det}}_{C|AE}$$

$$\rho^{\text{det}}_{B|AD} = \textstyle\sum_i \rho^{\text{det}}_{B|DA_i^L} \otimes I_{A_i^R}$$

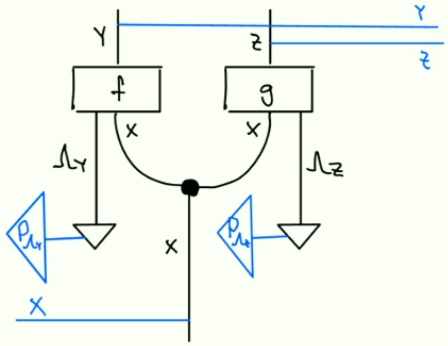
$$\rho^{\text{det}}_{C|AE} = \textstyle\sum_i I_{A_i^R} \otimes \rho^{\text{det}}_{C|A_i^R E}$$

$$P_{YZ|X} = \sum_{\lambda_Y \lambda_Z} P^{\text{det}}_{YZ|X\lambda_Y\lambda_Z} P_{\lambda_Y \lambda_Z}$$

$$\rho_{BC|A} = \text{Tr}_{\lambda_B\lambda_C}\left(\rho^{\text{det}}_{BC|A\lambda_B\lambda_C}\rho_{\lambda_B\lambda_C}\right)$$

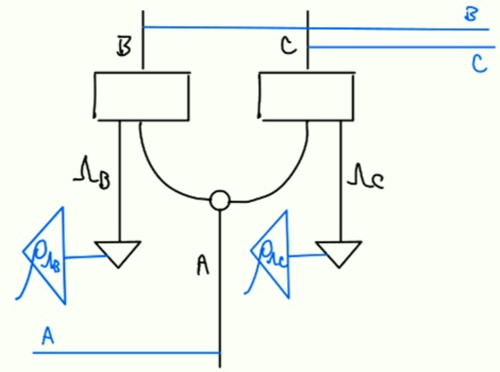
$$P^{\text{det}}_{YZ|X\lambda_Y\lambda_Z} = P^{\text{det}}_{Y|X\lambda_Y} P^{\text{det}}_{Z|X\lambda_Z}$$

$$P_{\lambda_Y\lambda_Z} = P_{\lambda_Y} P_{\lambda_Z}$$



$$\rho^{\text{det}}_{BC|A\lambda_B\lambda_C} = \rho^{\text{det}}_{B|A\lambda_B}\rho^{\text{det}}_{C|A\lambda_C}$$

$$\rho_{\lambda_B\lambda_C} = \rho_{\lambda_B} \otimes \rho_{\lambda_C}$$



$$P_{YZ|X} = \sum_{\lambda_Y,\lambda_Z} P^{\text{det}}_{Y|X\lambda_Y} P^{\text{det}}_{Z|X\lambda_Z} P_{\lambda_Y} P_{\lambda_Z}$$

$$= \left(\sum_{\lambda_Y} P^{\text{det}}_{Y|X\lambda_Y} P_{\lambda_Y}\right)\left(\sum_{\lambda_Z} P^{\text{det}}_{Z|X\lambda_Z} P_{\lambda_Z}\right)$$

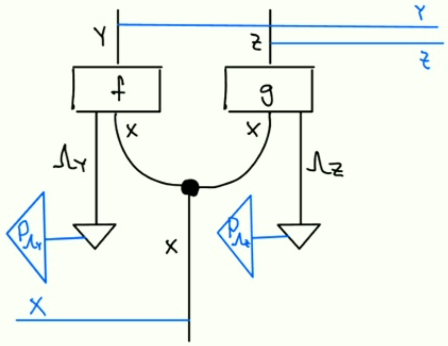
$$\rho_{YZ|X} = \text{Tr}_{\lambda_B\lambda_C}\left(\rho^{\text{det}}_{B|A\lambda_B}\rho^{\text{det}}_{C|A\lambda_C}(\rho_{\lambda_B}\otimes\rho_{\lambda_C})\right)$$

$$= \text{Tr}_{\lambda_B}(\rho^{\text{det}}_{B|A\lambda_B}\rho_{\lambda_B})\text{Tr}_{\lambda_C}(\rho^{\text{det}}_{C|A\lambda_C}\rho_{\lambda_C})$$

$$P_{YZ|X} = \sum_{\lambda_Y \lambda_Z} P^{\text{det}}_{YZ|X\lambda_Y\lambda_Z} P_{\lambda_Y \lambda_Z}$$

$$P^{\text{det}}_{YZ|X\lambda_Y\lambda_Z} = P^{\text{det}}_{Y|X\lambda_Y} P^{\text{det}}_{Z|X\lambda_Z}$$

$$P_{\lambda_Y \lambda_Z} = P_{\lambda_Y} P_{\lambda_Z}$$



$$P_{YZ|X} = \sum_{\lambda_Y, \lambda_Z} P^{\text{det}}_{Y|X\lambda_Y} P^{\text{det}}_{Z|X\lambda_Z} P_{\lambda_Y} P_{\lambda_Z}$$

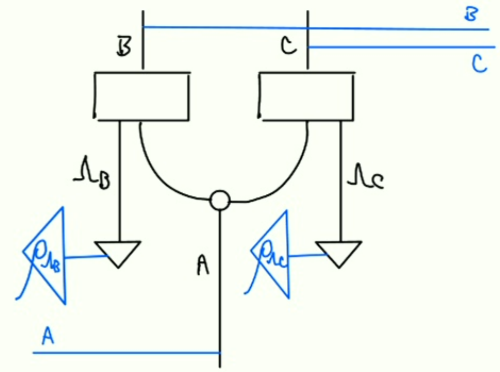
$$= \left(\sum_{\lambda_Y} P^{\text{det}}_{Y|X\lambda_Y} P_{\lambda_Y} \right) \left(\sum_{\lambda_Z} P^{\text{det}}_{Z|X\lambda_Z} P_{\lambda_Z} \right)$$

$$= P_{Y|X} P_{Z|X}$$

$$\rho_{BC|A} = \text{Tr}_{\lambda_B \lambda_C} \left(\rho^{\text{det}}_{BC|A\lambda_B\lambda_C} \rho_{\lambda_B \lambda_C} \right)$$

$$\rho^{\text{det}}_{BC|A\lambda_B\lambda_C} = \rho^{\text{det}}_{B|A\lambda_B} \rho^{\text{det}}_{C|A\lambda_C}$$

$$\rho_{\lambda_B \lambda_C} = \rho_{\lambda_B} \otimes \rho_{\lambda_C}$$



$$\rho_{YZ|X} = \text{Tr}_{\lambda_B \lambda_C} \left(\rho^{\text{det}}_{B|A\lambda_B} \rho^{\text{det}}_{C|A\lambda_C} (\rho_{\lambda_B} \otimes \rho_{\lambda_C}) \right)$$

$$= \text{Tr}_{\lambda_B} (\rho^{\text{det}}_{B|A\lambda_B} \rho_{\lambda_B}) \text{Tr}_{\lambda_C} (\rho^{\text{det}}_{C|A\lambda_C} \rho_{\lambda_C})$$

$$= \rho_{B|A} \rho_{C|A}$$

Generalizing to arbitrary causal structures

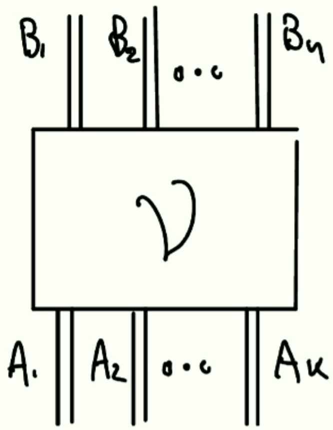
Reichenbach principle

If Z is a complete common cause of X and Y, then

$$P_{XY|Z\#} = P_{X|Z\#} P_{Y|Z\#}$$

If C is a complete common cause of A and B, then

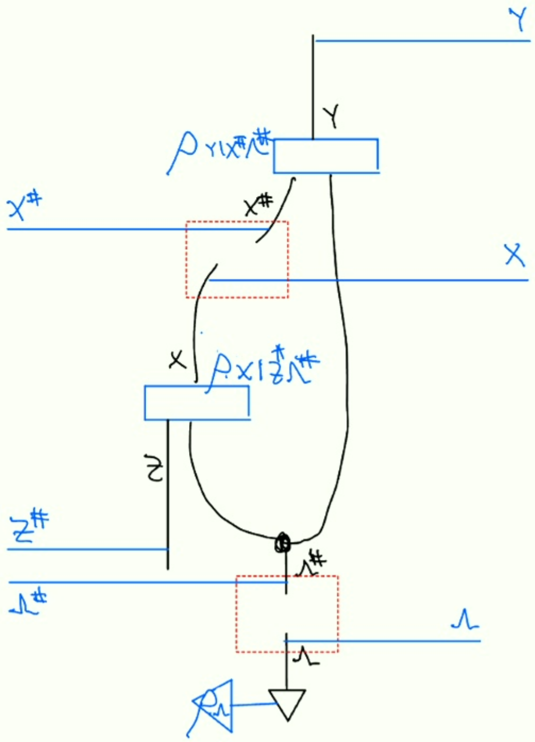
$$\rho_{AB|C\#} = \rho_{A|C\#} \rho_{B|C\#}$$



Let $\text{Pa}(B_i) = \{A_j : A_j \text{ influences } B_i\}$

Then $\rho^{\text{det}}_{B_1 \ldots B_n | A_1 \ldots A_k} = \prod_{i=1}^{n} \rho^{\text{det}}_{B_i | \text{Pa}(B_i)}$

where $[\rho^{\text{det}}_{B_i|\text{Pa}(B_i)}, \rho^{\text{det}}_{B_j|\text{Pa}(B_j)}] = 0 \;\; \forall i, j$



Markov condition for split-node intervention probing schemes

$$\rho_{XY\Lambda|Z^\# X^\# \Lambda^\#} = \rho_{Y|X^\# \Lambda^\#}\rho_{X|Z^\# \Lambda^\#}\rho_{\Lambda}$$

Next lecture:
Causal compatibility in quantum causal models