

Title: On Hikita-Nakajima conjecture for some quiver varieties and Slodowy slices

Speakers: Vasily Krylov

Series: Mathematical Physics

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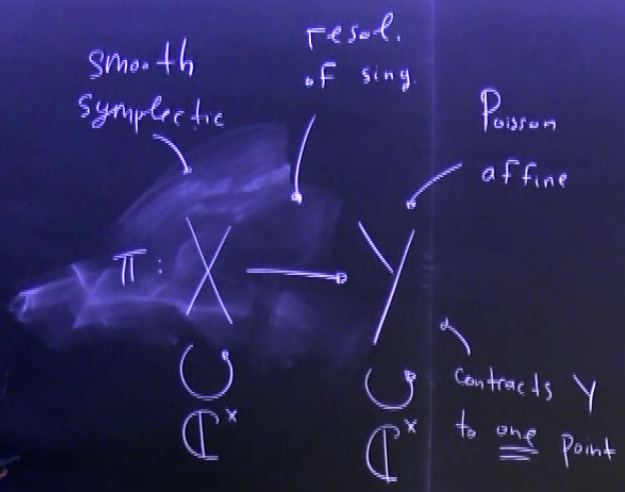
Abstract: Symplectic duality predicts that symplectic singularities should come in pairs. For example, Nakajima quiver varieties are conjecturally dual to BFN Coulomb branches (of the corresponding quiver theories). Another family of potentially symplectically dual pairs was described recently in the works of Losev, Mason-Brown, and Matvieievskyi: they describe symplectically duals to Slodowy slices to nilpotent orbits.

In this talk, we will discuss the Hikita-Nakajima conjecture that relates the geometry of symplectically dual varieties. We will restrict to the cases of certain quiver varieties and Slodowy slices and discuss the picture in these cases.

Based on the joint work with Pavel Shlykov (arXiv:2202.09934) and the work in progress with Do Kien Hoang and Dmytro Matvieievskyi.

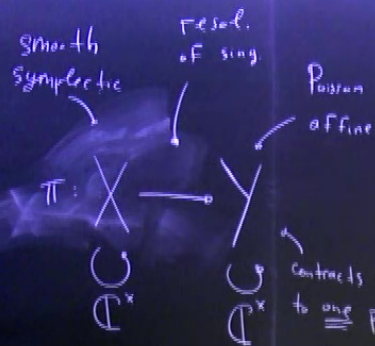
Zoom link: <https://pitp.zoom.us/j/98651907502?pwd=ODA1K3NKVHFLdkp6TEtaSnJXdThVZz09>

Conical symplectic resolutions



On Hikita-Nakashima conjecture for
 some quiver varieties and Slodowy slices
 (joint w. P. Shlykav,
 in progress w. K. Hoang, D. Matvieievskyi)

Conical / symplectic resolutions



On Hikita-Nakajima conjecture for some quiver varieties and Slodowy slices
 (joint w. P. Shrawan, in progress w. K. Liang, D. Maulik)

$$\textcircled{1} \{ (A, \ell) \mid A(a^i) \in \ell, A(\ell) = 0 \}$$

$$\pi: \mathbb{T}^* \mathbb{P}^1 \rightarrow \mathcal{N}_{\text{SL}_2} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, a^2 + bc = u \right\}$$

$$(A, \ell) \mapsto \bar{A}$$

$$X = \mathbb{M}(n, r) = H^{-1}(0) //_{\text{st}} GL(V), \quad \mathbb{Y} = H^{-1}(0) //_{\text{st}} GL(V)$$

$$\textcircled{2} (\mathbb{M}(n, r), \mathbb{M}(r)) \quad H: \mathbb{M} \rightarrow \mathbb{Y}$$

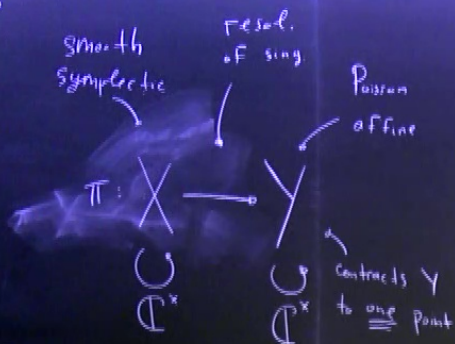
$$n, r \in \mathbb{Z}_{\geq 1}, \quad V = \mathbb{C}^n, \quad W = \mathbb{C}^r \quad (B_1, B_2, a, b) \mapsto [B_1]$$

$$M = \mathbb{M}(n, r) \cdot \text{End}(V)^{\oplus 2} \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W)$$

$$B_1 \subset V \xleftarrow{B_2} (B_1, B_2) \quad a \quad b$$

$$a \begin{pmatrix} W \\ b \end{pmatrix} \xleftarrow{M} GL(V)$$

Conical / symplectic resolutions



On Hikita-Nakajima conjecture for some equivariant varieties and Slodowy slices
 (joint w. P. Shrawan, in progress w. F. Liang, D. Malinik)

$$\textcircled{2} \{ (A, \ell) \mid A(a^i) \in \ell, A(\ell) = 0 \}$$

$$\pi: T^*\mathbb{P}^1 \rightarrow \mathcal{N}_{\text{SL}_2} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, a^2 + bc = u \right\}$$

$$(\tilde{A}, \ell) \mapsto \tilde{A}$$

$$T^*(G/B) \rightarrow \mathcal{N} \quad \text{--- Springer res}$$

$$\textcircled{1} \text{HR}_n(\mathbb{K}/\Gamma) \rightarrow S^*(\mathbb{A}^n/\Gamma) = Y$$

$$\Gamma = \mathbb{Z}/\mathbb{Z} \cap \mathbb{A}^2, \mathbb{Z}^k \cdot (x, y) = \left(e^{\frac{2\pi i k}{n}} x, e^{-\frac{2\pi i k}{n}} y \right)$$

$$\{ (a, b, c) \mid a^2 = bc \} / S_n$$

$$X = \mathbb{A}^n$$

$$M(n, \Gamma) = H^{-1}(0) // GL(V), \quad M(n, \Gamma) = H^{-1}(0) // GL(V)$$

$$H: M \rightarrow \mathbb{A}^1$$

$$\textcircled{2} (M(n, \Gamma), M(\mathbb{Z}))$$

$$n, \Gamma \in \mathbb{Z}_{\geq 1}, \quad V = \mathbb{C}^n, \quad W = \mathbb{C}^\Gamma$$

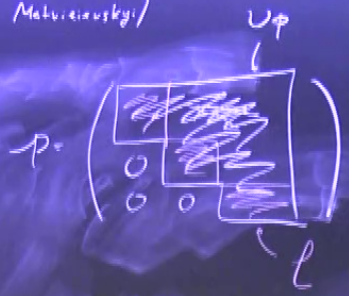
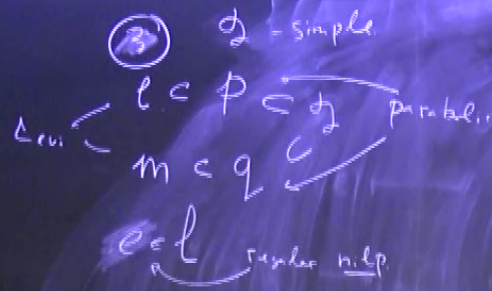
$$(B_1, B_2, a, b) \mapsto [B_1, B_2]$$

$$M = M(n, \Gamma) \cdot \text{End}(V)^{\oplus 2} \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W)$$

$$B_1 \subset V \subset B_2 \quad (B_1, B_2)$$

$$a \begin{pmatrix} W \\ V \end{pmatrix} b \leftarrow M \cap GL(V)$$

On Hikita-Nakajima conjecture for
 some quiver varieties and Slodowy slices
 (joint w. P. Shlykav,
 in progress w. K. Hoang, D. Matvieievskyi)



X
 " " \mathfrak{y}

$\underline{M(n,r)} = H^{-1}(0)^{st} / GL(V)$, $\underline{M(n,r)} = H^{-1}(0) // GL(V)$

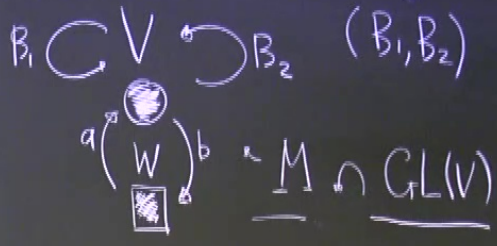
$H: M \rightarrow \mathfrak{sl}(V)$

$(B_1, B_2, a, b) \mapsto [B_1, B_2]$

② $(\underline{M(n,r)}, \underline{M(Q)})$

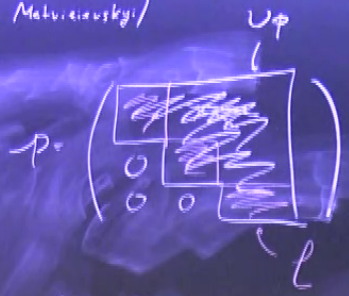
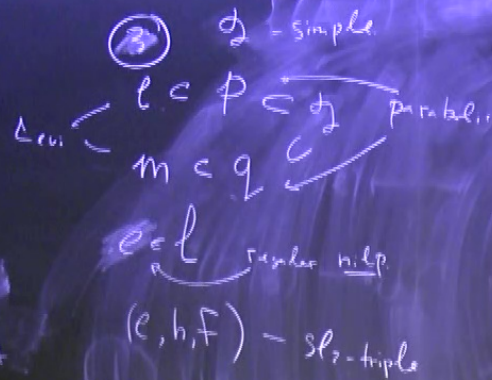
$n, r \in \mathbb{Z}_{\geq 1}$, $V = \mathbb{C}^n$, $W = \mathbb{C}^r$

$\underline{M} = \underline{M(n,r)} \cdot \underline{End(V)}^{\oplus 2} \oplus \underline{Hom(W,V)} \oplus \underline{Hom(V,W)}$



$Q_{V,W} \rightsquigarrow \underline{M(Q)}$

On Hikita-Nakajima conjecture for
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$Se := (e + \sum_{i=1}^l \alpha_i) \cdot \mathcal{N}$

$T^*(G/Q) \xrightarrow{\pi_2} \mathcal{N}$

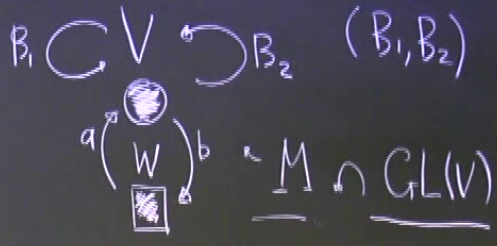
$G \times^{\mathfrak{g}} \mathcal{N}$

$X = \tilde{S}_e^{\mathfrak{g}} := \pi_1^{-1}(Se), Y = \text{Spec } \mathbb{C}[\tilde{S}_e^{\mathfrak{g}}]$

X
 $\mathcal{M}(n, \Gamma) = H^{-1}(0) \overset{st}{=} / GL(V), \mathcal{M}(n, \Gamma) = H^{-1}(0) // GL(V)$
 $H: \mathcal{M} \rightarrow \mathfrak{sl}(V)$
 $(B_1, B_2, a, b) \mapsto [B_1, B_2]$

② $(\mathcal{M}(n, \Gamma), \mathcal{M}(Q))$
 $n, \Gamma \in \mathbb{Z}_{\geq 1}, V = \mathbb{C}^n, W = \mathbb{C}^{\Gamma}$

$\mathcal{M} = \mathcal{M}(n, \Gamma) \cdot \text{End}(V)^{\oplus 2} \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W)$



$Q_{V, W} \rightsquigarrow \mathcal{M}(Q)$

Conical symplectic

Bural

$$E_x \quad q = \beta, \quad X = \tilde{S}_e^{\beta} = \tilde{S}_e$$

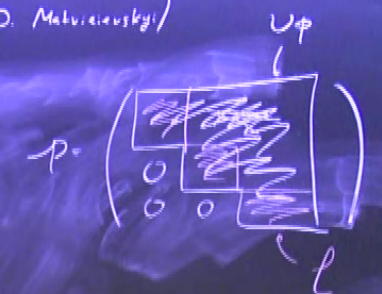
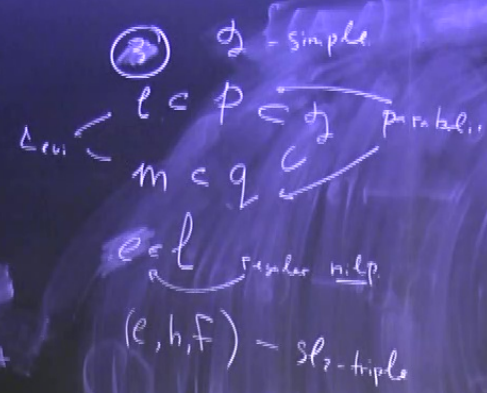
$$Y = S_e$$

$$p = \beta \Rightarrow e = 0, \quad \tilde{S}_e^q = T^*(G/Q)$$

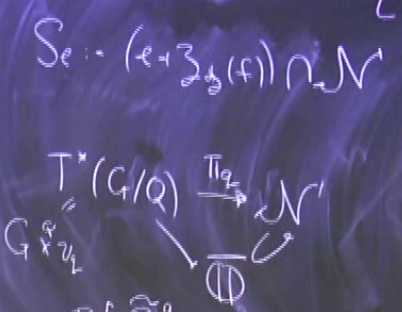
$$Y = \text{Spec } \mathbb{C}[\tilde{\mathbb{N}}]$$

coning of some nilp. orbit

On Hikita-Nakajima conjecture for some quiver varieties and Slodowy slices
 (joint w. P. Shlykova, in progress w. K. Huang, D. Mal'cevskiy)



$$X = \tilde{S}_e^q := \Pi_q^{-1}(S_e), \quad Y = \text{Spec } \mathbb{C}[\tilde{S}_e^q]$$



$$X$$

$$m(n, r) =$$

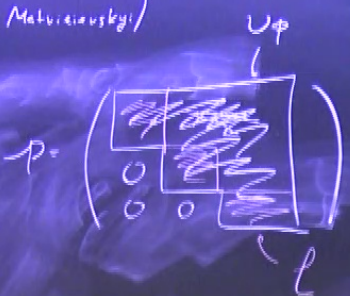
$$\textcircled{2} \quad (m(n, r) \in \mathbb{Z})$$

$$M = M$$

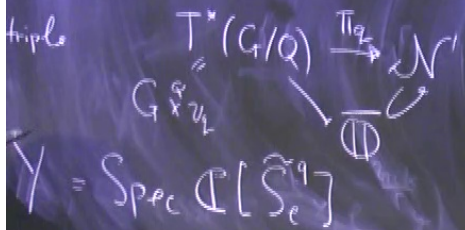
$$B \subset V$$

$$a \uparrow W$$

Kajima conjecture for
 necks and Slodowy slices
 P. Shlykav,
 w. K. Hoang, D. Mal'nevsky!

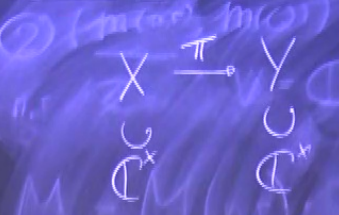


$$S_e := (e + \mathfrak{z}_{\mathfrak{g}}(f)) \cap \mathcal{N}'$$



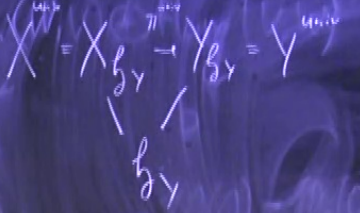
$$Y = \text{Spec } \mathbb{C}[\tilde{S}_e^q]$$

X
 $\mathfrak{m}(n) = \mathfrak{h}^*(\mathfrak{o}/\mathfrak{a}(n)), \mathfrak{m}(n) = \mathfrak{h}^*(\mathfrak{h}^*(\mathfrak{o}/\mathfrak{a}(n)) / \mathfrak{a}(n))$
Symplectic duality



$$T_Y \in \text{Aut}_{\mathbb{C}^*}^{l.s.}(Y), \mathfrak{h}_Y = \mathfrak{h}^*(X, \mathbb{C})$$

max. trans
 Exoticity



$$E_x: X \# T^*(G/P) = G \times^P U_P$$

$$X^{u,v} = G \times^P (U_P \oplus \mathfrak{z}(\mathfrak{e}))$$

$$(T^*P)^{u,v} = \{(A, d), A(d)cd\}$$

$\mathfrak{z}(\mathfrak{e})$

$\mathbb{C} \text{line in } \mathbb{C}^2$

Critical Symplectic

On Hikita-Nakajima conjecture for
 some quiver varieties and Slodowy slices
 (joint w. P. Shlykav,
 in progress w. K. Huang, D. Matvieievskyi)

$$X \rightarrow Y \xrightarrow{\text{Symplectic dual}} X' \rightarrow Y'$$

$$\begin{aligned} \ell_Y &= \ell_{Y'} \\ \mathfrak{h}_Y &= \mathfrak{h}_{Y'} \dots \\ X^{\text{Tr}} &= (X')^{\text{Tr}} \end{aligned}$$

- Ex) 1) $X = T^*(\mathbb{P}^n)$, $X' = T^*(\mathbb{P}^n)$
 2) $Y = \mathfrak{m}_0(\mathfrak{a})$, $Y' = \mathfrak{m}_0(\mathfrak{a}/(\mathfrak{a}, \mathbb{Z}))$
 3) $X = \tilde{\mathcal{S}}e_l$, $X' = T^*(G/P^v)$ *Langlands dual*
 4) $X = \tilde{\mathcal{S}}e_l^q$, $X' = \tilde{\mathcal{S}}e_{m'}^p$
 $m < q$
 $l < p$

X
 $\mathfrak{m}(n, r) = \mathfrak{m}(n, r)$
 Symplectic dual

$$\begin{aligned} X &\xrightarrow{\pi} Y \\ \mathbb{C}^x &\cup \mathbb{C}^y \\ \text{Ex: } X &= T^*(G/P) = \\ X^{m, n} &= G \times^P \mathbb{C}^n \\ (T^*(\mathbb{P}^1))^{m, n} &= \mathfrak{f}(A, d), \\ &e_{m, n, \mathbb{C}^2} \end{aligned}$$

Conical \mathbb{P}^n

Y - affine

$$A = \mathbb{C}[Y]$$

On Hikita-Nakajima conjecture for some quiver varieties and Slodowy slices
 (joint w. P. Shlyukov, in progress w. K. Huang, D. Matvieievskyi)

$$X \rightarrow Y \xrightarrow{\text{Symplectic dual}} X' \rightarrow Y'$$

$$\ell_Y = \ell_{Y'}$$

$$\ell_Y = \ell_{Y'}$$

$$X^{\text{Ty}} = (X')^{\text{Ty}}$$

Ex) $X = T^*(\mathbb{P}^1)$, $X' = T^*(\mathbb{P}^1)$

1) $Y = \mathfrak{m}_0(n, r)$, $Y' = S^n(A^r / (0, 1, z))$

2) $Y = \mathfrak{m}_0(0)$, $Y' = \mathfrak{m}(q)$

3) $X = \tilde{S}e_l$, $X' = T^*(G/P^r)$ Langlands dual

4) $X = \tilde{S}e_l^q$, $X' = \tilde{S}e_{m^r}$

$$\begin{matrix} m < q \\ l < p \end{matrix}$$

X
 $\mathfrak{m}(n, r) = \mathfrak{m}(0)$
 Symplectic dual

$$X \xrightarrow{\pi} Y$$

Ex: $X = T^*(G/P) =$

$$X^{m, n} = G \times^P$$

$$(T^*(\mathbb{P}^1))^{m, n} = \mathfrak{f}(A, d)$$

$$e_{m, n, d}$$

Cornell \mathbb{Q}^x

Y - affine

$E_x A = \mathbb{C}[Y]$

$A = \bigoplus_{i \in \mathbb{Z}} A_i$

$Y^{\mathbb{Q}^x} = \text{Spec } A / ((a_i)_{i \in \mathbb{Z}})$

$X \rightarrow Y$

Simple dual

$X' \rightarrow Y'$

$\ell_Y = \ell_{Y'}$

$\ell_Y = \ell_{Y'} \dots$

$X^T = (X')^T$

Lusztig - Matvievskii - Maron - Morita

On Hikita-Nakajima conjecture for some quiver varieties and Slodowy slices
(joint w. P. Shlykav, in progress w. K. Huang, D. Matvievskiy)

$E_x \circ X = T^*P', X' = T^*P'$

1) $Y = M_0(n, r), Y' = S^n(A^r / (a_i, \mathbb{Z}))$

2) $Y = M_0(a), Y' = M(a)$

3) $X = \tilde{S}e_r, X' = T^*(G/P^r)$ Langlands dual

4) $X = \tilde{S}e_q, X' = \tilde{S}e_m$
 $m < q$
 $l < p$

X
 Y
 $M(n, r) = \dots$

Hikita-Nakashima conjecture for
 some quiver varieties and Slodowy slices
 (joint w. P. Shlykav,
 in progress w. K. Heung, D. Mal'nevsky)

X
 \mathbb{A}^3
 $\mathbb{A}^3/\mathbb{C}^*$
 $\mathbb{A}^3/\mathbb{C}^*$
 $\mathbb{A}^3/\mathbb{C}^*$

$Y = \mathcal{N} + \{(a,b,c) \mid a^2+bc=0\}, \quad t \cdot (a,b,c) = (a, tb, t^2c)$

$\mathcal{N}^{\mathbb{C}^*} = \{(a,0,0)\}$

$\mathbb{C}[\mathcal{N}^{\mathbb{C}^*}] = \mathbb{C}[a,b,c]/(a^2+bc, b,c) = \mathbb{C}[a]/a^2$

On Hikita-Nakajima conjecture for
 some quiver varieties and Slodowy slices
 (joint w. P. Shlykav,
 in progress w. K. Hoang, D. Malviziavsky)

Conj (Hikita-Nakajima)

$$\exists H_{T_Y}^*(X, \mathbb{C}) \approx \mathbb{C}[(Y^{\vee})^{\text{univ}}]$$

$$H_{T_Y}^*(pt) = \mathbb{C}[\mathfrak{t}_Y] = \mathbb{C}[\mathfrak{b}_Y^*]$$

X

$$Y = \mathcal{N} = \{(a, b, c) \mid a^2 + bc = 0\}, \quad t \cdot (a, b, c)$$

$$\mathcal{N}^{\mathbb{C}^*} = \{(a, 0, 0)\}$$

$$\mathbb{C}[\mathcal{N}^{\mathbb{C}^*}] = \mathbb{C}[a, b, c] / (a^2 + bc, t \cdot c) = \mathbb{C}[a]$$

On Hikita-Nakashima conjecture for
 some quiver varieties and Slodowy slices
 (joint w. P. Shlykav,
 in progress w. K. Hoang, D. Mal'nevsky)

Conj (Hikita-Nakashima)

$$\exists H_{T_Y}^*(X, \mathbb{C}) \cong \mathbb{C}[(Y')^{(c)}]$$

$$H_{T_Y}^*(pt) = \mathbb{C}[\xi_Y] = \mathbb{C}[\xi_{Y'}]$$

$$\exists H^*(X, \mathbb{C}) \cong \mathbb{C}[(Y')^{(c)}]$$

X
Y

$$Y = \mathcal{N} = \{(a, b, c) \mid a^2 + bc = 0\}, \quad t \cdot (a, b, c)$$

$$\mathcal{N}^{\mathbb{C}^*} = \{(0, 0, 0)\}$$

$$\mathbb{C}[\mathcal{N}^{\mathbb{C}^*}] = \mathbb{C}[a, b, c] / (a^2 + bc, t \cdot c) = \mathbb{C}[a, b]$$

Nakajima conjecture for
 variables and Slodowy slices
 (w. P. Shlykov,
 joint w. K. Hoang, D. Matvieievskyi)

$$U \times \mathbb{A}^1 \rightarrow T_{Y'} \xrightarrow{U(\mathbb{A}^1)}$$

$$\mathbb{A}^1 \simeq \mathbb{A}^1[(Y')^{(r)}]$$

$$\mathbb{A}^1 = \mathbb{A}^1[\mathbb{F}_q] = \mathbb{A}^1[\mathbb{F}_q^*]$$

$$\mathbb{A}^1[(Y')^{(r)}]$$

X

Thm [K; Shlykov]

HN conj. holds for $X = \mathcal{M}(n, r) = \mathcal{M}$
 $(Y' = S^h(A^2/2I_2))$

Thm [KTWWY \Rightarrow]

HN conj. holds for $X = \mathcal{M}(Q) = \mathcal{M}$, ADE quiver
 $W \neq 0 \Rightarrow \mathbb{A}^1$ -minuscule.

On Hikita-Nakajima conjecture for
 some quiver varieties and Slodowy slices
 (joint w. P. Shlykav,
 in progress w. K. Hoang, D. Malinikovsky)

Conical slice

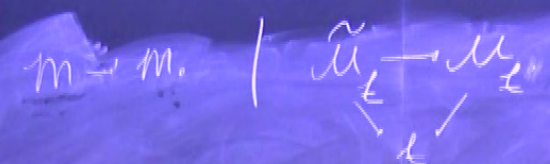
$Y = \text{affine}$

$E \rightarrow A = \mathbb{C}[Y]$

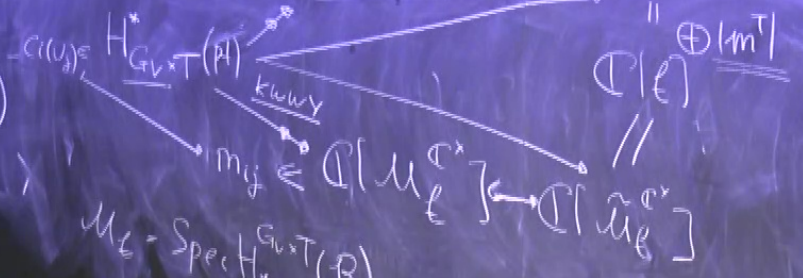
$A = \bigoplus_{i \in \mathbb{Z}} A_i$

$Y^{\text{aff}} = \text{Spec } A / ((a_i)_{i \in \mathbb{Z}})$

$G_V = \text{PGL}_n$



$\underline{c_i(v_j)} \in H_T^*(m) \leftarrow H_T^*(m^T)$



$M_\ell = \text{Spec } H_x^{G_V \times T}(R)$

$H_x^{G_V \times T}(R) = H_{G_V \times T}^*(\mathbb{P}^1)$

X

Thm [K]

HN Conj.

Thm [K]

HN Conj.

Kajima conjecture for
 replet and Slodowy slices
 (P. Shiyaku,
 w. K. Hoang, D. Matvieievskyi)

X
 $\mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1 \times \mathbb{A}^1$
 $(t, s) \mapsto (t, st)$
 $\mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1 \times \mathbb{A}^1$
 $(t, s) \mapsto (t, st)$

Values of m_{ij} at $\tilde{M}_t^{\mathbb{A}^1}$
 are equal to T-characters
 of \mathcal{U}_j at m^T

$$\frac{H_T^*(m)}{c(\mathcal{U}_j)}$$

Hilb
 $m(n, r)$

$$(B_1, B_2, a, b) \mapsto \mathcal{L}(d_1, \dots, d_n)$$

d_i -degree of B_1, B_2

HN

On Hikita-Nakajima conjecture for
 some quiver varieties and Slodowy slices
 (joint w. P. Shlykav,
 in progress w. K. Huang, D. Matvieievskyi)

$$\underline{H_{T \times \mathbb{C}^*}^*(X)} \simeq \mathbb{C}(B_0)(A_{\text{univ}})$$

$\mathbb{C}[t]$
 quotient of $\mathbb{C}[y_{i,j}]$

Values of m
 are equal
 of \mathcal{U}_j

Hilb

$m(n, r)$

(B, B_1)

Conjecture for
 Slodowy slices
 (D. Matvieievskyi)

X
 $\mathbb{C}[G/P] \cong \mathbb{C}[G/P] // G$
 Slodowy slices
 $(e) = 0$

$\mathfrak{g} = \mathfrak{so}_7$
 $\mathfrak{e} = (3,3,1)$
 $X = \tilde{S}_e$

$\dim B_e = 2$
 $B_e \cong \tilde{S}_e$
 $H^*(B_e) = H^*(\tilde{S}_e)$

B_e has 4 top dim
 $\dim(\text{Slice}(H^*(B_e))) \stackrel{\text{comp}}{\geq} 4$

$\mathfrak{g} = \mathfrak{sp}_6$
 $\mathfrak{e} = (2,2,2)$
 $T^*(G/P) \rightarrow \overline{\mathbb{D}}_e$
 $X' \quad Y'$

$\mathbb{C}[\overline{\mathbb{D}}_e^c] = \mathbb{C}[\overline{\mathbb{D}}_e \cap \mathbb{P}_{0,y}^2]$

slice
 \downarrow
 X, X_1, X_2
 \cap
 $\mathbb{C}[X_1, X_2, X_3]$
 $\int_S \frac{\mathbb{C}[X_1, X_2, X_3]}{\mathbb{C}[X_1, X_2, X_3]}$

On Hikita-Nakajima conjecture for
 some equivariant varieties and Slodowy slices
 (joint w. P. Shnytkov,
 in progress w. K. Huang, D. Matvieievskyi)

$$\textcircled{1} \mathbb{C}[\mathfrak{h}_{3e}] \rightarrow \mathbb{C}[\overline{\mathbb{W}}_e^{e^*}]$$

NOT surjective

$$\textcircled{2} S(H^*(\tilde{\mathfrak{S}}_e)) \rightarrow H^*(\tilde{\mathfrak{S}}_e) \cap \pi_0(Z_G(e)) = \mathbb{Z}/2\mathbb{Z}$$

$$f_{3e} \downarrow \quad \uparrow$$

$$H^*(T^*(\mathfrak{g}/\mathfrak{p}))$$

$$\mathbb{C}[\overline{\mathbb{W}}_e^{e^*}] \xrightarrow{(\kappa, \kappa^*)} H^*(\tilde{\mathfrak{S}}_e) \xrightarrow{\mathbb{Z}/2\mathbb{Z}} H^*(\tilde{\mathfrak{S}}_e)$$

Kajima conjecture for
 varieties and S-dowry slices
 P. Shlykav,
 w. K. Hoang, D. Matvieievskyi

$$\text{Im}(H_{T_e}^*(T^*(G/B)) - H_{T_e}^*(S_e)) \simeq \mathbb{C}[(\overline{\mathcal{D}}_e)^{\text{univ}}]^{\mathbb{C}^*}$$

$$\pi_0(Z_G(e)) = \mathbb{Z}/2\mathbb{Z}$$

$$\mathbb{C}[(\overline{\mathcal{D}}_e)^{\mathbb{C}^*}] \xrightarrow{(\text{Kir})} H^*(\tilde{S}_e) \xrightarrow{\mathbb{Z}/2\mathbb{Z}} H^*(\tilde{S}_e)$$

On Hikita-Nakajima conjecture for
 some quiver varieties and Slodowy slices
 (joint w. P. Shiyaku,
 in progress w. K. Huang, D. Matvieievskyi)

ee Thm (K-Huang-Matvieievskyi)

$$\frac{\text{Im}(H_{T_e}^*(T^*(G/P)) - H_{T_e}^*(\tilde{S}_e))}{\mathcal{L} / \tilde{S}_e} \cong \text{Im}(\mathbb{C}[b_j] \otimes \mathbb{C}[l_c] \xrightarrow{\varphi} \mathbb{C}[(T^*(G/P))^{l_{c'}}]^{G'})$$

$\mathcal{L} / \tilde{S}_e$

$T^*(G/P)^{l_{c'}}$
 \downarrow
 $\mathbb{Z} \times \mathbb{Z}(l')$