Title: Local Quantum Codes from Subdivided Manifolds

Speakers: Elia Portnoy

Series: Perimeter Institute Quantum Discussions

Date: March 22, 2023 - 11:00 AM

URL: https://pirsa.org/23030111

Abstract: For n>=3, we demonstrate the existence of quantum codes which are local in dimension n with V qubits, distance V^{(n-1)/n}, and dimension V^{(n-2)/n}, up to a polylog(V) factor. The distance is optimal up to the polylog factor. The dimension is also optimal for this distance up to the polylog factor. The proof combines the existence of asymptotically good quantum codes, a procedure to build a manifold from a code by Freedman-Hastings, and a quantitative embedding theorem by Gromov-Guth.

Zoom link: https://pitp.zoom.us/j/96227350980?pwd=TEFtamRmTXg3dnBMNEhKUkpHbmVoZz09

Perimeter talk 2
Mar 17, 2023 at 4:31 PM

Local Codes from Subdivided Manifolds

Acknowledgements Larry Guth

Matt Hastings, Mike Freedman, Misha Gromov

A. Leverrier, G. Zémor, ...

Def: A CSS code $\mathcal{C}$ is a 3-term chain complex of $\mathbb{F}_2$ vector spaces with distinguished basis elements.

$$\boxed{\begin{array}{c} C_2 \xrightarrow{H_2^T} C_2 \xrightarrow{H_1} C_1 \\ \\ H_1 \circ H_2^T = 0 \end{array}}$$

Def: A CSS code $\mathcal{C}$ is a 3-term chain complex of $\mathbb{F}_2$ vector spaces with distinguished basis elements.

$$\boxed{\begin{array}{c} C_2 \xrightarrow{H_2^T} C_1 \xrightarrow{H_1} C_0 \\[1em] H_1 \circ H_2^T = 0 \end{array}}$$

- $\dim(\mathcal{C}) = \dim(\ker H_1 / \operatorname{Im} H_2^T)$

- $\operatorname{dist}(\mathcal{C}) = \min \{ |v| : v \in \ker H_1 / \operatorname{Im} H_2^T$ or $v \in \ker H_2 / \operatorname{Im} H_1^T \}$

Hamming size

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<u>Def</u> A code is local if we can place its qubits i.e. the basis of C_1, into

$$Q_V = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : |x_1| + |x_2| + |x_3| \leq V^{1/3}\}$$

where $V = \text{size}(\mathcal{C})$ and so that each check only involves qubits inside a ball of radius $O(1)$ in Q_V.

Thm (Panteleev-Kalachev, Leverrier-Zémor)

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Thm (Panteleev-Kalachev, Leverrier-Zémor)

For any $V \geq 1$ there is a $\mathcal{C}'$

with $\text{size}(\mathcal{C}') > V$

and $\text{dist}(\mathcal{C}') \gtrsim \text{size}(\mathcal{C}')$

" $\mathcal{C}'$ is asymtotically good "

Thm 1: There is a local code with

$$\text{size}(\mathcal{C}) = V \,\text{polylog}(V)$$

$$\text{dist}(\mathcal{C}) \gtrsim V^{2/3}$$

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$$\mathrm{dist}(\mathcal{C}) \approx V$$

$$\dim(\mathcal{C}) \approx V^{1/3}$$

Conjecture: For any $d \geq 3$, we have a local* code $\mathcal{C}$ s.t

$$\mathrm{Size}(\mathcal{C}) = V \,\mathrm{polylog}(V)$$

$$\mathrm{dist}(\mathcal{C}) \approx V^{1/2}$$

$$\dim(\mathcal{C}) \approx 2^{d}$$

and energy barrier $\approx V^{\left(\frac{1}{2} - \frac{1}{d}\right)}$

Geometry of Manifolds

Let M be a triangulated d-dimensional manifold

We let vol(M) = # of d-simplices in M

A k-cycle is a set of k-simplices with 0 boundary.

A k-boundary is a set of

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We let $\mathrm{vol}(M)$ = # of d-simplices in M

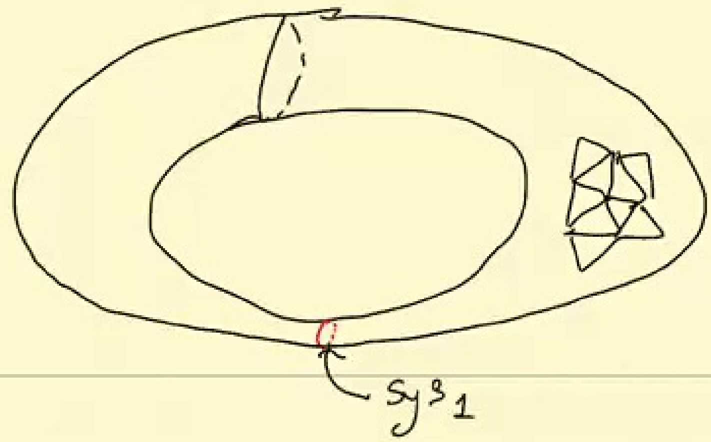
A k-cycle is a set of k-simplices with 0 boundary.

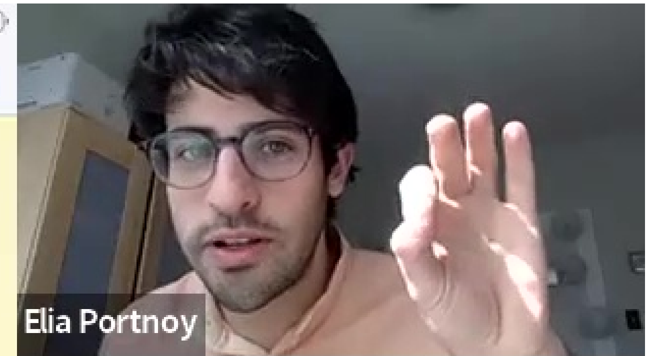
A k-boundary is a set of k-simplices which is a boundary of a set of $k+1$ simplices.

$\mathrm{Sys}_k(M)$ is the minimum number of k-simplicies needed to represent a k-cycle that

to represent a k-cycle
is not a k-boundary

Example





Example

Let $S^i(r) :=$ sphere of dim i and radius r triangulated by unit simplices.

$$M = S^{k}\left(V^{\frac{1}{2k}}\right) \times S^{d-k}\left(V^{\frac{1}{2(d-k)}}\right)$$

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sys_1

Example

let $S^i(r) :=$ sphere of dim i and radius r triangulated by unit simplices.

$$M = S^k\left(V^{\frac{1}{2k}}\right) \times S^{d-k}\left(V^{\frac{1}{2(d-k)}}\right)$$

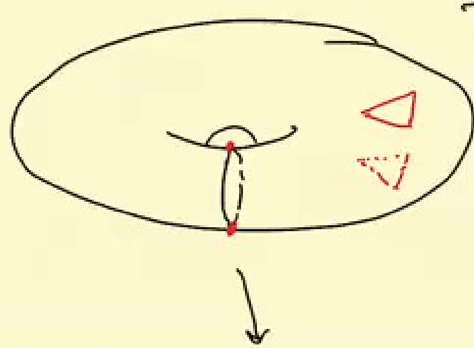
$$Vol(M) \approx V$$

$$Sys_k(M) \approx \sqrt{V}$$

We say $F: M \longrightarrow \mathbb{R}^3$ is coarse if the image of each simplex in M has diameter $O(1)$ and pre-image of each unit ball intersects $O(1)$ simplices

$T^2 \subset \mathbb{R}^3$

Example



Project torus to plane

= "toric code is local"

Dictionary

Code C	Triangulated d-manifold M $(d \geq 11)$
size	$\mathrm{vol}\ M$
qubit	$\approx$ k-simplex
distance	$\min(\mathrm{sys}_k(M), \mathrm{sys}_{d-k}(M))$ (where $4 \leq k \leq d-4$)
dimension	$\dim(H_k(M))$
LDPC	each vertex in M is adjacent to $O(1)$ simplices

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We say $F : M \longrightarrow \mathbb{R}^3$

is coarse if the image

of each simplex in M has

diameter $O(1)$ and pre-image of

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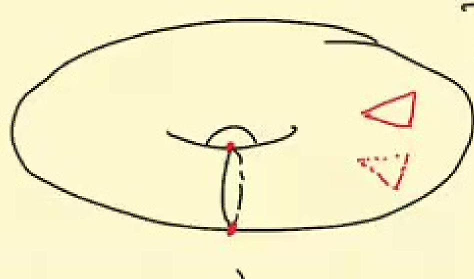
Example

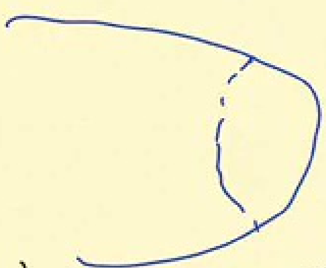


$T^2 \subset \mathbb{R}^3$

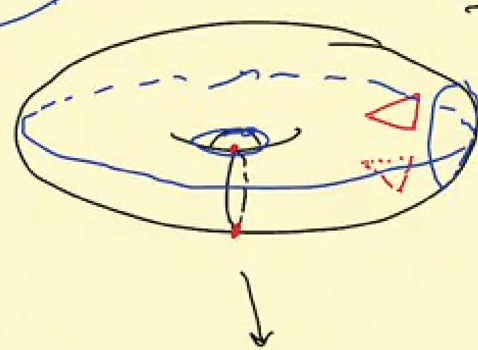
Project

= toric code

We say $F : M \longrightarrow \mathbb{R}^3$ is **coarse** if the image of each simplex in M has diameter $O(1)$ and pre-image of each unit ball intersects $O(1)$ simplices



Example



$T^2 \subset \mathbb{R}^3$

Project torus to

$\simeq$ toric code is local?

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Code C'	Triangulated d-manifold M $(d \geq 11)$
size	vol M
qubit	$\approx$ k-simplex
distance	$\min(sys_k(M), sys_{d-k}(M))$ (where $4 \leq k \leq d-4$)
dimension	$\dim(H_k(M))$
LDPC	each vertex in M is adjacent to $O(1)$ simplices
	There is a coarse

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dimension	$\dim(H_k(M))$
LDPC	each vertex in M is adjacent to $O(1)$ simplices
local	There is a coarse map from M to a ball of volume $\mathrm{vol}(M)$ in $\mathbb{R}^3$.

$$\boxed{\text{Codes}} \xrightarrow{\text{Freedman-Hastings 2020}} \boxed{\text{manifolds}}$$

Homological Code

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dimension	$\dim(H_k(M))$
LDPC	each vertex in M is adjacent to $O(1)$ simplices
local	There is a coarse map from M to a ball of volume $\mathrm{vol}(M)$ in $\mathbb{R}^3$.

$\left(|x-\frac{\sqrt{v}}{2}|, |y-\frac{\sqrt{v}}{2}|\right)$

$$\boxed{\text{Codes}} \xrightarrow{\text{Freedman-Hastings 2020}} \boxed{\text{manifolds}}$$

Homological Code

Code $\mathcal{C}$	Triangulated d-manifold M $(d \geq 11)$
size	vol M
qubit	$\approx$ k-simplex
distance	$\min(\mathrm{sys}_k(M), \mathrm{sys}_{d-k}(M))$ (where $4 \leq k \leq d-4$)
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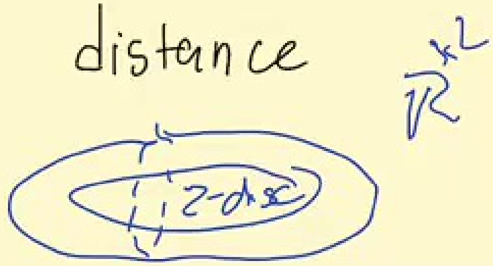

Proof Idea for thm 1

Step 1: C = asymp. good code $\rightsquigarrow$

M, $\mathrm{vol}\, M = V$, $\mathrm{sys}_k(M) \approx \mathrm{sys}_{d-k}(M) \approx V$

Step 2: Find a coarse map, P, from M to a 2-dimensional complex X

Step 3: Find a map, I, from X to a ball in $\mathbb{R}^3$

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Step 3: Find a map, I, from

X to a ball in $\mathbb{R}^3$

with small preimages,

but image of each simplex

is large.

Step 4: Subdivide M to $\tilde{M}$

Subdivide X to $\tilde{X}$ until,

$$\tilde{M} \xrightarrow[\text{coarse}]{P} \tilde{X} \xrightarrow[\text{coarse}]{I} \mathbb{R}^3$$

Turn $\tilde{M}$ into a

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new code $\mathcal{E}$

"X is related to the combinatorial structure you use to create the code e.g 2-sided Caley complex"

Rough picture of M

Basis element of C_0 $\rightsquigarrow$ red piece

Basis element of C_1 $\rightsquigarrow$ green piece

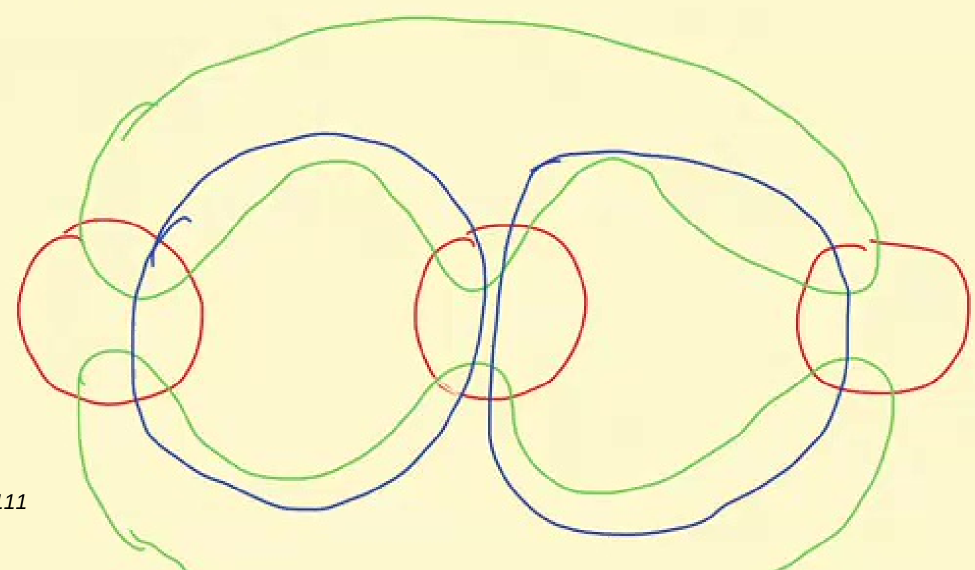
Rough picture of M

Basis element of C_0 $\rightsquigarrow$ red piece

Basis element of C_1 $\rightsquigarrow$ green piece

Basis element of C_2 $\rightsquigarrow$ blue piece

Glued together using data of H_2, H_1



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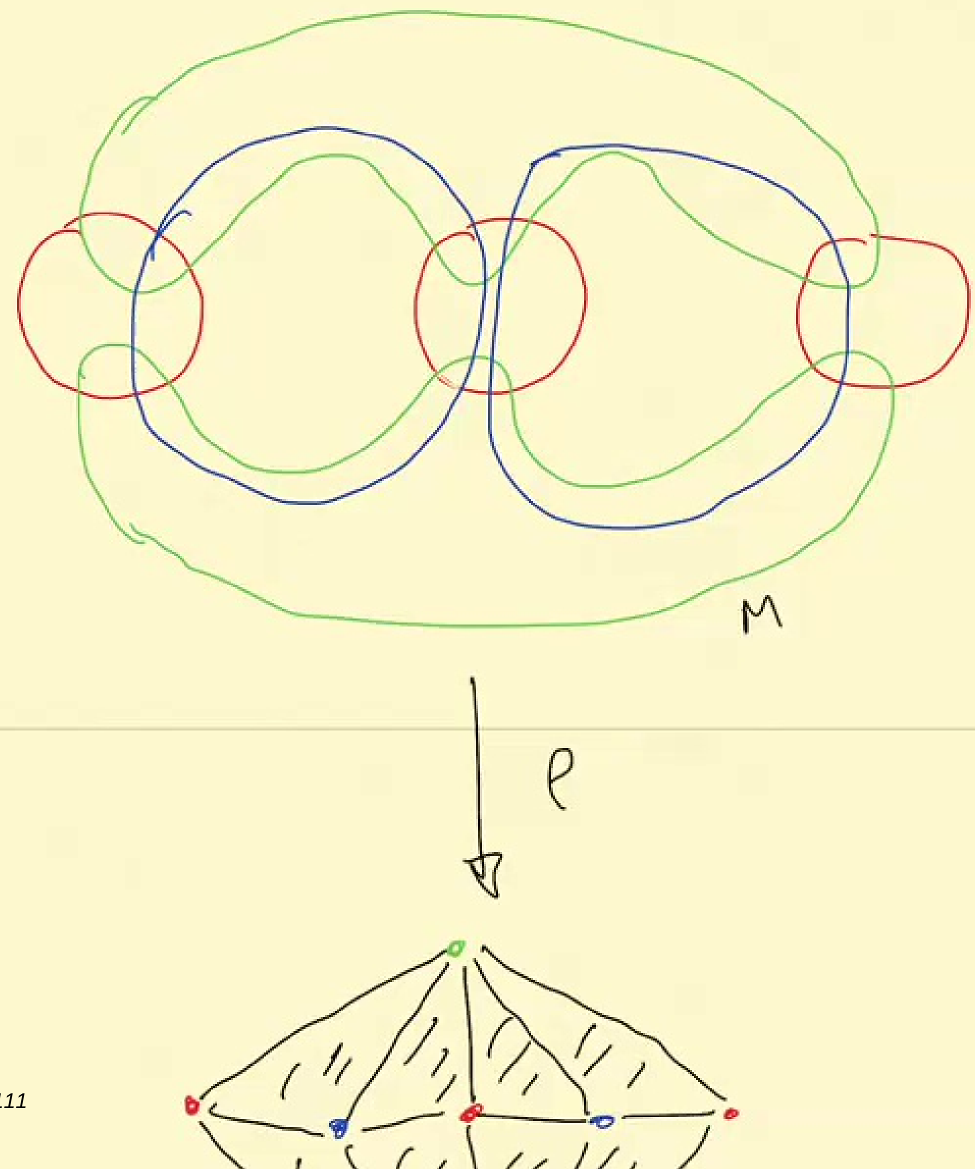
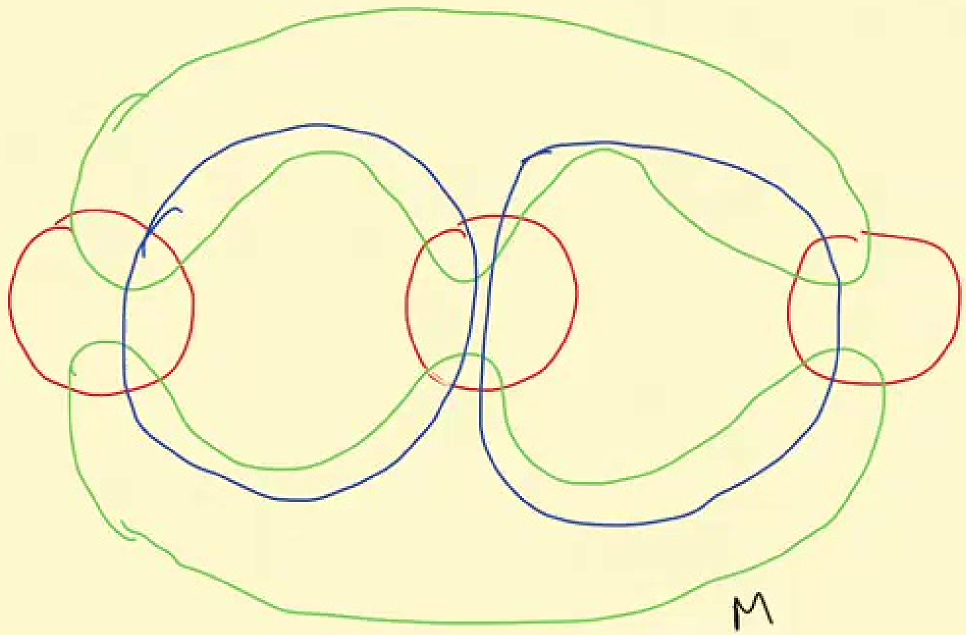
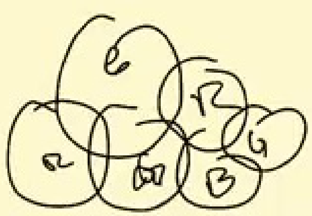
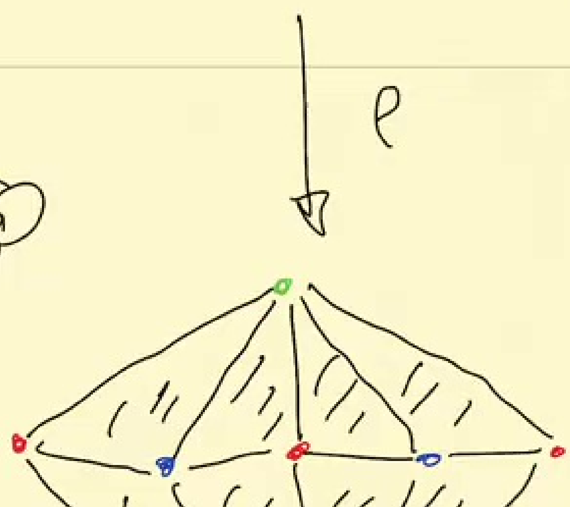
Glued together using data of H_2, H_1



Glued together using data of H_2, H_1







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How to find map from X to $\mathbb{R}^3$

Use method of Gromov-Guth 2011 to get map with small pre-images:

Map each vertex of X to a random point in the boundary of a ball of volume V^3.

Extend I linearly to simplices

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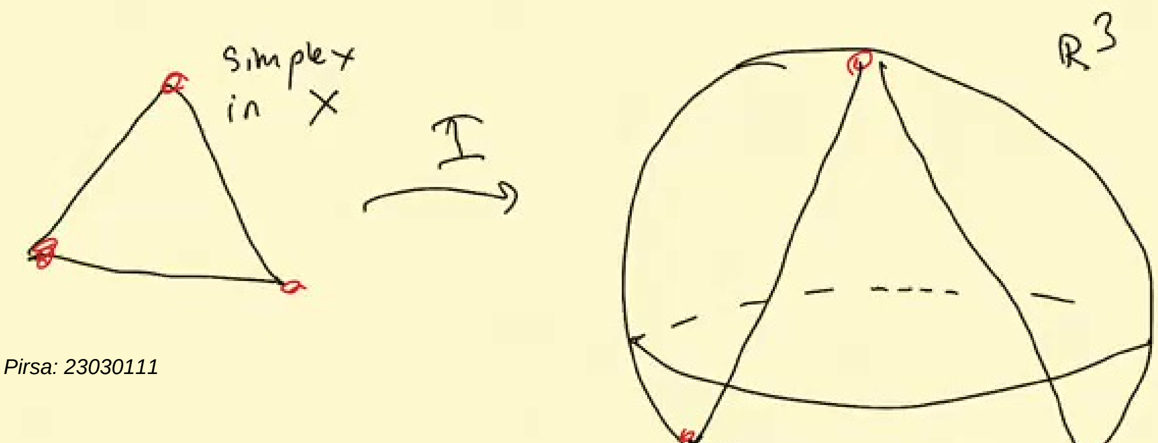
Use method of Gromov-Guth 2011 to get map with small pre-images:

Map each vertex of X to a random point in the boundary of a ball of volume V^3.

Extend I linearly to simplices



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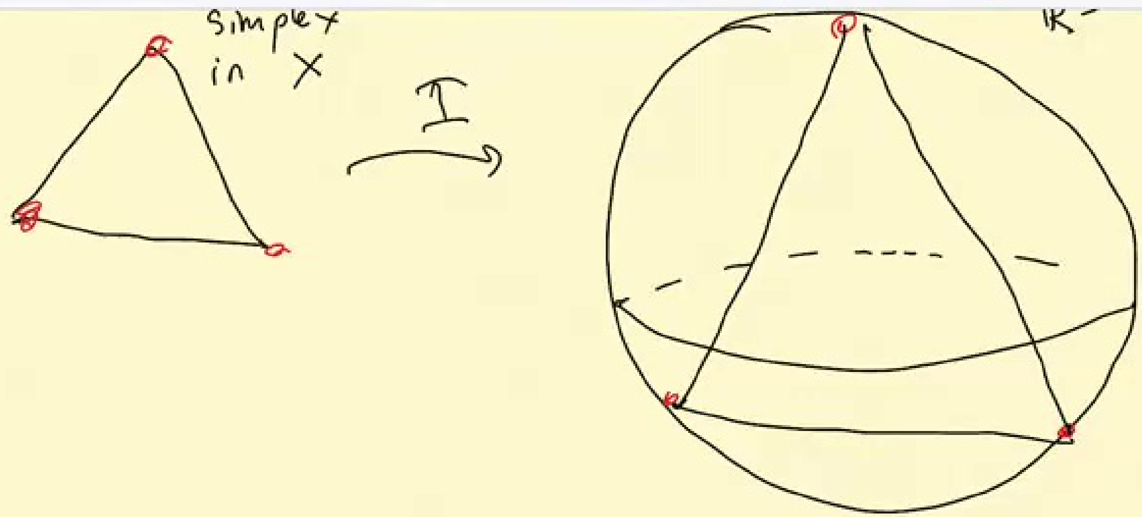
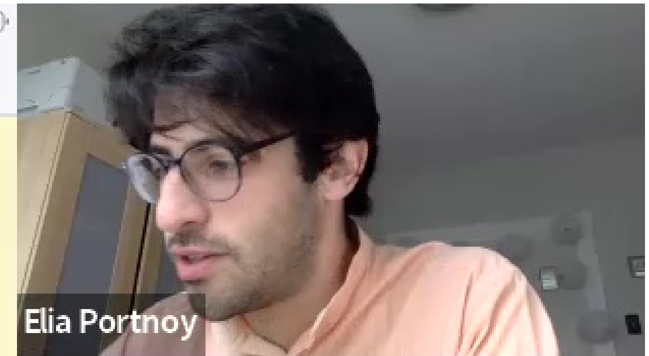


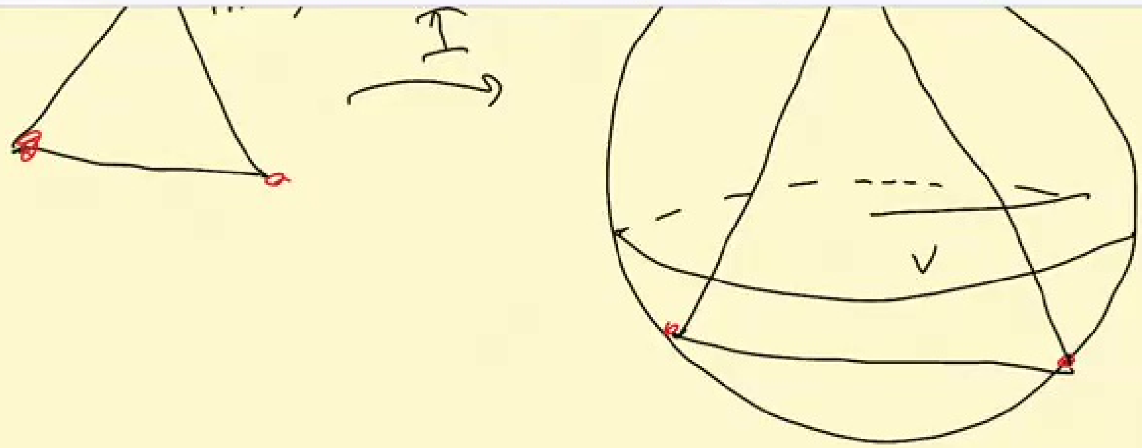


$$\mathrm{Prob}\left[\,\Delta \text{ intersects } I^{-1}\big(\overset{\text{fixed}}{\text{unit ball}}\big)\right] \lesssim V^{-1}$$

$$\mathbb{E}\left(\#\{ I^{-1}(\text{fixed unit ball}) \}\right) \leq C$$

$$\mathrm{Prob}\left[\#\{ I^{-1}\big(\overset{\text{fixed}}{\text{unit ball}}\big) \} \geq C \log V \right] \leq e^{-C \log V}$$

Chernoff Bound



$$\mathrm{Prob}\left[\, \Delta \text{ intersects } I^{-1}\left(\overset{\text{fixed}}{\text{unit ball}}\right)\right] \lesssim V^{-1}$$

$$\mathbb{E}\left(\#\left\{ I^{-1}(\text{fixed unit ball}) \right\}\right) \leq C$$

$$\mathrm{Prob}\left[\#\left\{ I^{-1}\left(\overset{\text{fixed}}{\text{unit ball}}\right)\right\} \geq C \log V \right] \leq e^{-C \log V}$$

Chernoff Bound

$$\mathrm{Prob}\left[\#\left\{ I^{-1}(\text{any unit ball}) \right\} \geq c \log V \right] < 1$$

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$$\text{Prob}\left[\,\Delta \text{ intersects } I^{-1}\big(\overset{\text{fixed}}{\text{unit ball}}\big)\right] \lesssim V^{-1}$$

$$\mathbb{E}\left(\#\{I^{-1}(\text{fixed unit ball})\}\right) \le C$$

$$\text{Prob}\left[\#\{I^{-1}\big(\overset{\text{fixed}}{\text{unit ball}}\big)\} \ge C\log V\right] \le e^{-C\log V} \approx V^{-C}$$

↑ Chernoff Bound

$$\text{Prob}\left[\#\{I^{-1}(\text{any unit ball})\} \ge c\log V\right] < 1$$

. . .

where poly log V comes from.

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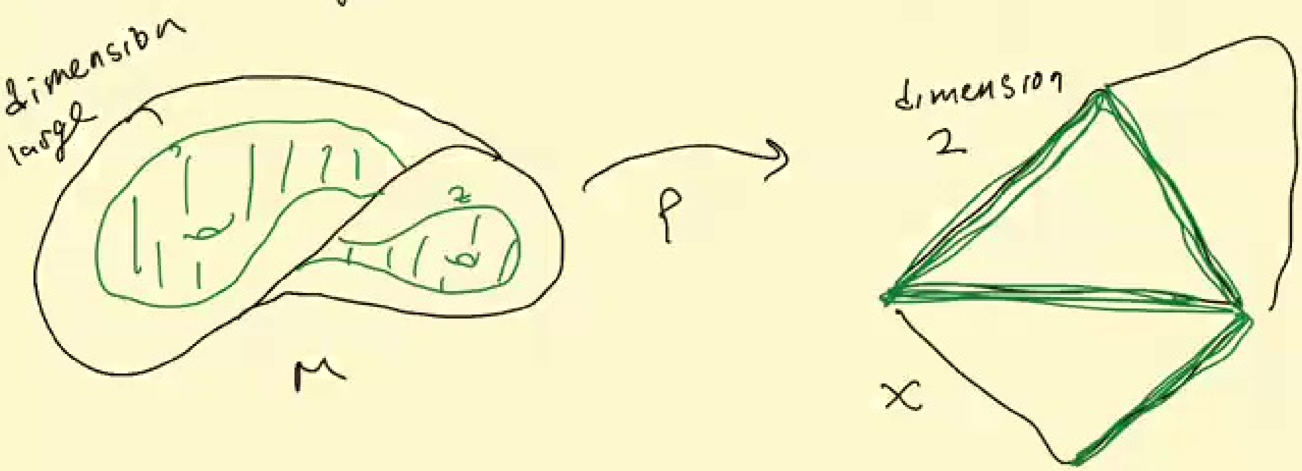
comes from.

What happens to systole of M after we subdivide?

Let $Z \subset M$ be a k-cycle representing $sys_k(M)$. $\mathrm{Vol}(Z) \approx V$



Most of Z gets mapped to most of the edges in X

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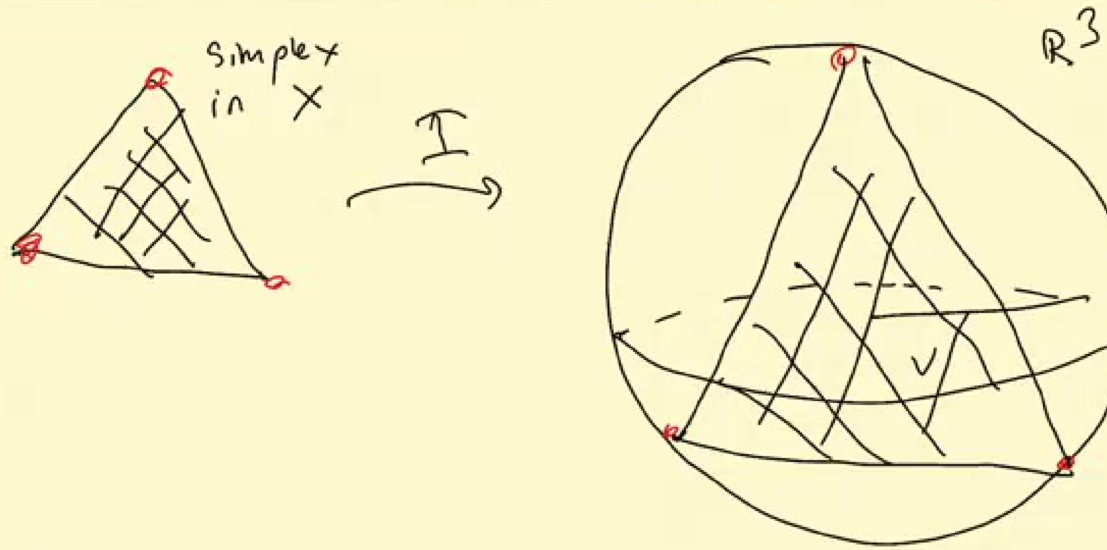

$$\text{Prob}\left[\,\Delta \text{ intersects } I^{-1}\overset{\text{fixed}}{(\text{unit ball})}\right] \lesssim V^{-1}$$

$$\mathbb{E}\left(\#\{I^{-1}(\text{fixed unit ball})\}\right) \leq C$$

$$\text{Prob}\left[\#\{I^{-1}\overset{\text{fixed}}{(\text{unit ball})}\} \geq C \log V\right] \leq \underline{\underline{e^{-C\log V}}}$$

$$\approx \underline{V^{-C}}$$

↑ Chernoff Bound

"X is related to the combinatorial structure you use to create the code e.g 2-sided Caley Complex"

Rough picture of M

Basis element of $C_0 \rightsquigarrow$ red piece

Basis element of $C_1 \rightsquigarrow$ green piece

Basis element of $C_2 \rightsquigarrow$ blue piece

Glued together using data of H_2, H_1

Basis element of $C_2 \rightsquigarrow$ blue piece

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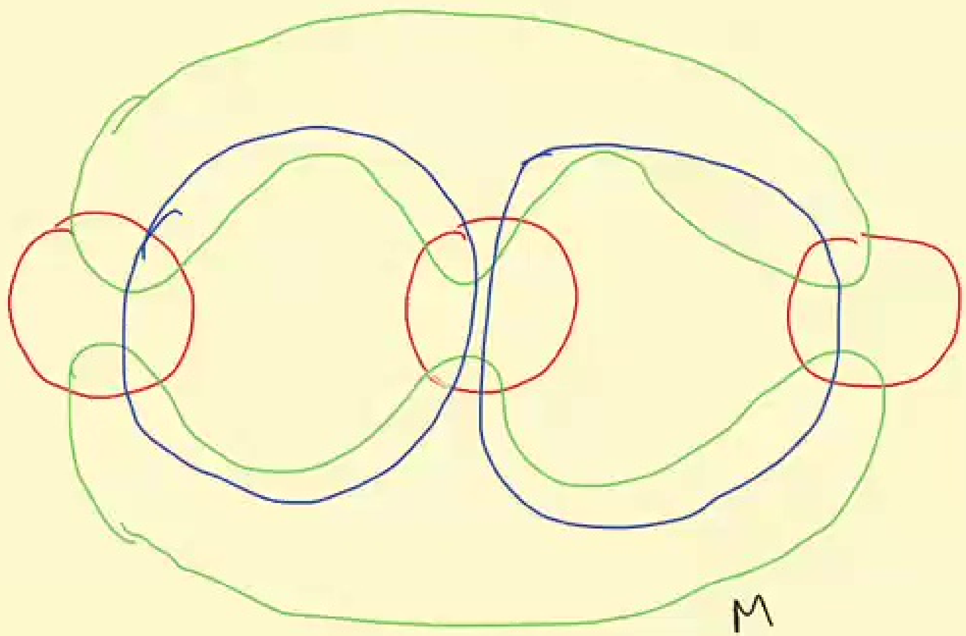

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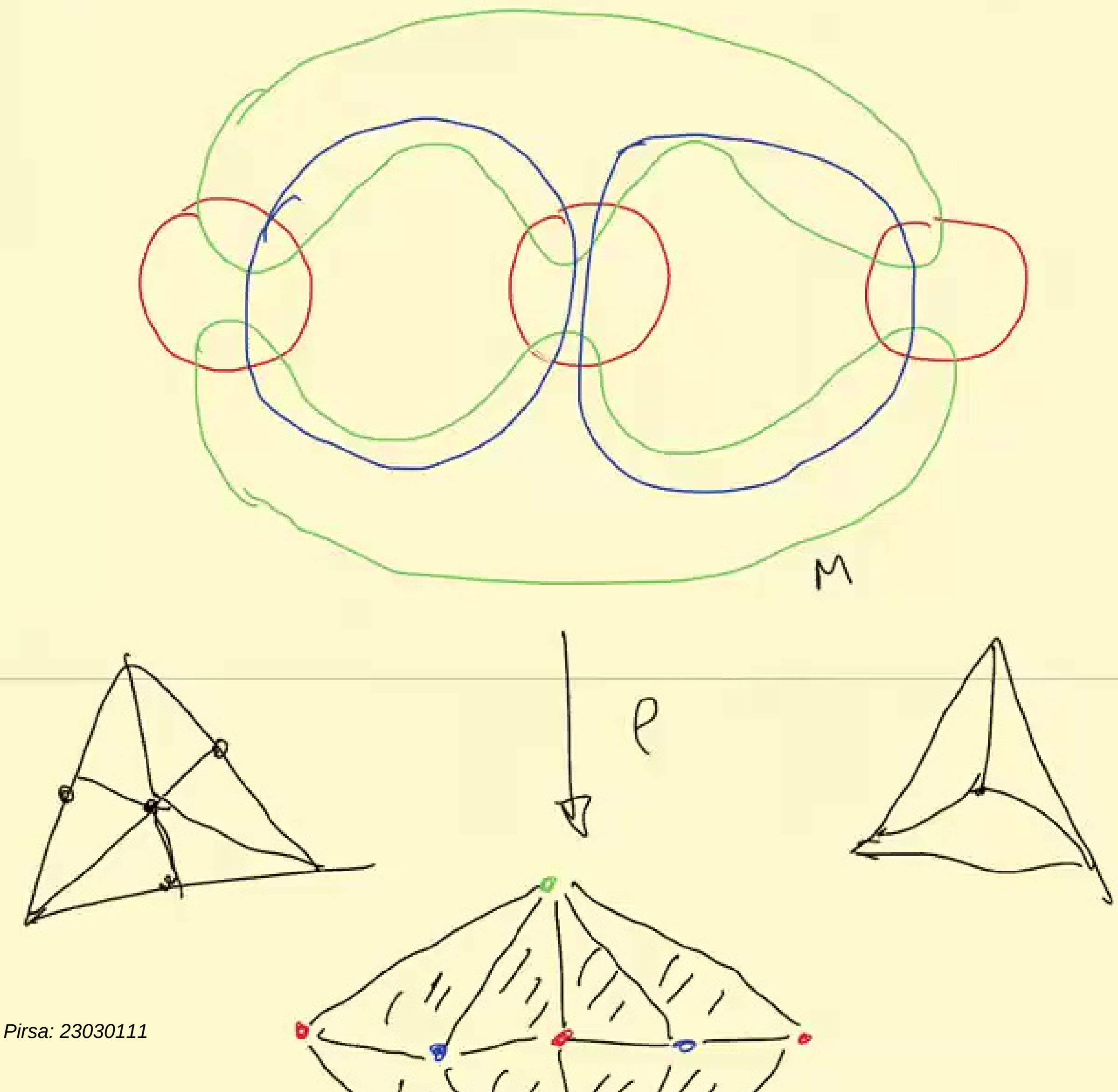

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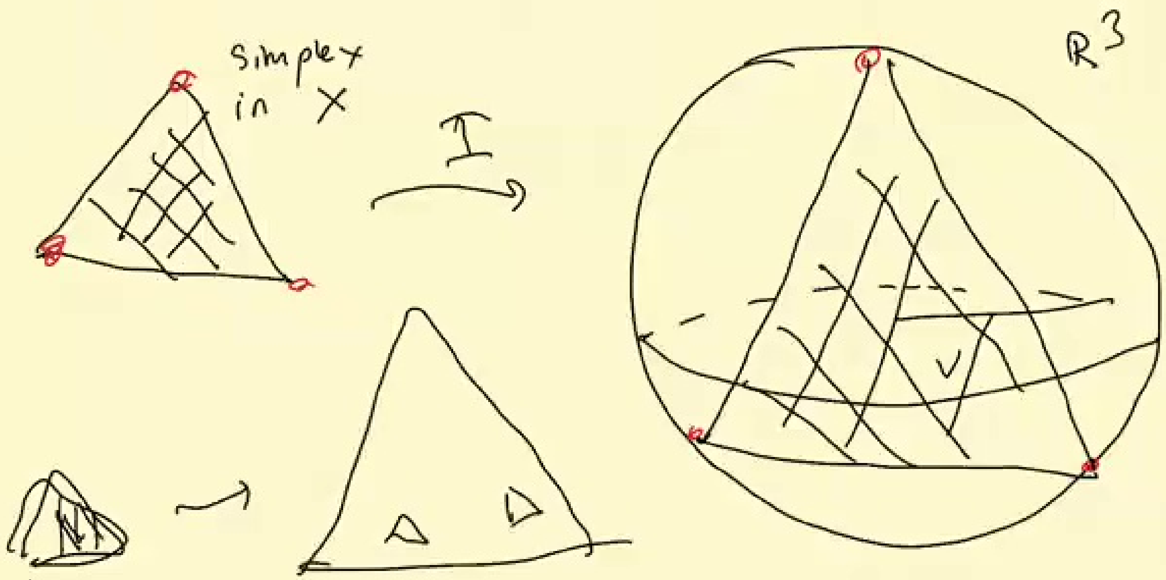
Map each vertex of X to a random point in the boundary of a ball of volume V^3.

Extend I linearly to simplices



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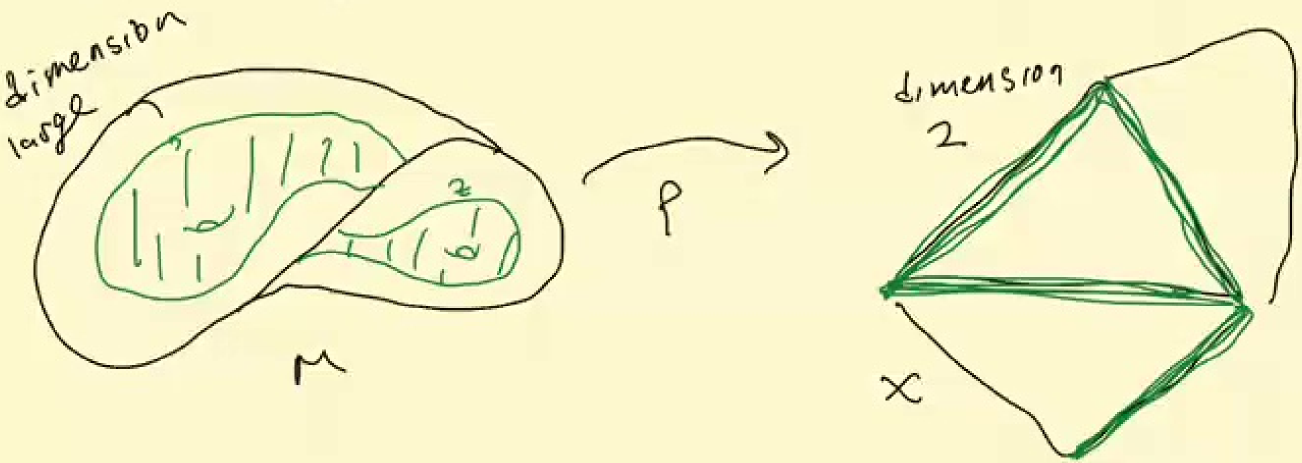
Let $Z \subset M$ be a k-cycle representing $\mathrm{sys}_k(M)$. $\mathrm{Vol}(Z) \approx V$



Most of Z gets mapped to most of the edges in X

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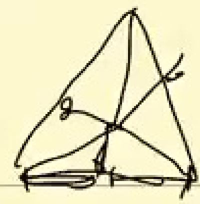
$$\mathrm{Prob}\left[\#\{ I^{-1}(\text{any unit ball}) \} \gtrsim c \log V \right] < 1$$

$\cdots$



where poly log V comes from.

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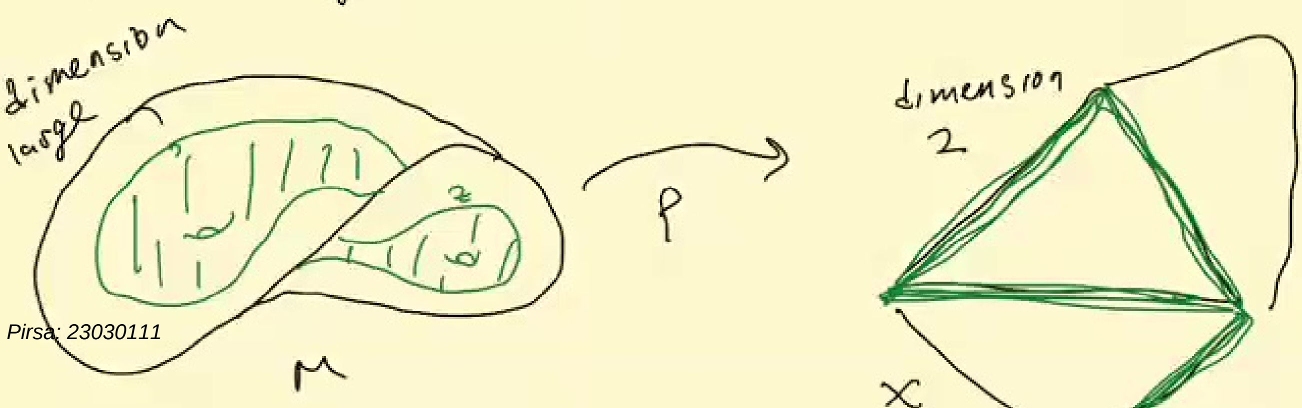

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after we subdivide?

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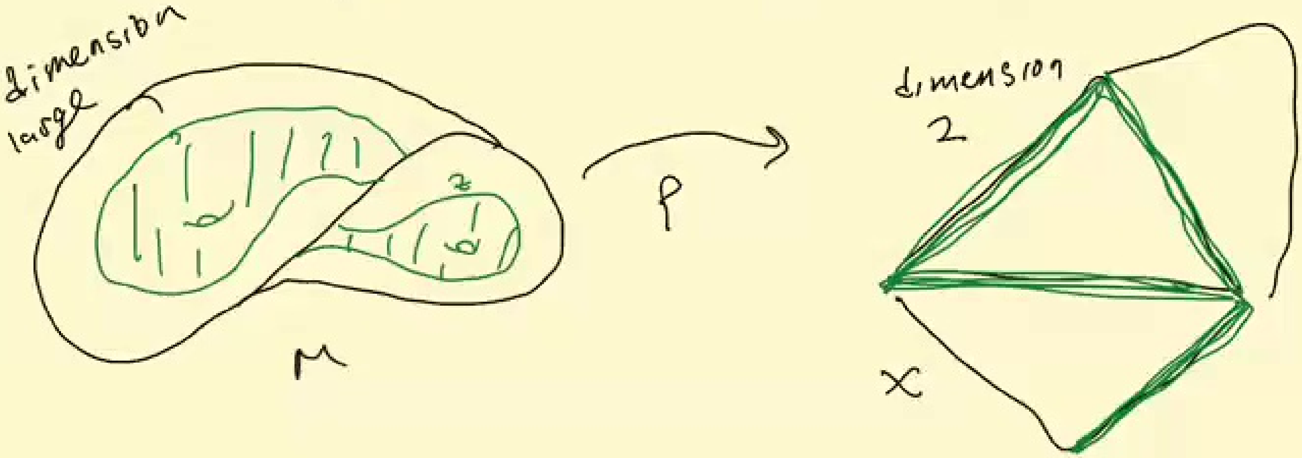

Most of Z gets mapped to most of the edges in X.

We can think about the

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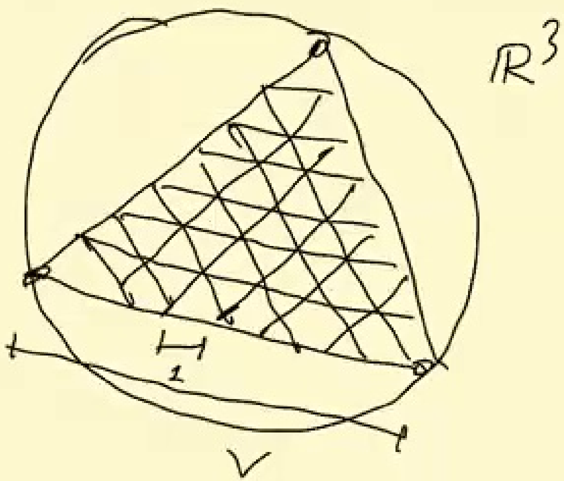
"pull-back" of a subdivison of X



To get $\tilde{X}$:

Subdivide each triangle of X into V^2 pieces

Subdivide each edge of X

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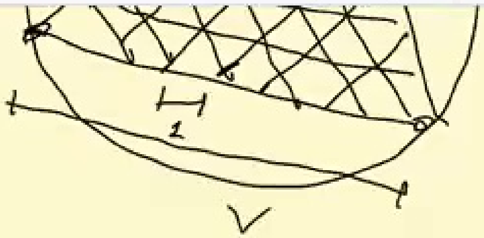

To get $\tilde{X}$:

Subdivide each triangle of X

into V^2 pieces

Subdivide each edge of X

into V pieces

$$\mathrm{Vol}(\tilde{M}) \approx \mathrm{Vol}(\tilde{X}) \approx V^3$$

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into V^2 pieces

Subdivide each edge of X

into V pieces

$$\mathrm{Vol}(\tilde{M}) \approx \mathrm{Vol}(\tilde{X}) \approx V^3$$

$$\mathrm{Sys}_{\kappa}(\tilde{M}) \approx \#\text{ edges in } \tilde{X} \approx V^2$$

So after $\tilde{M} \rightsquigarrow \mathcal{C}$

$$\mathrm{size}(\mathcal{C}) \approx V^3$$

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$$\dim(\mathcal{C}) \approx V^{1/3}$$

Conjecture: For any $d \geq 3$, we have a local* code $\mathcal{C}$ s.t

$$\text{size}(\mathcal{C}) = V\,\text{polylog}(V)$$

$$\text{dist}(\mathcal{C}) \approx V^{1/2}$$

$$\dim(\mathcal{C}) \approx 2^{d}$$

and energy barrier $\approx V^{\left(\frac{1}{2}-\frac{1}{d}\right)}$

$$\text{size}(\mathcal{C}) = V \,\text{polylog}(V)$$

$$\text{dist}(\mathcal{C}) \approx V^{2/3}$$

$$\dim(\mathcal{C}) \approx V^{1/3}$$

---

subdivided (d,d)-toric code

<u>Conjecture:</u> For any $d \geq 3$, we have a local* code $\mathcal{C}$ s.t

$$\text{Size}(\mathcal{C}) = V \,\text{polylog}(V)$$

$$\text{dist}(\mathcal{C}) \approx V^{1/2}$$