Title: Elliptic Trace Map on Chiral Algebras

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Abstract: Trace map on deformation quantized algebra leads to the algebraic index theorem. We investigate a two-dimensional chiral analogue of the algebraic index theorem via the theory of chiral algebras developed by Beilinson and Drinfeld. We construct a trace map on the elliptic chiral homology of the free beta gamma-bc system using the BV quantization framework. As an example, we compute the trace evaluated on the unit constant chiral chain and obtain the formal Witten genus in the Lie algebra cohomology. This talk is based on joint work with Si Li.

Zoom link: https://pitp.zoom.us/j/91920988875?pwd=dUE1WDI5Q21KWTUxTGNsM21nblh5Zz09

Elliptic trace map on chiral algebras

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Perimeter Institute for Theoretical Physics , March, 2023
Based on arXiv:2112.14572, with Si Li.

Talking: Perimeter-B

Overview

- Motivation/ Introduction
- VOA and Chiral algebra
- Elliptic chiral homology
- Batalin-Vilkovisky (BV) quantization
- Chiral homology and quantum master equation.
- Witten genus.

Talking: Perimeter-B

Motivation

Let (M, ω) be a symplectic manifold. $C^\infty(M)$ is a Poisson algebra

$$\{f, g\} = \sum_{i,j} \omega^{ij} (\partial_i f)(\partial_j g).$$

A deformation quantization is defined to be an $\hbar$-linear associative product $\star$ (usually called the star product) on $C^\infty(M)[\![\hbar]\!]$ satisfying

- Locality: $\star$ is represented by bi-differential operators
- Classical limit: $\forall f, g \in C^\infty(M)$

$$f \star g = fg + O(\hbar)$$

- 1st-order noncommutativity: $\forall f, g \in C^\infty(M)$

$$\frac{1}{\hbar}(f \star g - g \star f) = \{f, g\} + O(\hbar).$$

Talking:

Motivation

Given a deformation quantization $\mathcal{A}_\hbar(M) = (C^\infty(M)[\![\hbar]\!], \star_\hbar)$, there exists a unique linear map

$$\mathrm{Tr} : \mathcal{A}_\hbar(M) \to \mathbb{R}(\!(\hbar)\!)$$

satisfying

- Trace property: $\mathrm{Tr}(f \star g) = \mathrm{Tr}(g \star f)$
- Normalization:

$$\mathrm{Tr}(f) = \frac{1}{\hbar^n} \int_M \frac{\omega^n}{n!}(f + O(\hbar)) \quad n = \dim M/2.$$

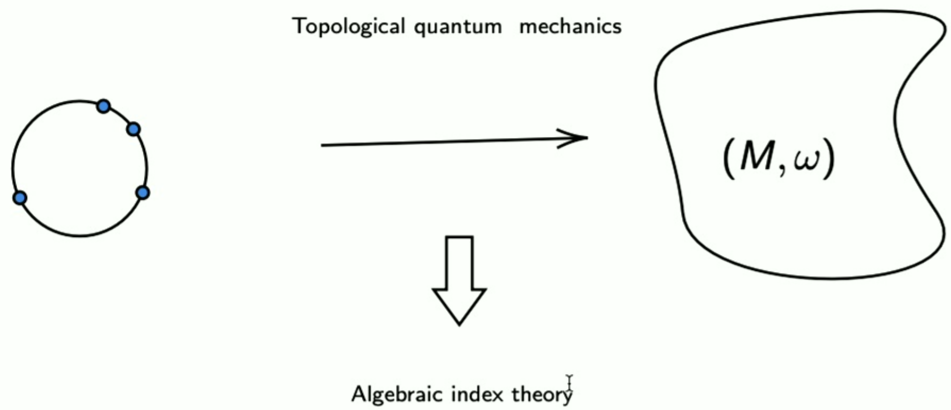
Then

$$\mathrm{Index} = \mathrm{Tr}(1) = \int_M e^{\omega/\hbar} \hat{A}(M).$$

This is the algebraic index theorem which was first formulated by Fedosov and Nest-Tsygan as the algebraic analogue of Atiyah-Singer index theorem.

Motivation

[Grady-Li-Li, JDG 2017][G-Li-Xu, CMP 2020] establish a rigorous connection between the effective BV quantization theory for topological quantum mechanics and the algebraic index theorem.



$$Tr : HH_{\bullet}(\mathcal{A}_{\hbar}(M)) \to \mathbb{R}((\hbar))$$

$$Tr(1) = \int_M e^{\omega/\hbar} \hat{A}(M)$$ (**Fedosov** and **Nest-Tsygan**)

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Motivation

Here we recall the definition of the Hochschild homology. Let $\mathcal{A}$ be an associative algebra over $\mathbb{K}$. Consider the $\mathbb{K}-$module

$$C_n(\mathcal{A}) := \mathcal{A} \otimes \mathcal{A}^{\otimes n}.$$

The Hochschild boundary is the $\mathbb{K}$-linear map $b : \mathcal{A} \otimes \mathcal{A}^{\otimes n} \to \mathcal{A} \otimes \mathcal{A}^{\otimes n-1}$ given by the formula

$$b(a_0, a_1, \dots, a_n) := (a_0 a_1, a_2, \dots, a_n)$$

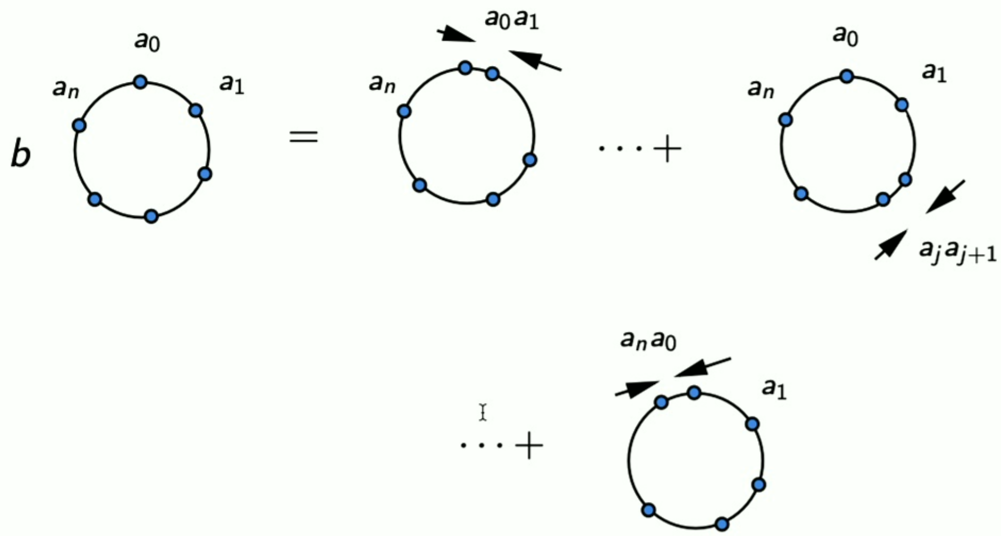
$$+ \sum_{i=1}^{n-1} (-1)^i (a_0, a_1, \dots, a_i a_{i+1}, \dots, a_n)$$

$$+ (-1)^n (a_n a_0, a_1, \dots, a_{n-1}).$$

Talking: Perimeter-B

Motivation

The Hochschild boundary can be represented by the following diagram:



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Talking: Perimeter-B

Motivation

We take $\mathcal{A} = (\mathcal{A}_\hbar(M), \star_\hbar)$ and $\mathbb{K} = \mathbb{R}((\hbar))$.
The trace map $\mathrm{Tr} : HH_\bullet(\mathcal{A}) \to \mathbb{R}((\hbar))$ can be viewed as the partition function of the 1d TQM system.

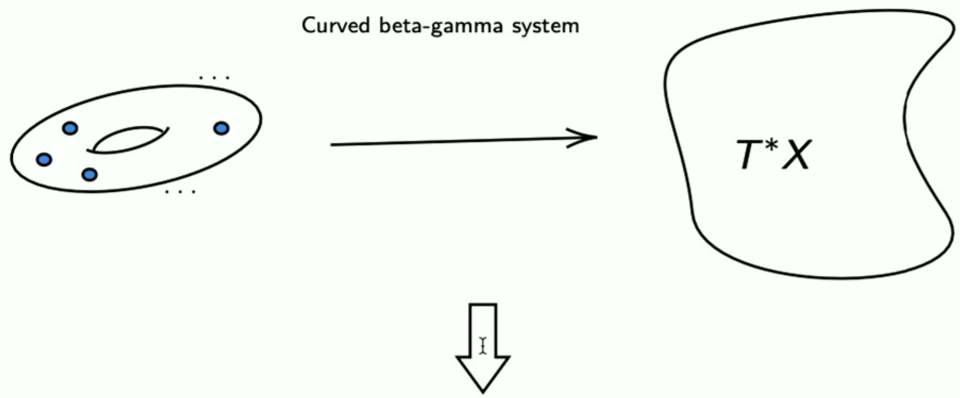
In this talk, we will focus on the trace map in 2d CFT where the Hochschild homology is replaced with the chiral homology.

Talking: Perimeter-B

Motivation

Witten's "Index Theorem" on loop space

Replace S^1 with an elliptic curve E_τ. (**Witten**: index of Dirac operators on loop space.)

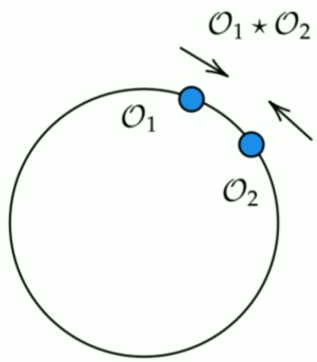


2d Chiral analogue of algebraic index

1d TQM Vs 2d Chiral CFT

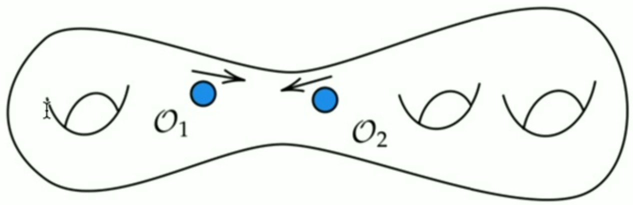
1d TQM	2d Chiral CFT
S^1	Σ
Associative algebra	Vertex operator algebra
Hochschild homology	Chiral homology

Associative product



Operator product expansion

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_{n\geq 0} \frac{\mathcal{O}_{1(n)}\mathcal{O}_2(w)}{(z-w)^{n+1}} + \text{reg}$$



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Vertex operator algebra

A vertex algebra is a vector space V with structures

- state-field correspondence

$$V \to \mathrm{End}(V)[[z, z^{-1}]]$$

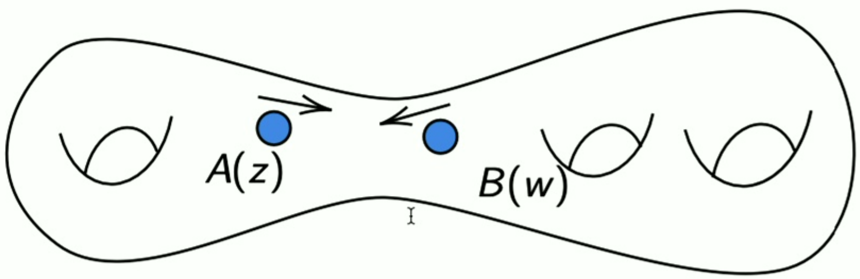
$$a \to Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$

- vacuum: $|0\rangle \in V \to Y(0|\rangle, z) = 1 \in \mathrm{End}(V)[[z, z^{-1}]]$.
- translation covariance: $[L_{-1}, Y(a, z)] = \partial_z Y(a, z)$, $L_{-1} \in \mathrm{End}(V)$.
- locality: $(z - w)^N Y(a, z) Y(b, w) = (-1)^{p(a)p(b)} (z - w)^N Y(b, w) Y(a, z)$ for $N \gg 0$.

Operator product expansion (OPE)

We can define OPE's of fields by

$$A(z)B(w) = \sum_{n\in\mathbb{Z}} \frac{(A_{(n)} \cdot B)(w)}{(z-w)^{n+1}}$$

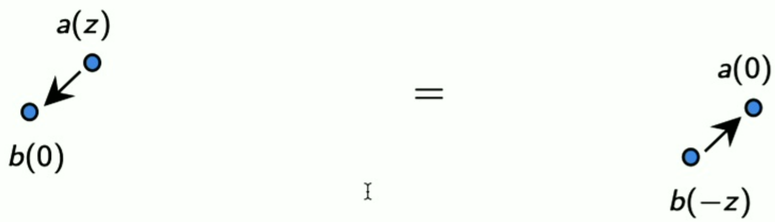


Talking:

Antisymmetry

We have

$$a_{(n)}b = -(-1)^{p(a)p(b)} \sum_{j=0}^{\infty} (-1)^{j+n} \frac{(L_{-1}^{j})}{j!} (b_{(n+j)}a).$$



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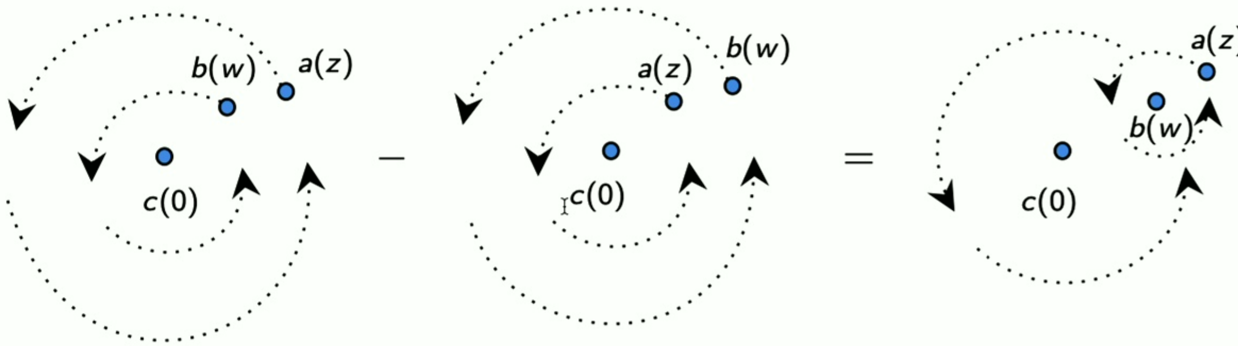
The Borcherds identity

We have

$$\sum_{j\geq 0}\binom{m}{j}(a_{(n_j)}b)_{(m+n-j)}c$$

$$=\sum_{j\geq 0}(-1)^j\binom{n}{j}\left(a_{(m+n-j)}b_{(k+j)}c-(-1)^{n+p(a)p(b)}b_{(n+k-j)}a_{(m+j)}c\right).$$



Talking:

VOA examples: $\beta\gamma - bc$ system.

Let $\mathbf{h} = \mathbf{h}_{\bar{0}} \oplus \mathbf{h}_{\bar{1}}$ be a (super) vector space equipped with an even symplectic pairing

$$\langle -, - \rangle : \wedge^2 \mathbf{h} \to \mathbb{C}$$

We obtain a vertex algebra structure on the free differential ring

$$\mathcal{V}^{\beta\gamma - bc}(\mathbf{h}) \cong \mathbb{C}[[\partial^k a^i]][[\hbar]], \quad a^i \text{ is a basis of } \mathbf{h}, k \geq 0$$

The OPE's are generated by

$$a(z)b(w) \sim \hbar \frac{\langle a, b \rangle}{(z - w)}, \quad \forall a, b \in \mathbf{h}.$$

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VOA examples: $\beta\gamma$ system.

When $\mathbf{h} = \mathbf{h}_{\bar{0}} = \mathbb{C}^N \oplus \mathbb{C}^N$ is purely bosonic (there is no bc), and the symplectic pairing is given by

$$\langle (u_1, v_1), (u_2, v_2) \rangle = u_1 v_2 - v_1 u_2.$$

We denote it by $\mathcal{V}_N^{\beta\gamma}$. $\mathcal{V}_N^{\beta\gamma}$ can be viewed as a chiral version of the Weyl algebra on $T^*\mathbb{C}^N$.

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Talking: Perimeter-B

Chiral algebra

Let X be a smooth complex curve.

Definition

(Beilinson and Drinfeld) Let $\mathcal{A}$ be a $\mathcal{D}_X$-module. A chiral algebra structure on $\mathcal{A}$ is a $\mathcal{D}_{X^2}$-module map:

$$\mu : (\mathcal{A} \boxtimes \mathcal{A})(*\Delta_{\{1,2\}}) \to \Delta_*(\mathcal{A}),$$

that satisfies the following two conditions:

- **Antisymmetry:**
  If $f(z_1, z_2) \cdot a \boxtimes b$ is a local section of $\mathcal{A} \boxtimes \mathcal{A}(*\Delta_{\{1,2\}})$, then

  $$\mu(f(z_1, z_2) \cdot a \boxtimes b) = -(-1)^{p(a)p(b)} \sigma_{1,2} \mu(f(z_2, z_1) \cdot b \boxtimes a),$$

  where $\sigma_{1,2}$ acts on $\Delta_* \mathcal{A} = \mathcal{A} \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to X^2} = \mathcal{A} \otimes_{\mathcal{D}_X} (\mathcal{O}_X \otimes_{\mathcal{O}_{X^2}} \mathcal{D}_{X^2})$ by permutating two factors of X^2.

Talking:

Chiral algebra

- Jacobi identity:
  If $a \boxtimes b \boxtimes c \cdot f(z_1, z_2, z_3)$ is a local section of $\mathcal{A}^{\boxtimes 3}(*\Delta_{\{1,2,3\}})$, then the element

$$\mu(\mu(f(z_1, z_2, z_3) \cdot a \boxtimes b) \boxtimes c)$$
$$+ (-1)^{p(a)\cdot(p(b)+p(c))}\sigma_{1,2,3}\mu(\mu(f(z_2, z_3, z_1) \cdot b \boxtimes c) \boxtimes a)$$
$$+ (-1)^{p(c)\cdot(p(a)+p(b))}\sigma_{1,2,3}^{-1}\mu(\mu(f(z_3, z_1, z_2) \cdot c \boxtimes a) \boxtimes b) = 0,$$

  here $\sigma_{1,2,3}$ denotes the cyclic permutation action on $\Delta_* \mathcal{A} = \mathcal{A} \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to X^3}$.

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From VOA to chiral algebra

From a quasi-conformal vertex algebra one can get a chiral algebra on a smooth complex curve X [E.Frenkel and D.Ben-Zvi].

The set of pairs (x, t_x) consisting of a point $x \in X$ and a formal local coordinate t_x at x is an $\mathrm{Aut}_{\mathcal{O}}$-torsor over X. Applying the associated bundle construction to the $\mathrm{Aut}_{\mathcal{O}}$-module V yields vector bundle

$$\mathcal{V} = \mathrm{Aut}_X \times_{\mathrm{Aut}_{\mathcal{O}}} V, \quad \mathcal{O} \cong \mathbb{C}[[t]]$$

over X.

Talking: Perimeter-B

From VOA to chiral algebra

The bundle $\mathcal{V}$ carries a connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_X} \omega_X$, thus a left $\mathcal{D}_X$-module structure. The connection is defined relative to a choice of local coordinate by

$$\nabla_{\partial_z} = \partial_z + L_{-1},$$

but is independent of this choice. The corresponding right $\mathcal{D}_X$-module is $\mathcal{V}^r = \mathcal{V} \otimes_{\mathcal{O}_X} \omega_X$.

VOA: antisymmetry, Bocherds identity		Chiral algebra: antisymmetry, Jacobi identity
V	$\rightsquigarrow$	$\mathcal{V}^r$

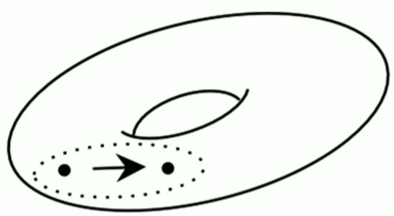
Chiral algebra: Distribution supported on the small diagonal

Locally

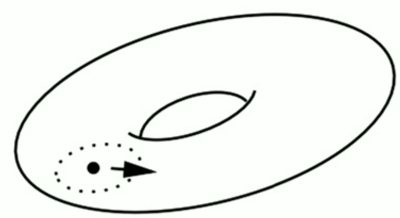
$$\mu : \mathcal{V}^r \boxtimes \mathcal{V}^r(*\Delta_{\{1,2\}}) \to \Delta_* \mathcal{V}^r,$$

$$\mu(\frac{f(z_1, z_2)}{(z_1 - z_2)^k} v_1 dz_1 \boxtimes v_2 dz_2)$$

$$= \sum_n \sum_{l \geq 0} \frac{1}{(n+k-l)!} (\partial_{z_1}^{n+k-l} f(z_1, z_2))|_{z_1=z_2=w} v_{1(n)} v_2 dw \otimes_{\mathcal{D}_{X_{12}}} \frac{1}{l!} \partial_{z_1}^l,$$



$\mathcal{A} \boxtimes \mathcal{A}(*\Delta)$



$\Delta_* \mathcal{A}$

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Elliptic chiral homology

- In [Zhu, JAMS 1994], Zhu study the space of genus 1 conformal block (the 0-th elliptic chiral homology) and establish the modular invariance for certain class of VOA.
- Beilinson and Drinfeld define the chiral homology for general algebraic curves using the Chevalley-Cousin complex.
- Recently, [Ekeren-Heluani,2018,2021] write down the chiral complexes computing the degree 0 and the degree 1 chiral homology explicitly.
- In [Rozenblyum,2021]: the chiral homology of the integrable quotient of the affine Kac-Moody chiral algebra at positive level is concentrated in degree 0.

We now review the construction of Beilinson and Drinfeld.

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Chiral homology: $\mathcal{D}$-modules on $X^{\mathcal{S}}$

Denote by $\mathcal{S}$ the category of finite non-empty sets and surjective maps.

Definition

A right $\mathcal{D}$-module M on $X^{\mathcal{S}}$ is a rule that assigns to $I \in \mathcal{S}$ a right $\mathcal{D}_{X^I}$-module M_{X^I} and to $\pi : J \twoheadrightarrow I$ a morphism of $\mathcal{D}-$modules

$$\theta^{(\pi)} = \theta_M^{(\pi)} : \Delta_*^{(\pi)} M_{X^I} \to M_{X^J}.$$

Compatibility condition: for $K \xrightarrow{\pi_2} J \xrightarrow{\pi_1} I$

$$\begin{array}{ccc} \Delta_*^{(\pi_2)}\Delta_*^{(\pi_1)} M_{X^I} & \xrightarrow{\Delta_*^{(\pi_2)}(\theta^{(\pi_1)})} & \Delta_*^{(\pi_2)} M_{X^J} \\ \| & & \downarrow \theta^{\pi_2} \\ \Delta_*^{(\pi_1\pi_2)} M_{X^I} & \xrightarrow{\theta^{(\pi_1\pi_2)}} & M_{X^K} \end{array}$$

And $\theta^{(\mathrm{id})} = \mathrm{id}$.

Talking:

Chiral homology: embedding $\Delta_*^{(S)}$

Let $\mathcal{M}(X)$ be the category of $\mathcal{D}_X$-modules. There is an exact fully faithful embedding

$$\boxed{\Delta_*^{(\mathcal{S})} : \mathcal{M}(X) \hookrightarrow \mathcal{M}(X^{\mathcal{S}})}$$

defined by

$$(\Delta_*^{(\mathcal{S})} M)_{X^I} := \Delta_*^{(I)} M, \quad \theta^{(\pi)} = \mathrm{id}_{\Delta_*^{(J)} M}.$$

Here

$$\Delta^{(I)} : X \to X^I$$

is the diagonal embedding.

Talking: Perimeter-B

Chiral homology: tensor structure $\otimes^{\mathrm{ch}}$ on $\mathcal{M}(X^{\mathcal{S}})$

The category $\mathcal{M}(X^{\mathcal{S}})$ carries a tensor product $\otimes^{\mathrm{ch}}$ defined as follows. Let $M_i, i \in I$, be a finite non-empty family of objects of $\mathcal{M}(X^{\mathcal{S}})$. One has

$$(\otimes_I^{\mathrm{ch}} M_i)_{X^J} := \bigoplus_{J \twoheadrightarrow I} j_*^{[J/I]} j^{[J/I]*} \boxtimes_I (M_i)_{X^{J_i}},$$

where the arrows $\theta^{(\pi)}$ are obvious ones. Here $j^{[J/I]}$ is defined as follows.

Definition

For $\pi : J \twoheadrightarrow I$ let $j^{[J/I]} : U^{J/I} \hookrightarrow X^J$ be the complement to all the diagonals that are transversal to $\Delta^{(J/I)} : X^I \hookrightarrow X^J$. Therefore one has

$$U^{[J/I]} = \{(x_j) \in X^J : x_{j_1} \neq x_{j_2} \quad \text{if} \quad \pi(j_1) \neq \pi(j_2)\}.$$

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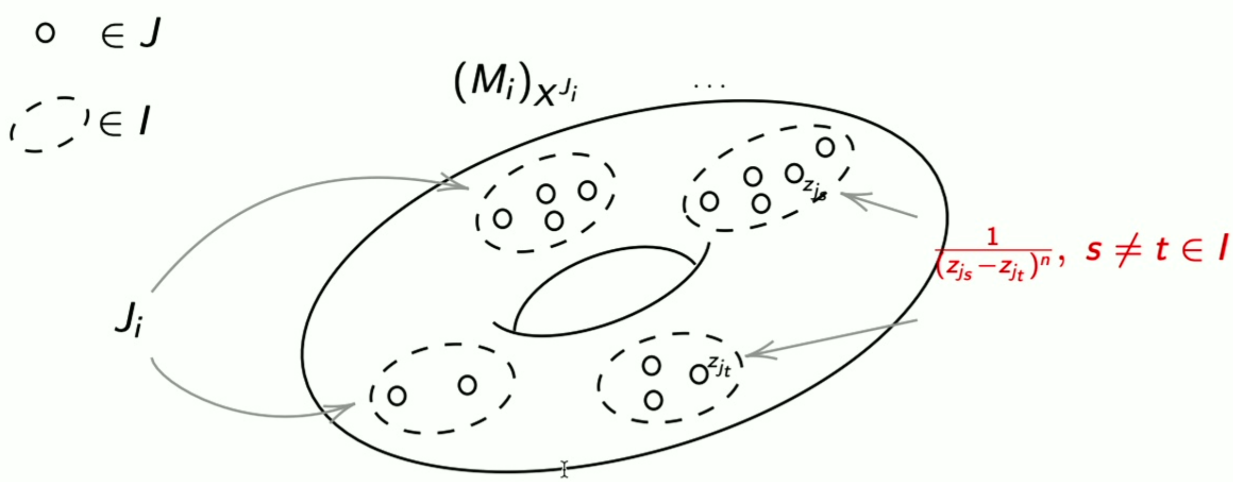
Chiral homology: tensor structure $\otimes^{\mathrm{ch}}$ on $\mathcal{M}(X^{\mathcal{S}})$

Here is the picture for $j_*^{[J/I]} j^{[J/I]*} \underset{I}{\boxtimes} (M_i)_{X^{J_i}}$



$j_*^{[J/I]} j^{[J/I]*} \underset{I}{\boxtimes} (M_i)_{X^{J_i}}$

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Talking: Perimeter-B

Chiral homology: chiral algebra as Lie object

Recall that a chiral algebra $\mathcal{A}$ is a right $\mathcal{D}_X$-module satisfying antisymmetry and Jacobi identity. Then $\Delta_*^{(\mathcal{S})}\mathcal{A}$ is a Lie algebra object in the tensor category $(\mathcal{M}(X^{\mathcal{S}}), \otimes^{\mathrm{ch}})$.

We consider the reduced Chevalley-Eilenberg complex

$$(C(\mathcal{A}), d_{\mathrm{CE}}) = (\bigoplus_{\bullet>0} \mathrm{Sym}^{\bullet}_{\otimes^{\mathrm{ch}}}(\Delta_*^{(\mathcal{S})}\mathcal{A}[1]), d_{\mathrm{CE}}),$$

which is a complex in $\mathcal{M}(X^{\mathcal{S}})$.

The chiral homology (complex) $C^{\mathrm{ch}}(X, \mathcal{A})$ is defined by $R\Gamma_{\mathrm{DR}}(X^{\mathcal{S}}, C(\mathcal{A}))$, where

Dolbeault Resolution ↘ $R\Gamma_{DR}\,(X^{\mathcal{S}}, C(\mathcal{A}))$ ↙ Chevalley-Eilenberg

↑ Spencer Resolution

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Chiral homology of the unit chiral algebra ω_X

Beilinson and Drinfeld prove that the trace map

$$C^{\mathrm{ch}}(X, \omega_X) \xrightarrow{\mathrm{tr}_{\mathrm{BD}}} \mathbb{C}$$

is a quasi-isomorphism. It reflects the fact that the Ran space of X is weakly contractible.

Consider the natural map

$$\iota : \Omega^{0,n}(X^n, \omega_{X^n}[n](*\Delta)) \to C^{\mathrm{ch}}(X, \omega_X).$$

We prove that the composition $\mathrm{tr}_{\mathrm{BD}} \circ \iota$ coincides with the regularized integral [Li-Zhou, 2020]. In other words, the trace map $\mathrm{tr}_{\mathrm{BD}}$ can be viewed as a homological renormalization.

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Talking: Perimeter-B

BV formalism

Batalin-Vilkovisky (BV) formalism arises in physics as general method to quantize gauge theory. We briefly discuss

- BV-algebra: the algebraic structure in quantizing gauge theory.
- master equation: the output of BV quantization.
- renormalization: mathematical framework for performing BV quantization in (perturbative) quantum field theory.

Remark

There are many ways to do renormalization/ regularization in QFT such as : dimensional regularization, zeta regularization and etc. Costello build a mathematical framework of renormalization and effective theory using heat kernels. In our case (2d CFT), there is a canonical way to renormalize the integrals of singular forms by the trace map constructed by Beilinson and Drinfeld. This trace map turns out to be equivalent to the regularized integral studied by [Li-Zhou, 2020].

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Talking: Perimeter-B

BV formalism: BV algebra

A Batalin-Vilkovisky (BV) algebra is a pair $(O_{\mathrm{BV}}, \Delta)$ where

- O_{BV} is a $\mathbb{Z}$-graded commutative associative unital algebra.
- $\Delta_{\mathrm{BV}} : O_{\mathrm{BV}} \to O_{\mathrm{BV}}$ is a linear operator of degree 1 such that $\Delta_{\mathrm{BV}}^2 = 0$.
- Define $\{-,-\} : O_{\mathrm{BV}} \otimes O_{\mathrm{BV}} \to O_{\mathrm{BV}}$ by

$$\{a, b\} := \Delta_{\mathrm{BV}}(ab) - (\Delta_{\mathrm{BV}} a)b - (-1)^{|a|} a \Delta_{\mathrm{BV}} b, \ \ a, b \in O_{\mathrm{BV}}.$$

Then $\{-,-\}$ satisfies the following graded Leibniz rule

$$\{a, bc\} := \{a, b\}c + (-1)^{(|a|+1)|b|} b\{a, c\}, \quad a, b, c \in O_{\mathrm{BV}}.$$

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BV formalism: quantum master equation

Let $(O_{\mathrm{BV}}, \Delta_{\mathrm{BV}})$ be a BV algebra. Let $I = I_0 + I_1\hbar + \cdots \in O_{\mathrm{BV}}[[\hbar]]$.

Definition

I is said to satisfy quantum master equation (QME) if

$$\boxed{\hbar\Delta_{\mathrm{BV}} e^{I/\hbar} = 0.}$$

This is equivalent to

$$\hbar\Delta_{\mathrm{BV}} I + \frac{1}{2}\{I, I\} = 0.$$

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BV formalism: quantum master equation

In general, if $(C_\bullet, d)$ is a $\mathbb{C}[[\hbar]]$-chain complex, we say a $\mathbb{C}[[\hbar]]$-linear map

$$\langle - \rangle : C_\bullet \to O_{\mathrm{BV}}((\hbar))$$

satisfies QME if

$$\boxed{(d + \hbar\Delta_{\mathrm{BV}})\langle - \rangle = 0.}$$

Remark

If we take $(C_\bullet, d) = (\mathbb{C}[[\hbar]], 0)$, the map $I(-)$

$$I(-) : \mathbb{C}[[\hbar]] \to O_{\mathrm{BV}}((\hbar)), \; I(c) = c e^{I/\hbar}$$

satisfies QME if and only if I itself satisfies QME.

Talking: Perimeter-B

BV formalism

Roughly speaking, the effective BV quantization theory T contains

- "Factorization algebra" (in the sense of Costello-Gwilliam): Obs_T, a $\mathbb{C}[[\hbar]]$-module equipped with certain algebraic structure.
- "Factorization homology": $(C_\bullet(\mathrm{Obs}_T), d)$, a chain complex.
- A BV algebra $(O_T, \Delta_{\mathrm{BV}})$.
- A linear map, "the partition function map"

$$\mathbf{Tr}_T : C_\bullet(\mathrm{Obs}_T) \to O_T((\hbar))$$

satisfies QME , which means it is a chain map

$$(d + \hbar\Delta_{\mathrm{BV}})\mathbf{Tr}_T = 0.$$

Talking: Perimeter-B

Example: 1d TQM

In 1d TQM, we have

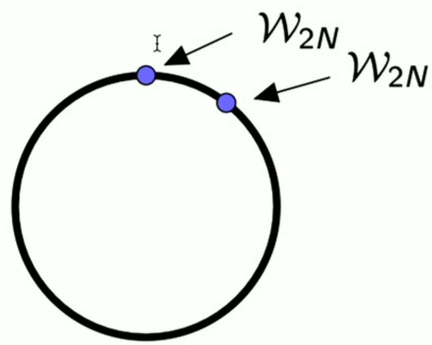
- "Factorization algebra" : $\mathrm{Obs}_{1d} = \mathcal{W}_{2N} = \mathcal{W}(\mathbb{R}^{2N}, \omega)$, a $\mathbb{C}[[\hbar]]$-module equipped with the Moyal product.
- "Factorization homology": $(C_\bullet(\mathrm{Obs}_{1d}), d) = (C_\bullet(\mathcal{W}_{2N}), b)$, the Hochschild chain complex.

More precisely, local observables on S^1 form the Weyl algebra

$$\mathcal{W}_{2n} = \left(\mathbb{C}[[p_i, q^i]][[\hbar]], \star\right)$$

where $\star$ is the Moyal-Weyl product

$$(f \star g)(p, q) := f(p, q) e^{\hbar\left(\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i} - \frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}\right)} g(p, q).$$



BV formalism: 1d TQM

- The BV algebra $(O_{1d}, \Delta_{\mathrm{BV}}) = (\mathbb{C}[[y^i, dy^i = \theta^i]], \sum_{i,j} \omega^{ij} \partial_{y^i} \partial_{\theta^j})$.
- There exist a $\mathbb{C}[[\hbar]]$-map

$$\mathbf{Tr}_{1d} : C_\bullet(\mathcal{W}_{2N}) \to O_{1d}((\hbar))$$

satisfying QME

$$(b + \hbar\Delta_{\mathrm{BV}})\mathbf{Tr}_{1d} = 0$$

We can glue the above construction on a symplectic manifold (M, ω) using Fedosov connection to get

$$\widehat{\mathbf{Tr}}(-) : HH_\bullet(\mathcal{A}_\hbar(M)) \to \mathbb{R}((\hbar))$$

and

$$\widehat{\mathbf{Tr}}(1) = \int_M e^{\omega/\hbar} \hat{A}(M).$$

$\widehat{\mathbf{Tr}}(-)$ has an explicit formula by [Feigin-Felder-Shoikhet, DMJ 2003].

Talking: Perimeter-B

- Motivation/ Introduction
- VOA and Chiral algebra
- Elliptic chiral homology
- Batalin-Vilkovisky (BV) quantization
- Chiral homology and quantum master equation
- Witten genus.

Talking:

BV formalism: 2d Chiral CFT

The VOA $\mathcal{V}^{\beta\gamma-bc}$ of $\beta\gamma - bc$ system is the chiral analogue of Weyl/Clifford algebra. It gives rise to a chiral algebra (in the sense of Beilinson and Drinfeld) $\mathcal{A}^{\beta\gamma-bc}$ on a Riemann surface $X = \Sigma$.

The factorization homology (complex)

$$(C_\bullet(\mathcal{V}^{\beta\gamma-bc}(\mathbf{h})), d) \quad \text{in the BV formalism}$$

will be the chiral chain complex $C^{\mathrm{ch}}(X, \mathcal{A}^{\beta\gamma-bc})$.

Talking: Perimeter-B

BV formalism: 2d Chiral CFT

Now we want to extend our previous construction to the case of 2d chiral CFT(for example, $\beta\gamma - bc$ system). We still have

- "Factorization algebra" : $\mathrm{Obs}_{2d} = \mathcal{V}^{\beta\gamma - bc}(\mathbf{h})$, a $\mathbb{C}[[\hbar]]$-module equipped with the VOA structure.
- "Factorization homology":

$$(C_\bullet(\mathrm{Obs}_{2d}), d) = (C_\bullet(\mathcal{V}^{\beta\gamma - bc}(\mathbf{h})), d),$$

  is the chiral chain complex: $C^{\mathrm{ch}}(X, \mathcal{A}^{\beta\gamma - bc})$.
- A BV algebra $(O_{2d}, \Delta_{\mathrm{BV}})$.

We want to construct a $\mathbb{C}[[\hbar]]$-linear map

$$\mathbf{Tr}_{2d} : C_\bullet(\mathcal{V}^{\beta\gamma - bc}(\mathbf{h})) \to O_{2d}((\hbar))$$

satisfying QME

$$(d + \hbar\Delta_{\mathrm{BV}})\mathbf{Tr}_{2d} = 0.$$

Talking: Perimeter-B

Theorem (G-Li)

Let X be an elliptic curve E_τ. We can construct an explicit map

$$\mathbf{Tr}_{2d} : C^{\mathrm{ch}}(X, \mathcal{A}^{\beta\gamma-bc}) \to O_{2d}((\hbar))$$

satisfying

$$QME : \quad (d + \hbar\Delta_{\mathrm{BV}})\mathbf{Tr}_{2d} = 0.$$

Furthermore, the chain map $\mathbf{Tr}_{2d}$ is a quasi-isomorphism.

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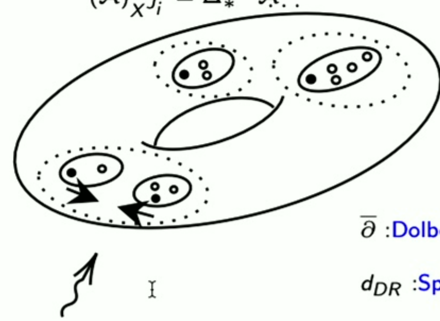
Talking: Perimeter-B

Proof of the main theorem

A typical element in the chiral chain complex can be written as a smooth differential form:

$$C^{\mathrm{ch}}(X,\mathcal{A}) \supset \Omega^{0,\bullet}\left(X^J, \Delta_*^{(J/I)}\left(\mathcal{A}[1]^{\boxtimes I}(*\Delta_I)\right) \otimes \wedge^\bullet \Theta_{X^J}\right):$$

$$(\mathcal{A})_{X^{J_i}} = \Delta_*^{(J_i)}\mathcal{A}$$



$\overline{\partial}$:Dolbeault Resolution

d_{DR} :Spencer Resolution

$d_{ch} = \mu[1]$:Chevalley-Eilenberg

Talking: Perimeter-B

Proof of the main theorem

Using Feynman diagrams, one can construct a chain map:

$$\mathcal{W} : C^{\mathrm{ch}}(X, \mathcal{A}^{\beta\gamma-bc}) \to C^{\mathrm{ch}}(X, \omega_X) \otimes O_{2d}((\hbar)),$$

that is, we have

$$\mathcal{W}(d(-)) = (d + \hbar\Delta_{\mathrm{BV}})\mathcal{W}(-).$$

Now we apply the trace map of Beilinson and Drinfeld:

$$\mathrm{tr}_{\mathrm{BD}} : C^{\mathrm{ch}}(X, \omega_X) \otimes O_{2d}((\hbar)) \to O_{2d}((\hbar)),$$

satisfying

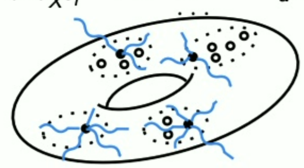
$$\mathrm{tr}_{\mathrm{BD}} d(-) = \mathrm{tr}_{\mathrm{BD}}(\bar{\partial} + d_{\mathrm{DR}} + d_{\mathrm{ch}})(-) = 0, \quad \text{Stokes Theorem.}$$

Talking:

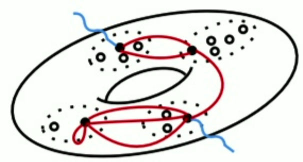
Here is a picture for $\mathcal{W}$:

$$(\mathcal{A})_{X^{J_i}} = \Delta_*^{(J_i)} \mathcal{A}$$

$$\Omega^{0,\bullet}\left(X^J, \Delta_*^{(J/I)}\left(\mathcal{A}[1]^{\boxtimes I}(\Delta_I)\right) \otimes \wedge^\bullet \Theta_{X^J}\right),$$
$$d = \overline{\partial} + d_{DR} + d_{ch}$$



Feynman diagrams



$$\Omega^{0,\bullet}\left(X^J, \Delta_*^{(J/I)}\left(\omega_X[1]^{\boxtimes I}(\Delta_I)\right) \otimes \wedge^\bullet \Theta_{X^J}\right) \otimes \mathcal{O}_{2d},$$
$$d = \overline{\partial} + d_{DR} + d_{ch} + \hbar\Delta$$

Finally, we define

$$\mathbf{Tr}_{2d} = \mathrm{tr}_{\mathrm{BD}} \circ \mathcal{W}(-).$$

It satisfies

$$\text{QME}: \quad (d + \hbar\Delta_{\mathrm{BV}})\mathbf{Tr}_{2d} = 0.$$

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Talking: Perimeter-B

Proof of the main theorem: summary

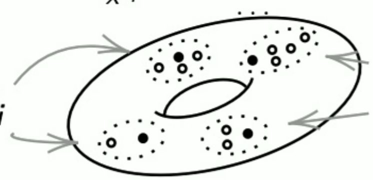
$\circ \in J$

$\because \in I$

J_i

$(\mathcal{A})_{X^{J_i}} = \Delta_*^{(J_i)} \mathcal{A}$

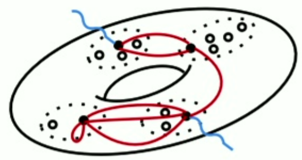
$\Omega^{0,\bullet}\left(X^j, \Delta_*^{(J/I)}\left(\mathcal{A}[1]^{\boxtimes I}(*\Delta_I)\right) \otimes \wedge^\bullet \Theta_{X^J}\right),$
$d = \bar{\partial} + d_{DR} + d_{ch}$



$\frac{1}{(z_{j_s} - z_{j_t})^n}, \ s \neq t \in I$

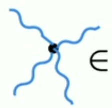
Feynman diagrams

$\fint$



$\Omega^{0,\bullet}\left(X^j, \Delta_*^{(J/I)}\left(\omega_X[1]^{\boxtimes I}(*\Delta_I)\right) \otimes \wedge^\bullet \Theta_{X^J}\right) \otimes \mathcal{O}_{2d},$
$d = \bar{\partial} + d_{DR} + d_{ch} + \hbar\Delta$

Regularized integrals$=\mathrm{tr}_{\mathrm{BD}}$



$\in \mathcal{O}_{2d}$

$$\fint_{X^I} \bar{\partial}(-) = \sum_{I'} \fint_{X^{I'}} \operatorname*{Res}_{I,I'} \rightsquigarrow \text{QME}$$

Stokes formula

- Motivation/ Introduction
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- Witten genus

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Talking: Perimeter-B

Witten genus

Corollary

For any Lie subalgebra $\mathfrak{g}$ of innner derivations of $\mathcal{A}^{\beta\gamma-bc}$, the above construction can be extended to the Lie algebra cochain, that is, an element

$$\mathbf{Tr}_{\mathfrak{g},2d}(-)\{-\} \in C_{\mathrm{Lie,BV}} = C_{\mathrm{Lie}}\big(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(C^{\mathrm{ch}}(X,\mathcal{A}),\mathbf{k})\big)\otimes_{\mathbf{k}} O_{2d}((\hbar))$$

satisfying

$$\mathbf{Tr}_{\mathfrak{g},2d}(-)\{-\}|_{C^0_{\mathrm{Lie,BV}}} = \mathbf{Tr}_{2d}(-)$$

and

$$(\partial_{\mathrm{Lie}} + d + \hbar\Delta_{\mathrm{BV}})\mathbf{Tr}_{\mathfrak{g},2d}(-)\{-\} = 0.$$

Here $\mathbf{k} := \mathbb{C}((\hbar))$.

Talking: Perimeter-B

Witten genus

If we take $\mathcal{A}^{\beta\gamma-bc}$ to be the free $\beta\gamma$-system $\mathcal{A}_N^{\beta\gamma}$ of rank N and $\mathfrak{g}$ to be $\widetilde{W}_N$ which is an extension of the Lie algebra of formal vector fields W_N in N variables. Then there is an element

$$\theta \in \mathrm{C}^1_{\mathrm{Lie}}(\widetilde{W}_N, \mathbb{C}) \otimes O_{\mathrm{BV}}$$

such that the chain $e^{-\frac{\pi}{i\hbar}\theta}\mathbf{Tr}_{\widetilde{W}_N,2d}(\underline{\mathbf{1}})\{-\}$ can be identified with an element in $\mathrm{C}_{\mathrm{Lie}}(W_N, \mathrm{GL}_N; \Omega_{\hat{\mathbb{O}}_N})$ which is cohomologous to the formal Witten genus

$$e^{-\frac{\pi}{i\hbar}\theta}\mathbf{Tr}_{\widetilde{W}_N,2d}(\underline{\mathbf{1}})\{-\} = \mathrm{Wit}_N(\tau) \quad \text{in } H_{\mathrm{Lie}}(W_N, \mathrm{GL}_N; \Omega_{\hat{\mathbb{O}}_N}).$$

Talking: Perimeter-B

Remark

This set-up follows closely the philosophy of Gelfand-Kazhdan formal geometry which appears in Kontsevich's work on deformation quantization and Cattaneo-Felder's work on Poisson σ-model.

Its use to formulate effective theory originates from Costello's work on Witten genera. Several examples are established in the literature along this line

- Grady-Gwilliam: TQM on $X = T^*M$
- Grady-Li-Li: TQM on a symplectic manifold X
- Gorbounov-Gwilliam-Williams: curved $\beta\gamma$-system.

Talking:

Algebraic Index Vs Elliptic Chiral Index

1d TQM	2d Chiral CFT
Associative algebra	Vertex operator algebra
Hochschild homology	Chiral homology
QME: $(\hbar\Delta_{\mathrm{BV}} + b)\mathbf{Tr}_{1d} = 0$	QME: $(\hbar\Delta_{\mathrm{BV}} + d)\mathbf{Tr}_{2d} = 0$
$\mathbf{Tr}_{1d}$ is a quasi-isomorphism	$\mathbf{Tr}_{2d}$ is a quasi-isomorphism
$\mathbf{Tr}_{1d}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n) =$integrals on the compactified configuration spaces of S^1	$\mathbf{Tr}_{2d}(\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n) =$regularized integrals of singular forms on Σ^n
Algebraic Index theory	Elliptic Chiral Algebraic Index

Talking: Perimeter-B

Future Work

- Higher genus: work in progress (with Kai Xu).
- Let M be a complex symplectic representation of a complex Lie group G. We can apply our method to study chiral symplectic bosons which is generated by $(P \times_G M) \otimes \omega_X^{\frac{1}{2}}$ (here P is holomorphic principal G-bundle on X). The space of chiral homology can be viewed as a quantization of Gaiotto's Lagrangian in the moduli space of G-Higgs bundles.
- We can study more general holomorphic symplectic varieties. The chiral algebras will become nonlinear version of chiral symplectic bosons.
- Fredholm interpretation of this result.
- 2d Chiral index theorem for families?
- Higher dimension?

Talking: Perimeter-B

Thank you!

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Talking: