Title: Large N von Neumann algebras and the renormalization of Newtonâ€™s constant

Speakers: Elliott Gesteau

Series: Quantum Fields and Strings

Date: March 21, 2023 - 2:00 PM

URL: https://pirsa.org/23030108

Abstract: In holography, the quantum extremal surface formula relates the entropy of a boundary state to the sum of two terms: the area term and the entropy of bulk fields inside the entanglement wedge. As the bulk effective field theory suffers from UV divergences, the second term must be regularized. It has been conjectured since the work of Susskind and Uglum that the renormalization of Newton's constant in the area term exactly cancels the difference between different choices of regularization for bulk entropy. In this talk, I will explain how the recent developments on von Neumann algebras appearing in the large N limit of holography allow to prove this claim within the framework of holographic quantum error correction, and to reinterpret it as an instance of the ER=EPR paradigm. This talk is based on the paper arXiv:2302.01938.

Zoom link: https://pitp.zoom.us/j/97435154387?pwd=OHYrRW9uSW5VeHRFUld1dmtVbmJiZz09

The Quantum Extremal Surface Formula

- The **Quantum Extremal Surface** (QES) Formula is one of the cornerstones of holography.

$$S(\rho) = \frac{A(\Sigma)}{4G_N} + S(\rho_{bulk}).$$

Σ is the quantum extremal surface associated to the subregion. It is defined by **extremizing** the RHS.

The Quantum Extremal Surface Formula

- The **Quantum Extremal Surface** (QES) Formula is one of the cornerstones of holography.

$$S(\rho) = \frac{A(\Sigma)}{4G_N} + S(\rho_{bulk}).$$

  Σ is the quantum extremal surface associated to the subregion. It is defined by **extremizing** the RHS.
- In the case of one side of a two-sided black hole, QES reduces to the calculation of **black hole entropy**.

$$S(\rho_L) = \frac{A(\Sigma)}{4G_N} + S(\rho_{L,bulk}).$$

Ambiguities in QES

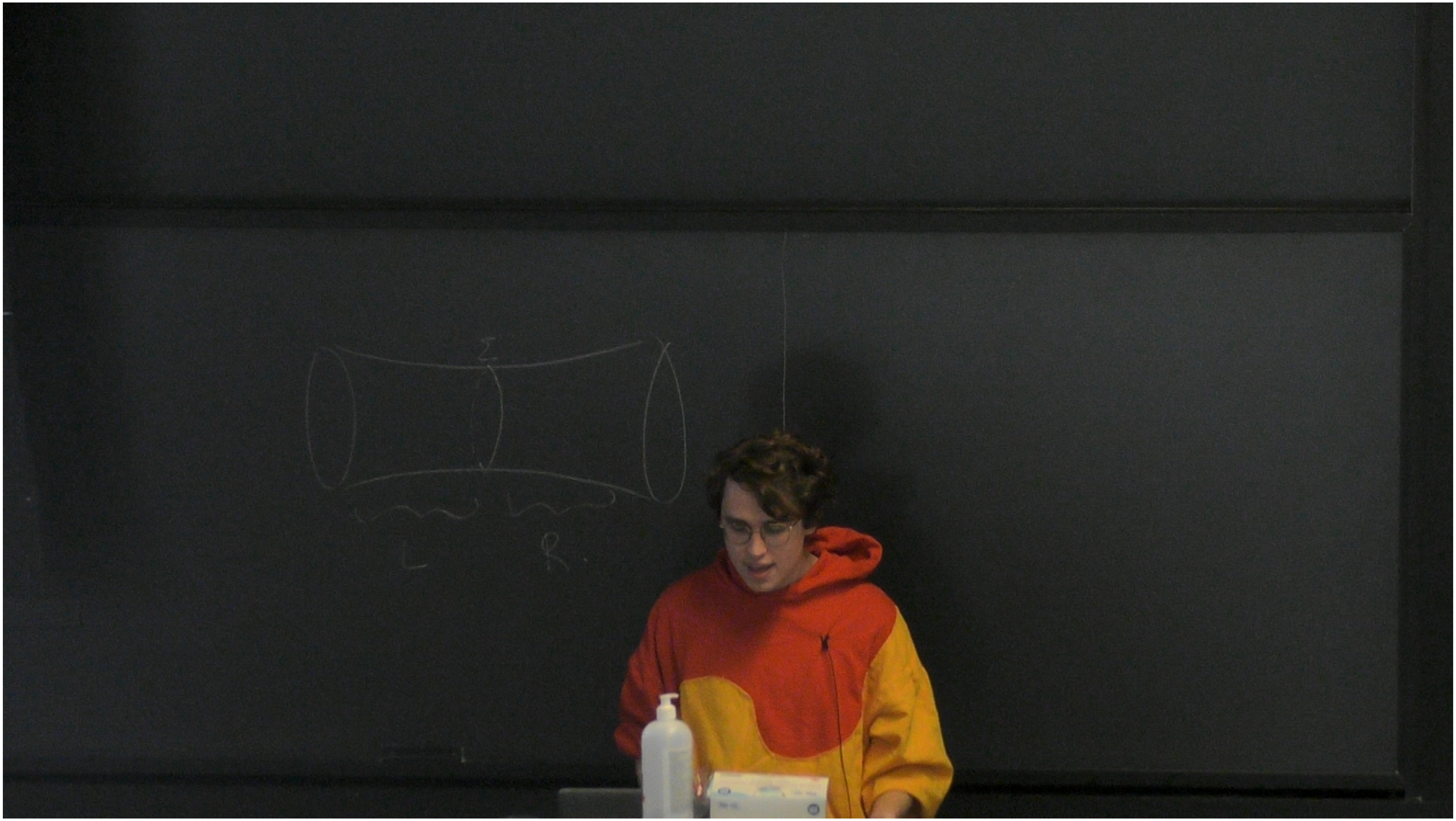
- Even though QES is fundamental, it is not so straightforward to properly **<u>define</u>** each of its terms!

Ambiguities in QES

- Even though QES is fundamental, it is not so straightforward to properly **define** each of its terms!
- In the effective field theory description, the entropy S_{bulk} of quantum fields across the horizon is **infinite**: needs to be **regulated**.

Ambiguities in QES

- Even though QES is fundamental, it is not so straightforward to properly **define** each of its terms!
- In the effective field theory description, the entropy S_{bulk} of quantum fields across the horizon is **infinite**: needs to be **regulated**.
- If G_N is zero or perturbatively small in the effective field theory the **area term** also blows up.


Σ
L
R.

Ambiguities in QES

- Even though QES is fundamental, it is not so straightforward to properly **define** each of its terms!
- In the effective field theory description, the entropy S_{bulk} of quantum fields across the horizon is **infinite**: needs to be **regulated**.
- If G_N is zero or perturbatively small in the effective field theory the **area term** also blows up.
- On the boundary if G_N is taken to be zero then the entropy term also blows up: G_N needs to be taken **small but nonzero**.

The Susskind—Uglum conjecture

- There seems to be an **arbitrariness** in the choice of UV cutoff in the EFT, but since the boundary quantity is UV-finite, QES itself **cannot be dependent** on this choice of cutoff.

The Susskind—Uglum conjecture

- There seems to be an **arbitrariness** in the choice of UV cutoff in the EFT, but since the boundary quantity is UV-finite, QES itself **cannot be dependent** on this choice of cutoff.
- **Susskind—Uglum conjecture**: The renormalization of the area term (i.e. Newton's constant) exactly **cancels** that of the bulk entropy term!

The Susskind—Uglum conjecture

- There seems to be an **arbitrariness** in the choice of UV cutoff in the EFT, but since the boundary quantity is UV-finite, QES itself **cannot be dependent** on this choice of cutoff.
- **Susskind—Uglum conjecture**: The renormalization of the area term (i.e. Newton's constant) exactly **cancels** that of the bulk entropy term!
- This talk: recent discussions on the **large N limit** of holography, as well as holographic **quantum error correction**, allow to formulate this conjecture precisely and prove it.

Outline

- I - Large N von Neumann algebras

Outline

- I - Large N von Neumann algebras
- II - Code subspace renormalization

Outline

- I - Large N von Neumann algebras
- II - Code subspace renormalization
- III - Proof of the Susskind—Uglum conjecture

Part I

Large N von Neumann algebras

The Leutheusser—Liu construction

- Another more abstract way of saying that entanglement of bulk fields in the EFT diverges at large N like that of a subregion in a QFT is that there is an **emergent type III_1** von Neumann algebra, recently identified by Leutheusser and Liu.

Page 15 sur 104

The Leutheusser—Liu construction

- Another more abstract way of saying that entanglement of bulk fields in the EFT diverges at large N like that of a subregion in a QFT is that there is an **emergent type III_1** von Neumann algebra, recently identified by Leutheusser and Liu.
- This algebra is constructed in the following way: introduce a **formal vacuum vector** $|\Omega\rangle$, and define the **Hilbert space** as being spanned by operators of the form $\mathrm{Tr}(X_1)\ldots\mathrm{Tr}(X_k)\,|\Omega\rangle$.

The Leutheusser—Liu construction

- Another more abstract way of saying that entanglement of bulk fields in the EFT diverges at large N like that of a subregion in a QFT is that there is an **emergent type III_1** von Neumann algebra, recently identified by Leutheusser and Liu.
- This algebra is constructed in the following way: introduce a **formal vacuum vector** $|\Omega\rangle$, and define the **Hilbert space** as being spanned by operators of the form $\mathrm{Tr}(X_1)\ldots\mathrm{Tr}(X_k)\,|\Omega\rangle$.
- The inner product of the Hilbert space is defined from the limits of the correlation functions of single trace operators:

$$\langle\Omega|\, A^\dagger B\, |\Omega\rangle = \langle A^\dagger B\rangle_\beta \,.$$

This is known as the **GNS construction**.

The Leutheusser—Liu construction

- Another more abstract way of saying that entanglement of bulk fields in the EFT diverges at large N like that of a subregion in a QFT is that there is an **emergent type III_1** von Neumann algebra, recently identified by Leutheusser and Liu.
- This algebra is constructed in the following way: introduce a **formal vacuum vector** $|\Omega\rangle$, and define the **Hilbert space** as being spanned by operators of the form $\mathrm{Tr}(X_1)\ldots\mathrm{Tr}(X_k)\,|\Omega\rangle$.
- The inner product of the Hilbert space is defined from the limits of the correlation functions of single trace operators:

$$\langle\Omega|\, A^\dagger B\,|\Omega\rangle = \langle A^\dagger B\rangle_\beta\,.$$

This is known as the **GNS construction**.

The Leutheusser—Liu construction

- Another more abstract way of saying that entanglement of bulk fields in the EFT diverges at large N like that of a subregion in a QFT is that there is an **emergent type III_1** von Neumann algebra, recently identified by Leutheusser and Liu.
- This algebra is constructed in the following way: introduce a **formal vacuum vector** $|\Omega\rangle$, and define the **Hilbert space** as being spanned by operators of the form $\mathrm{Tr}(X_1)\ldots\mathrm{Tr}(X_k)\,|\Omega\rangle$.
- The inner product of the Hilbert space is defined from the limits of the correlation functions of single trace operators:

$$\langle\Omega|\, A^\dagger B\, |\Omega\rangle = \langle A^\dagger B\rangle_\beta\,.$$

  This is known as the **GNS construction**.
- The von Neumann algebra is defined as the **bicommutant** of the single trace operators on one side of the thermofield double, and is dual to operators in the EFT.

Properties of the large N algebra

- **Below** the Hawking—Page temperature, the large N algebra has **type** I: this means that the large N Hilbert space **factorizes** between the right and the left. There is no Einstein—Rosen bridge, just two entangled copies of **thermal AdS**.

Properties of the large N algebra

- **Below** the Hawking—Page temperature, the large N algebra has **type I**: this means that the large N Hilbert space **factorizes** between the right and the left. There is no Einstein—Rosen bridge, just two entangled copies of **thermal AdS**.
- **Above** the Hawking—Page temperature, the gauge theory deconfines and the large N algebra has **type III_1** (still type I at any finite N!): entanglement pattern of quantum field theory. A geometric **Einstein—Rosen** bridge appears between the right and the left.

Properties of the large N algebra

- **Below** the Hawking—Page temperature, the large N algebra has **type I**: this means that the large N Hilbert space **factorizes** between the right and the left. There is no Einstein—Rosen bridge, just two entangled copies of **thermal AdS**.
- **Above** the Hawking—Page temperature, the gauge theory deconfines and the large N algebra has **type III_1** (still type I at any finite N!): entanglement pattern of quantum field theory. A geometric **Einstein—Rosen** bridge appears between the right and the left.
- This can be shown **rigorously** (paper to appear with L. Santilli) from the fact that the **spectral density** of the large N generalized free fields becomes continuous above the Hawking–Page temperature.

The bulk to boundary map

- It is a bit tricky to think about holographic **quantum error correction** in that context.

The bulk to boundary map

- It is a bit tricky to think about holographic **quantum error correction** in that context.
- The code should map the $N = \infty$ type III_1 von Neumann algebra M^L, or some perturbative correction of it, to the **large but finite** N type I von Neumann algebra $\mathcal{B}(\mathcal{H}_N^L)$ on the boundary.


∞>N>>1
Σ
L
R

The bulk to boundary map

- It is a bit tricky to think about holographic **quantum error correction** in that context.
- The code should map the $N = \infty$ type III_1 von Neumann algebra M^L, or some perturbative correction of it, to the **large but finite** N type I von Neumann algebra $\mathcal{B}(\mathcal{H}_N^L)$ on the boundary.
- Then one shouldn't trust the map when operators have energy that starts scaling parametrically with N and break the EFT: the code works **pointwise** at large N but not **uniformly**.

Asymptotically isometric codes

- Faulkner and Li recently formalized this by introducing the notion of **asymptotically isometric code**, from the large N Hilbert space to the finite N boundary Hilbert space.

$$V_N^\dagger V_N - Id \underset{N\to\infty}{\longrightarrow} 0,$$

$$\forall A, \gamma_N(A) V_N - V_N A \underset{N\to\infty}{\longrightarrow} 0.$$

Asymptotically isometric codes

- Faulkner and Li recently formalized this by introducing the notion of **asymptotically isometric code**, from the large N Hilbert space to the finite N boundary Hilbert space.

$$V_N^\dagger V_N - Id \underset{N\to\infty}{\longrightarrow} 0,$$

$$\forall A, \gamma_N(A) V_N - V_N A \underset{N\to\infty}{\longrightarrow} 0.$$

- These conditions are imposed for the **weak and strong** operator topologies respectively, but NOT for the norm topology. This is an abstract way of saying that only **pointwise** convergence is required, rather than **uniform** convergence.

Asymptotically isometric codes

- Asymptotic conservation of **modular flow** (JLMS) can be derived from this approach.

Asymptotically isometric codes

- Asymptotic conservation of **modular flow** (JLMS) can be derived from this approach.
- However if one wants to derive something like QES, the full large N algebra cannot be considered: infinite entropy, breakdown of EFT at each fixed N. Instead it must be **regulated**.

Asymptotically isometric codes

- Asymptotic conservation of **modular flow** (JLMS) can be derived from this approach.
- However if one wants to derive something like QES, the full large N algebra cannot be considered: infinite entropy, breakdown of EFT at each fixed N. Instead it must be **regulated**.
- How do we do this? Single out small (for example finite-dimensional) **subalgebras** of the large N algebra.

Part II

Code subspace renormalization

Type I and bounded entropy

- What does a good regulated subalgebra look like?

Type I and bounded entropy

- What does a good regulated subalgebra look like?
- We want the regulated algebra to match the **bulk entropy term** in the large N limit.

Type I and bounded entropy

- What does a good regulated subalgebra look like?
- We want the regulated algebra to match the **bulk entropy term** in the large N limit.
- In order to hope for a finite entropy, the regulated algebra must have **type I**: either finite-dimensional or $\mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ a separable Hilbert space.

Type I and bounded entropy

- What does a good regulated subalgebra look like?
- We want the regulated algebra to match the **bulk entropy term** in the large N limit.
- In order to hope for a finite entropy, the regulated algebra must have **type I**: either finite-dimensional or $\mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ a separable Hilbert space.
- **Schmidt decompositions** can be defined for states on these algebras, and **von Neumann entropy** is defined in the usual way.

Entanglement wedge reconstruction

- Entanglement wedge reconstruction is thought to be **equivalent** to the QES formula.

Entanglement wedge reconstruction

- Entanglement wedge reconstruction is thought to be **equivalent** to the QES formula.

Entanglement wedge reconstruction

- Entanglement wedge reconstruction is thought to be **equivalent** to the QES formula.
- It is a statement of **complementary recovery**: in the case of the two-sided BH any operator in region L can be reconstructed on the left boundary, and any operator in region R can be reconstructed on the right boundary.

Entanglement wedge reconstruction

- Entanglement wedge reconstruction is thought to be **equivalent** to the QES formula.
- It is a statement of **complementary recovery**: in the case of the two-sided BH any operator in region L can be reconstructed on the left boundary, and any operator in region R can be reconstructed on the right boundary.
- It can be proven (in a pointwise sense) for asymptotically isometric codes.

Entanglement wedge reconstruction

- Entanglement wedge reconstruction is thought to be **equivalent** to the QES formula.
- It is a statement of **complementary recovery**: in the case of the two-sided BH any operator in region L can be reconstructed on the left boundary, and any operator in region R can be reconstructed on the right boundary.
- It can be proven (in a pointwise sense) for asymptotically isometric codes.
- How do we now regulate the bulk algebra while still being able to formulate entanglement wedge reconstruction?

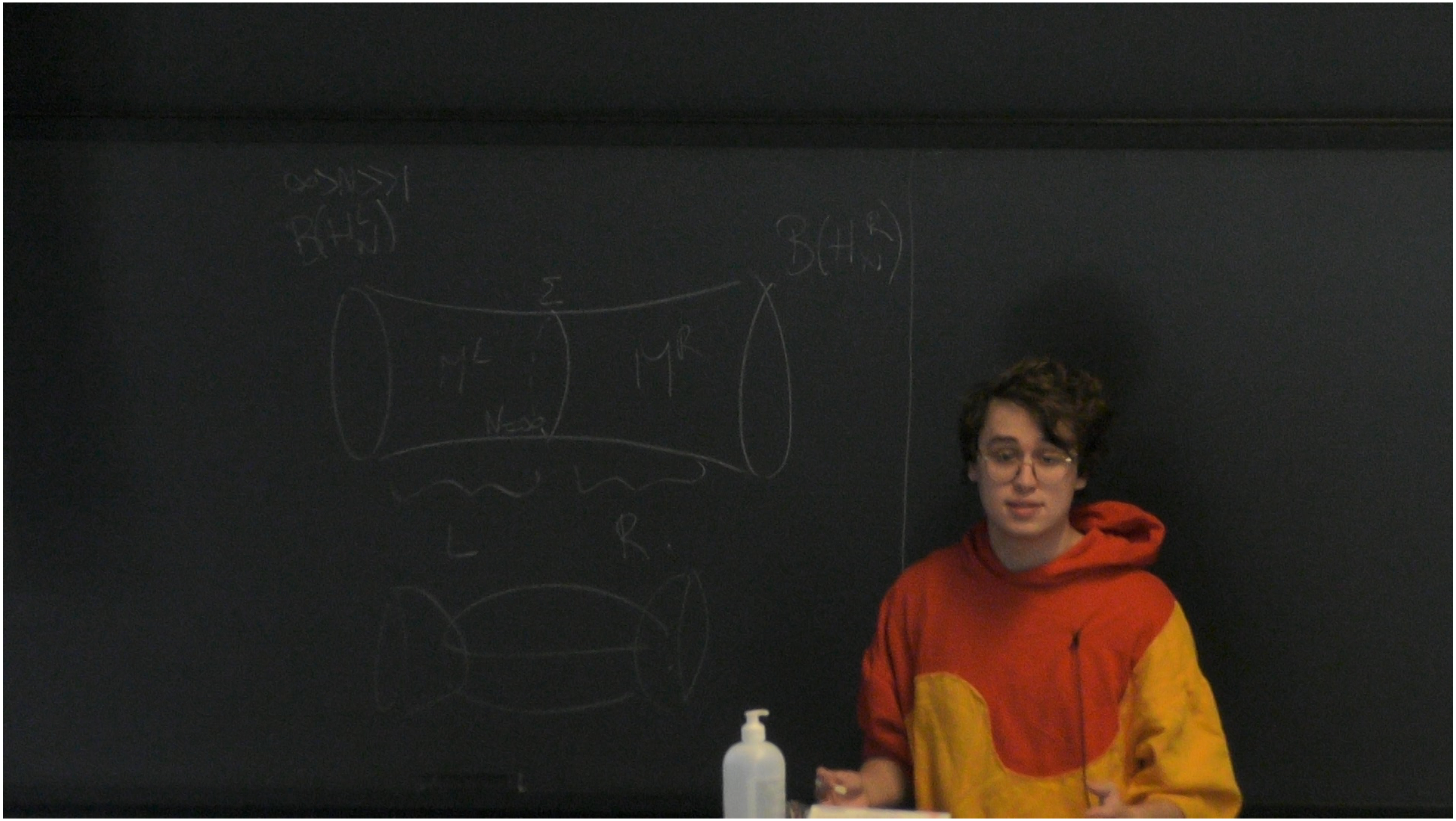

∞>N>>1
B(H_N^L)
B(H_N^R)
Σ
M^L
M^R
N=∞
L
R

Entanglement wedge reconstruction

- Entanglement wedge reconstruction is thought to be **equivalent** to the QES formula.
- It is a statement of **complementary recovery**: in the case of the two-sided BH any operator in region L can be reconstructed on the left boundary, and any operator in region R can be reconstructed on the right boundary.
- It can be proven (in a pointwise sense) for asymptotically isometric codes.
- How do we now regulate the bulk algebra while still being able to formulate entanglement wedge reconstruction?
- Then we also want compatibility with **complementary recovery**.

Constraints from complementary recovery

- We want to find a way to **regulate** the large N algebra that respects the fundamental structure of complementary recovery.


∞≫N≫1
B(H_N^L)
B(H_N^R)
Σ
M^L
M^R
L
R

Constraints from complementary recovery

- We want to find a way to **regulate** the large N algebra that respects the fundamental structure of complementary recovery.
- We want something like:

$$
\begin{array}{ccccc}
M^L & \xrightarrow{\;\prime\;} & M^R & \qquad & \mathcal{H} \\
{\scriptstyle ?}\downarrow & & {\scriptstyle ?}\downarrow & & \downarrow \\
M^L_\lambda & \xrightarrow{\;\prime\;} & M^R_\lambda & & \mathcal{H}_\lambda
\end{array}
$$

Constraints from complementary recovery

- We want to find a way to **regulate** the large N algebra that respects the fundamental structure of complementary recovery.
- We want something like:

$$
\begin{array}{ccccc}
M^L & \xrightarrow{\;'\;} & M^R & \qquad & \mathcal{H} \\
{\scriptstyle ?}\downarrow & & {\scriptstyle ?}\downarrow & & \downarrow \\
M^L_\lambda & \xrightarrow{\;'\;} & M^R_\lambda & \qquad & \mathcal{H}_\lambda
\end{array}
$$

- This is a **nontrivial constraint**: what should "?" be?

Takesaki's theorem

- Fortunately mathematicians (Takesaki) solved the problem for us in 1972. The following statements are equivalent:

Page 43 sur 104

Takesaki's theorem

- Fortunately mathematicians (Takesaki) solved the problem for us in 1972. The following statements are equivalent:
- The **commutant structure** is preserved.
- **Modular flow** is conserved: for $n \in M^L_\lambda$ and $|\psi\rangle \in \mathcal{H}_\lambda$,

$$\Delta_{M^L}(\psi)^{it} n \Delta_{M^L}(\psi)^{-it} = \Delta_{M^L_\lambda}(\psi)^{it} n \Delta_{M^L_\lambda}(\psi)^{-it}.$$

Takesaki's theorem

- Fortunately mathematicians (Takesaki) solved the problem for us in 1972. The following statements are equivalent:
- The **commutant structure** is preserved.
- **Modular flow** is conserved: for $n \in M^L_\lambda$ and $|\psi\rangle \in \mathcal{H}_\lambda$,

$$\Delta_{M^L}(\psi)^{it} n \Delta_{M^L}(\psi)^{-it} = \Delta_{M^L_\lambda}(\psi)^{it} n \Delta_{M^L_\lambda}(\psi)^{-it}.$$

- There exists a faithful normal **conditional expectation** from the large N algebra onto the subalgebra that leaves the state $|\psi\rangle \in \mathcal{H}_\lambda$ **invariant**.

Page 46 sur 104

Conditional expectations

- A **conditional expectation** from a von Neumann algebra M^L onto a (unital) subalgebra M^L_λ is a map $\mathcal{E}_\lambda : M^L \longrightarrow M^L_\lambda$ satisfying for $n_1, n_2 \in M^L_\lambda$ and $m \in M^L$,

$$\mathcal{E}_\lambda(n_1 m n_2) = n_1 \mathcal{E}_\lambda(m) n_2.$$

Conditional expectations

- A **conditional expectation** from a von Neumann algebra M^L onto a (unital) subalgebra M^L_λ is a map $\mathcal{E}_\lambda : M^L \longrightarrow M^L_\lambda$ satisfying for $n_1, n_2 \in M^L_\lambda$ and $m \in M^L$,

$$\mathcal{E}_\lambda(n_1 m n_2) = n_1 \mathcal{E}_\lambda(m) n_2.$$

- If a state $|\psi\rangle$ is in the space $\mathcal{H}_\lambda$ of **invariant** states under a conditional expectation, then its Tomita–Takesaki **modular data** associated to both the large and small algebras coincide :

$$J_{M^L}(\psi)|_{\mathcal{H}_\lambda} = J_{M^L_\lambda}(\psi), \quad \Delta_{M^L}(\psi)|_{\mathcal{H}_\lambda} = \Delta_{M^L_\lambda}(\psi).$$

Conditional expectations

- A **conditional expectation** from a von Neumann algebra M^L onto a (unital) subalgebra M^L_λ is a map $\mathcal{E}_\lambda : M^L \longrightarrow M^L_\lambda$ satisfying for $n_1, n_2 \in M^L_\lambda$ and $m \in M^L$,

$$\mathcal{E}_\lambda(n_1 m n_2) = n_1 \mathcal{E}_\lambda(m) n_2.$$

- If a state $|\psi\rangle$ is in the space $\mathcal{H}_\lambda$ of **invariant** states under a conditional expectation, then its Tomita–Takesaki **modular data** associated to both the large and small algebras coincide :

$$J_{M^L}(\psi)|_{\mathcal{H}_\lambda} = J_{M^L_\lambda}(\psi), \;\; \Delta_{M^L}(\psi)|_{\mathcal{H}_\lambda} = \Delta_{M^L_\lambda}(\psi).$$

- From there the characterizations in terms of compatibility with **modular flow** and **commutant** follow.

Conditional expectations onto a type I factor

- In the case of a type I subfactor, conditional expectations allow to **factor** everything as a tensor product of low and high energy contributions.

Conditional expectations onto a type I factor

- In the case of a type I subfactor, conditional expectations allow to **factor** everything as a tensor product of low and high energy contributions.
- More precisely, if the factor M^L acts on a Hilbert space $\mathcal{H}$ and M^L_λ is a type I subfactor of M^L, then

$$M^L = M^L_\lambda \otimes M^{L,c}_\lambda .$$

Conditional expectations onto a type I factor

- In the case of a type I subfactor, conditional expectations allow to **factor** everything as a tensor product of low and high energy contributions.
- More precisely, if the factor M^L acts on a Hilbert space $\mathcal{H}$ and M^L_λ is a type I subfactor of M^L, then

$$M^L = M^L_\lambda \otimes M^{L,c}_\lambda .$$

- Let $\mathcal{E}_\lambda$ be a conditional expectation $M^L \longrightarrow M^L_\lambda$. $\mathcal{H}$ factorizes as

$$\mathcal{H} = \mathcal{H}_\lambda \otimes \mathcal{H}^c_\lambda ,$$

and the Hilbert space of **invariant states** under $\mathcal{E}_\lambda$ is of the form

$$\mathcal{H}^{inv} = \mathcal{H}_\lambda \otimes |\chi_\lambda\rangle .$$

Conditional expectations onto a type I factor

- In the case of a type I subfactor, conditional expectations allow to **factor** everything as a tensor product of low and high energy contributions.
- More precisely, if the factor M^L acts on a Hilbert space $\mathcal{H}$ and M^L_λ is a type I subfactor of M^L, then

$$M^L = M^L_\lambda \otimes M^{L,c}_\lambda .$$

- Let $\mathcal{E}_\lambda$ be a conditional expectation $M^L \longrightarrow M^L_\lambda$. $\mathcal{H}$ factorizes as

$$\mathcal{H} = \mathcal{H}_\lambda \otimes \mathcal{H}^c_\lambda,$$

and the Hilbert space of **invariant states** under $\mathcal{E}_\lambda$ is of the form

$$\mathcal{H}^{inv} = \mathcal{H}_\lambda \otimes |\chi_\lambda\rangle \,.$$

- The conditional expectation and χ_λ are the **same data**:

$$\mathcal{E}(X \otimes X^c) = \chi_\lambda(X^c)(X \otimes Id).$$

Code subspace renormalization schemes

- A **code subspace renormalization scheme** (CSRS) is the given data of:

Code subspace renormalization schemes

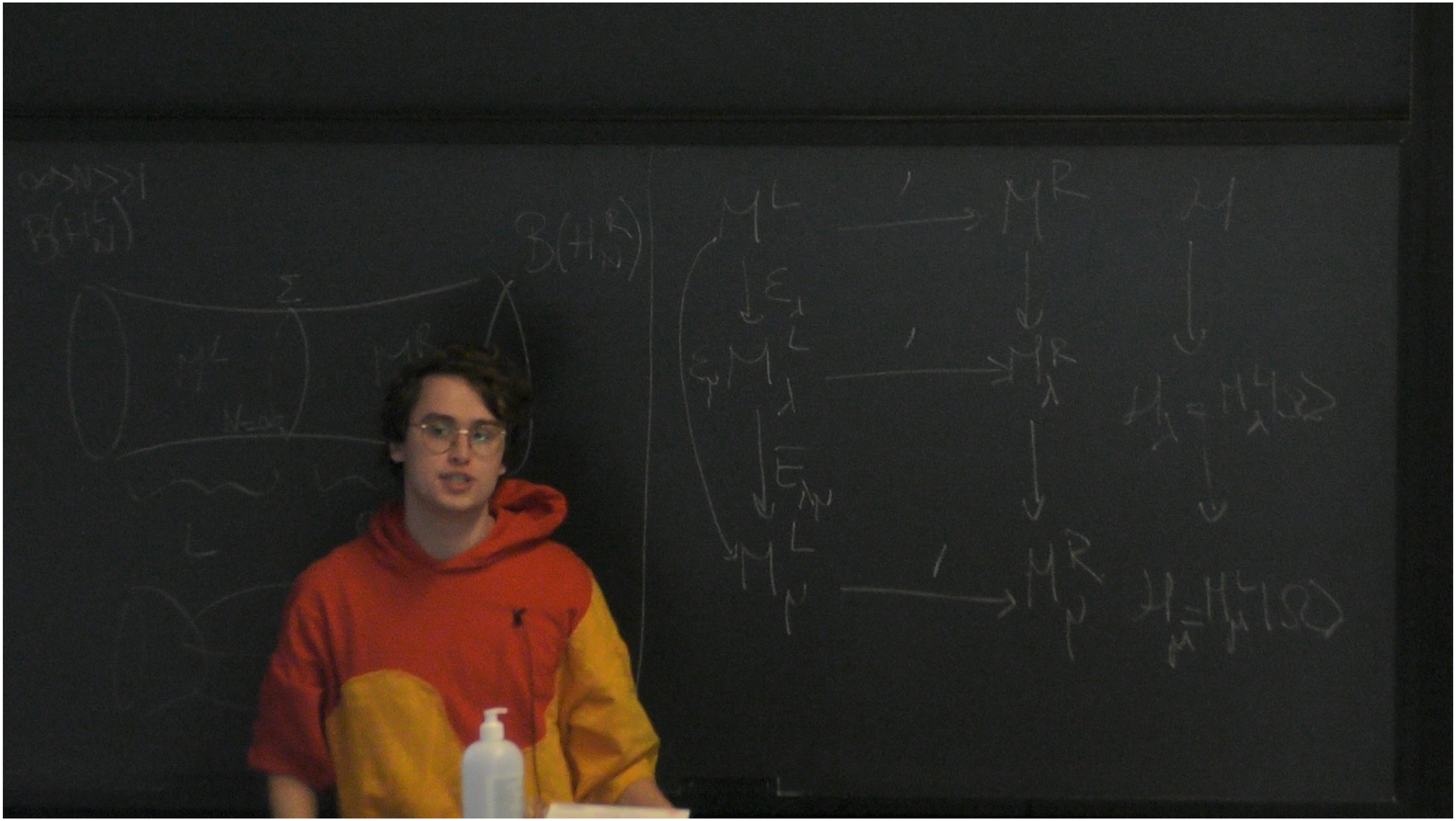
- A **code subspace renormalization scheme** (CSRS) is the given data of:
- A **von Neumann factor** M^L acting on a **Hilbert space** $\mathcal{H}$.
- A family of type I **subfactors** $(M^L_\lambda)_{\lambda \in \Lambda}$ of M indexed by a **poset** Λ.

Code subspace renormalization schemes

- A **code subspace renormalization scheme** (CSRS) is the given data of:
- A **von Neumann factor** M^L acting on a **Hilbert space** $\mathcal{H}$.
- A family of type I **subfactors** $(M_\lambda^L)_{\lambda\in\Lambda}$ of M indexed by a **poset** Λ.
- A family of faithful normal **conditional expectations** $\mathcal{E}_\lambda : M^L \longrightarrow M_\lambda^L$.

Code subspace renormalization schemes

- A **code subspace renormalization scheme** (CSRS) is the given data of:
- A **von Neumann factor** M^L acting on a **Hilbert space** $\mathcal{H}$.
- A family of type I **subfactors** $(M^L_\lambda)_{\lambda \in \Lambda}$ of M indexed by a **poset** Λ.
- A family of faithful normal **conditional expectations** $\mathcal{E}_\lambda : M^L \longrightarrow M^L_\lambda$.
- A family of faithful normal **conditional expectations** $E_{\lambda\mu} : M^L_\lambda \longrightarrow M^L_\mu,\ \lambda \geq \mu$.
- A reference cyclic separating state $|\Omega\rangle$ **invariant** under all these expectations.



Expressing entropies in a CSRS

- Consider two algebras M^L_λ, M^L_μ with $\lambda \geq \mu$:

$$M_\lambda = M_\mu \otimes M_{\lambda\mu}.$$

Expressing entropies in a CSRS

- Consider two algebras M^L_λ, M^L_μ with $\lambda \geq \mu$:

$$M_\lambda = M_\mu \otimes M_{\lambda\mu}.$$

- As it is **invariant** under the conditional expectations, the restriction of $|\psi\rangle \in \mathcal{H}_\mu$ to M^L decomposes into:

$$\psi = \psi_\mu \otimes \chi_{\lambda\mu} \otimes \chi_\lambda.$$

Expressing entropies in a CSRS

- Consider two algebras M^L_λ, M^L_μ with $\lambda \geq \mu$:

$$M_\lambda = M_\mu \otimes M_{\lambda\mu}.$$

- As it is **invariant** under the conditional expectations, the restriction of $|\psi\rangle \in \mathcal{H}_\mu$ to M^L decomposes into:

$$\psi = \psi_\mu \otimes \chi_{\lambda\mu} \otimes \chi_\lambda.$$

- Entropy can be expressed in terms of **3 pieces** (as long as they're all well-defined!):

$$\text{“}S(\psi, M_L)\text{”} = S(\psi_\mu, M^L_\mu) + S(\chi_{\lambda\mu}, M^L_{\lambda\mu}) + \text{“}S(\chi_\lambda, M^{L,c}_\lambda)\text{”}.$$

Expressing entropies in a CSRS

- Consider two algebras M^L_λ, M^L_μ with $\lambda \geq \mu$:

$$M_\lambda = M_\mu \otimes M_{\lambda\mu}.$$

- As it is **invariant** under the conditional expectations, the restriction of $|\psi\rangle \in \mathcal{H}_\mu$ to M^L decomposes into:

$$\psi = \psi_\mu \otimes \chi_{\lambda\mu} \otimes \chi_\lambda.$$

- Entropy can be expressed in terms of **3 pieces** (as long as they're all well-defined!):

$$\text{“}S(\psi, M_L)\text{”} = S(\psi_\mu, M^L_\mu) + S(\chi_{\lambda\mu}, M^L_{\lambda\mu}) + \text{“}S(\chi_\lambda, M^{L,c}_\lambda)\text{”}.$$

- It is the **middle term** that will be crucial in the proof of the Susskind–Uglum conjecture.

Link to exact quantum error correction

- A few years ago, Faulkner made the observation that the structure of conditional expectation underpins **exact** entanglement wedge reconstruction.
- The holographic map is identified with the **conditional expectation**, complementary recovery and JLMS with the **conservation of modular data**.

Link to exact quantum error correction

- A few years ago, Faulkner made the observation that the structure of conditional expectation underpins **exact** entanglement wedge reconstruction.
- The holographic map is identified with the **conditional expectation**, complementary recovery and JLMS with the **conservation of modular data**.
- Here, the interpretation is different: conditional expectations **integrate out** high energy degrees of freedom in the EFT.

Link to exact quantum error correction

- A few years ago, Faulkner made the observation that the structure of conditional expectation underpins **exact** entanglement wedge reconstruction.
- The holographic map is identified with the **conditional expectation**, complementary recovery and JLMS with the **conservation of modular data**.
- Here, the interpretation is different: conditional expectations **integrate out** high energy degrees of freedom in the EFT.
- Both perspectives are related: one can see this regulation as an **exact code** mapping a subalgebra of the code directly into the $N = \infty$ von Neumann algebra.

Link to exact quantum error correction

- A few years ago, Faulkner made the observation that the structure of conditional expectation underpins **exact** entanglement wedge reconstruction.
- The holographic map is identified with the **conditional expectation**, complementary recovery and JLMS with the **conservation of modular data**.
- Here, the interpretation is different: conditional expectations **integrate out** high energy degrees of freedom in the EFT.
- Both perspectives are related: one can see this regulation as an **exact code** mapping a subalgebra of the code directly into the $N = \infty$ von Neumann algebra.
- It makes sense that this exact structure remains in the large N limit, as obstructions to exactness of reconstruction come from **nonperturbative corrections**, which disappear in the large N algebra.

Part III

Proof of the Susskind—Uglum conjecture

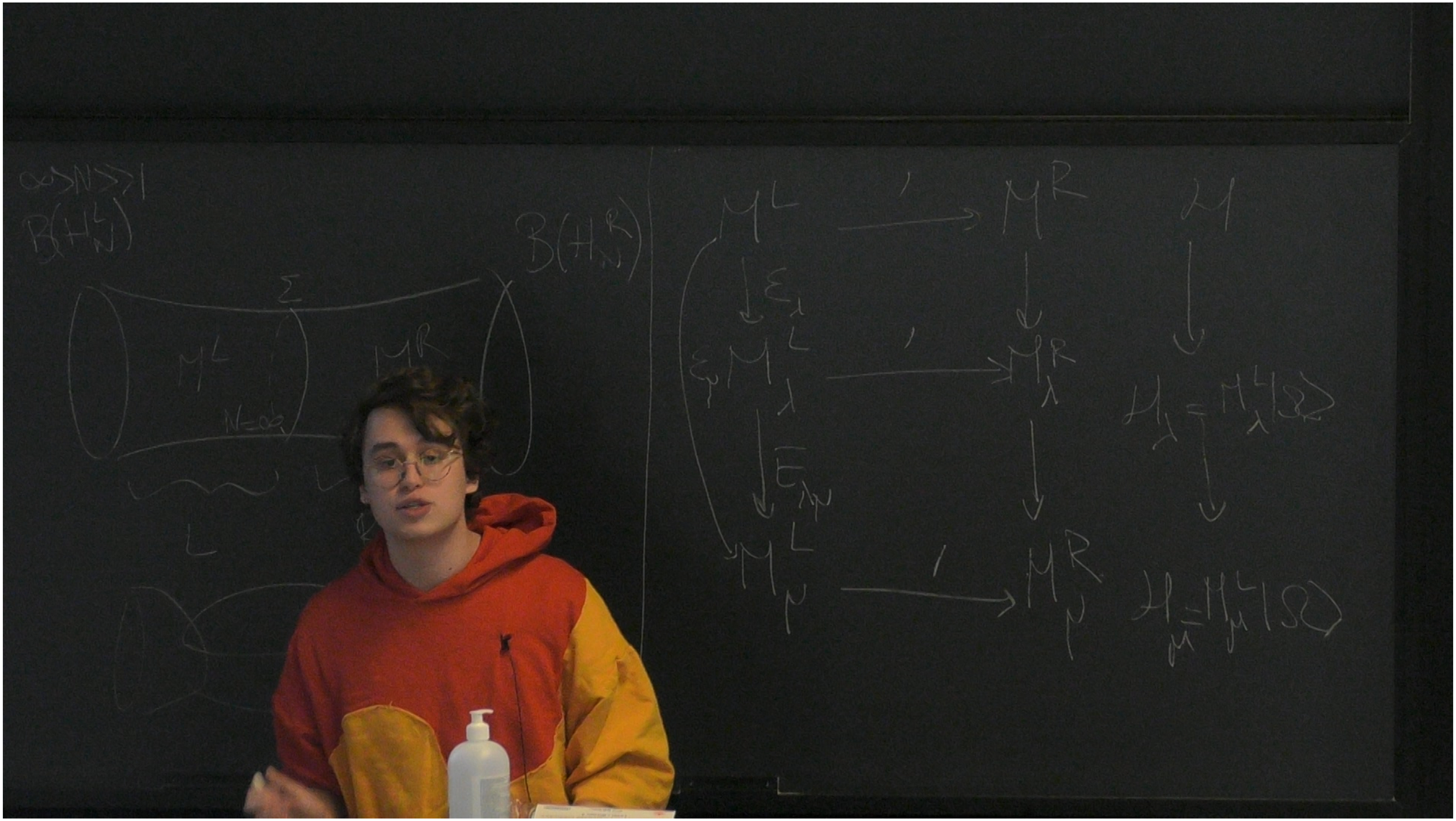
Back to asymptotically isometric codes

- We now have a consistent way of **renormalizing** the code subspace.

Back to asymptotically isometric codes

- We now have a consistent way of **renormalizing** the code subspace.
- Recall that there is an **encoding map** from the effective theory at $N = \infty$ to the finite N theory

$$V_N : \mathcal{H} \longrightarrow \mathcal{H}_N^L \otimes \mathcal{H}_N^R.$$


∞>N>>1
B(H_N^L)
B(H_N^R)
Σ
M^L
M^R
N=∞
L
E_λ
E_λμ
M_λ^L
M_λ^R
M_μ^L
M_μ^R
H_λ = M_λ^L|Ω⟩
H_μ = M_μ^L|Ω⟩

Back to asymptotically isometric codes

- We now have a consistent way of **renormalizing** the code subspace.
- Recall that there is an **encoding map** from the effective theory at $N = \infty$ to the finite N theory

$$V_N : \mathcal{H} \longrightarrow \mathcal{H}_N^L \otimes \mathcal{H}_N^R.$$

- However closeness to isometry and reconstruction properties can only be formulated **pointwise**.

Back to asymptotically isometric codes

- We now have a consistent way of **renormalizing** the code subspace.
- Recall that there is an **encoding map** from the effective theory at $N = \infty$ to the finite N theory

$$V_N : \mathcal{H} \longrightarrow \mathcal{H}_N^L \otimes \mathcal{H}_N^R.$$

- However closeness to isometry and reconstruction properties can only be formulated **pointwise**.
- Idea here: ask for **stronger** reconstruction properties, but only for **renormalized subalgebras**.

A smaller code

- Consider a **renormalized Hilbert space** $\mathcal{H}_\lambda$ of a CSRS, and a von Neumann factor M_λ^L acting on it. To simplify, we will assume everything is **finite dimensional** of dimension independent of N.

A smaller code

- Consider a **renormalized Hilbert space** $\mathcal{H}_\lambda$ of a CSRS, and a von Neumann factor M_λ^L acting on it. To simplify, we will assume everything is **finite dimensional** of dimension independent of N.
- Then we can write

$$\mathcal{H}_\lambda \cong \mathcal{H}_\lambda^L \otimes \mathcal{H}_\lambda^R,$$

$$M_\lambda^L \cong \mathcal{B}(\mathcal{H}_\lambda^L) \otimes Id.$$

Page 77 sur 104

A smaller code

- Consider a **renormalized Hilbert space** $\mathcal{H}_\lambda$ of a CSRS, and a von Neumann factor M_λ^L acting on it. To simplify, we will assume everything is **finite dimensional** of dimension independent of N.
- Then we can write

$$\mathcal{H}_\lambda \cong \mathcal{H}_\lambda^L \otimes \mathcal{H}_\lambda^R,$$

$$M_\lambda^L \cong \mathcal{B}(\mathcal{H}_\lambda^L) \otimes Id.$$

- The maps $V_N : \mathcal{H} \longrightarrow \mathcal{H}_N^L \otimes \mathcal{H}_N^R$ induce a natural map $\mathcal{H}_\lambda \longrightarrow \mathcal{H}_N^L \otimes \mathcal{H}_N^R$ under **embedding** of $\mathcal{H}_\lambda$ in $\mathcal{H}$.

A smaller code

- Consider a **renormalized Hilbert space** $\mathcal{H}_\lambda$ of a CSRS, and a von Neumann factor M_λ^L acting on it. To simplify, we will assume everything is **finite dimensional** of dimension independent of N.
- Then we can write

$$\mathcal{H}_\lambda \cong \mathcal{H}_\lambda^L \otimes \mathcal{H}_\lambda^R,$$

$$M_\lambda^L \cong \mathcal{B}(\mathcal{H}_\lambda^L) \otimes Id.$$

- The maps $V_N : \mathcal{H} \longrightarrow \mathcal{H}_N^L \otimes \mathcal{H}_N^R$ induce a natural map $\mathcal{H}_\lambda \longrightarrow \mathcal{H}_N^L \otimes \mathcal{H}_N^R$ under **embedding** of $\mathcal{H}_\lambda$ in $\mathcal{H}$.
- It is for this restriction of the map that we will ask for **uniform** reconstruction properties.

Areas in approximate codes

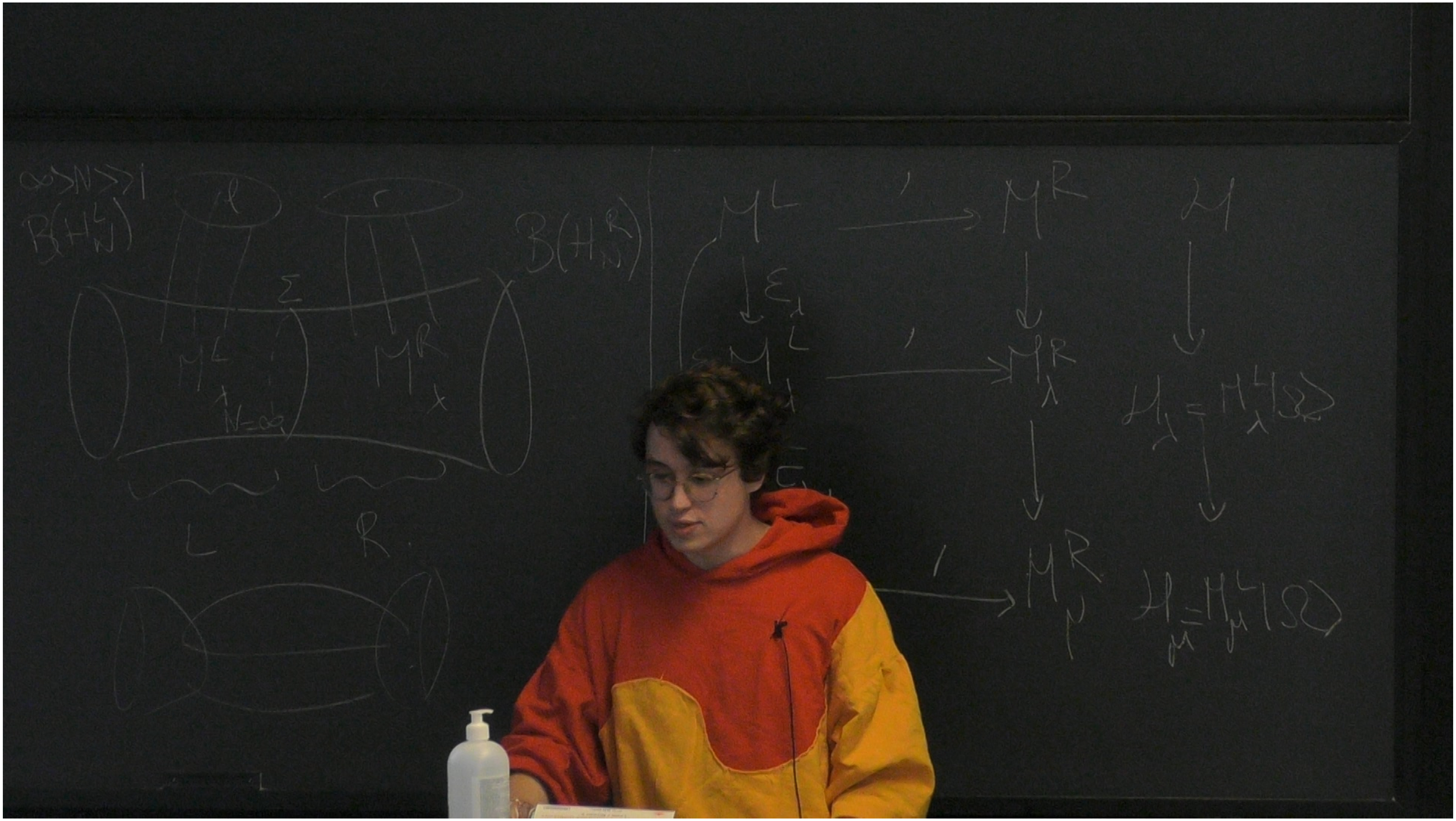
- I will follow an approach due to Akers–Penington. Consider the map $V_N : \mathcal{H}_\lambda^L \otimes \mathcal{H}_\lambda^R \longrightarrow \mathcal{H}_N^L \otimes \mathcal{H}_N^R$.

Areas in approximate codes

- I will follow an approach due to Akers–Penington. Consider the map $V_N : \mathcal{H}^L_\lambda \otimes \mathcal{H}^R_\lambda \longrightarrow \mathcal{H}^L_N \otimes \mathcal{H}^R_N$.
- Define the **Choi–Jamiolkowski state**

$$|CJ\rangle := (V_N \otimes Id)\,|MAX\rangle_{L\ell} \otimes |MAX\rangle_{Rr}\,,$$

where r, ℓ are reference systems of the same dimension as L, R.



L
R

Areas in approximate codes

- I will follow an approach due to Akers–Penington. Consider the map $V_N : \mathcal{H}_\lambda^L \otimes \mathcal{H}_\lambda^R \longrightarrow \mathcal{H}_N^L \otimes \mathcal{H}_N^R$.
- Define the **Choi–Jamiolkowski state**

$$|CJ\rangle := (V_N \otimes Id)\,|MAX\rangle_{L\ell} \otimes |MAX\rangle_{Rr}\,,$$

where r, ℓ are reference systems of the same dimension as L, R.

- The **area** associated to this subdivision is defined by the formula

$$A(\mathcal{H}_\lambda^L) = S(|CJ\rangle\,, L\ell).$$

Page 82 sur 104

Areas in approximate codes

- I will follow an approach due to Akers–Penington. Consider the map $V_N : \mathcal{H}_\lambda^L \otimes \mathcal{H}_\lambda^R \longrightarrow \mathcal{H}_N^L \otimes \mathcal{H}_N^R$.
- Define the **Choi–Jamiolkowski state**

$$|CJ\rangle := (V_N \otimes Id)\, |MAX\rangle_{L\ell} \otimes |MAX\rangle_{Rr}\,,$$

  where r, ℓ are reference systems of the same dimension as L, R.
- The **area** associated to this subdivision is defined by the formula

$$A(\mathcal{H}_\lambda^L) = S(|CJ\rangle\,, L\ell).$$

- There is only one value of area per code subspace, but the area changes depending on the **choice of code subspace**!

An approximate Ryu—Takayanagi formula

Following Akers and Penington, one can derive the following result:

- **Suppose** that for all unitary operators U_λ^L, U_λ^R in M_λ^L and M_λ^R, there exist unitary operators $\tilde{U}_\lambda^L$ and $\tilde{U}_\lambda^R$ (chosen in a measurable way) in $\mathcal{B}(\mathcal{H}_N^L)$ and $\mathcal{B}(\mathcal{H}_N^R)$ such that

$$\| V_N U_\lambda^R U_\lambda^L |_{\mathcal{H}_\lambda} - \tilde{U}_\lambda^R \tilde{U}_\lambda^L V_N |_{\mathcal{H}_\lambda} \| \leq \delta_N,$$

  where δ_N decays faster than any polynomial in $1/N$.

An approximate Ryu—Takayanagi formula

Following Akers and Penington, one can derive the following result:

- **Suppose** that for all unitary operators U_λ^L, U_λ^R in M_λ^L and M_λ^R, there exist unitary operators $\tilde{U}_\lambda^L$ and $\tilde{U}_\lambda^R$ (chosen in a measurable way) in $\mathcal{B}(\mathcal{H}_N^L)$ and $\mathcal{B}(\mathcal{H}_N^R)$ such that

$$\| V_N U_\lambda^R U_\lambda^L|_{\mathcal{H}_\lambda} - \tilde{U}_\lambda^R \tilde{U}_\lambda^L V_N|_{\mathcal{H}_\lambda}\| \leq \delta_N,$$

  where δ_N decays faster than any polynomial in $1/N$.
- **Then**, for all $|\Psi\rangle \in \mathcal{H}_\lambda$,

$$|S(|\Psi\rangle, M_\lambda^L) + A(\mathcal{H}_\lambda^L) - S(V_N|\Psi\rangle, \mathcal{B}(\mathcal{H}_N^L))| \underset{N\to\infty}{\longrightarrow} 0.$$

Running the RG flow

- The crucial point is that this formula is valid for **any choice** of cutoff λ (as long as it doesn't depend on N).

Running the RG flow

- The crucial point is that this formula is valid for **any choice** of cutoff λ (as long as it doesn't depend on N).
- Then, we have **both** formulas for $|\Psi\rangle \in \mathcal{H}_\mu$:

$$\left|S(|\Psi\rangle, M_\mu^L) + A(\mathcal{H}_\mu^L) - S(V_N|\Psi\rangle, \mathcal{B}(\mathcal{H}_N^L))\right| \underset{N\to\infty}{\longrightarrow} 0,$$

$$\left|S(|\Psi\rangle, M_\lambda^L) + A(\mathcal{H}_\lambda^L) - S(V_N|\Psi\rangle, \mathcal{B}(\mathcal{H}_N^L))\right| \underset{N\to\infty}{\longrightarrow} 0.$$

Running the RG flow

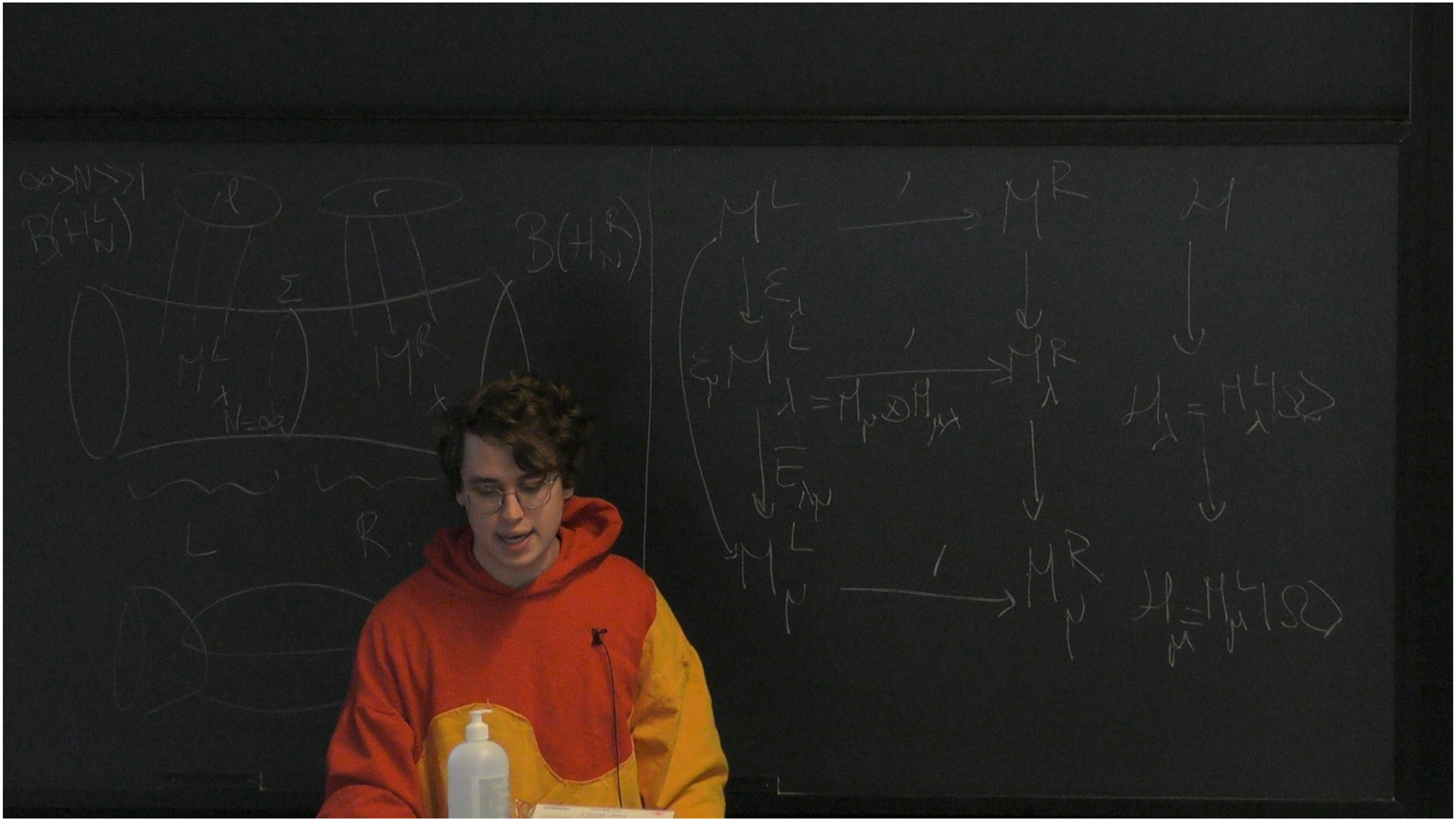
- The crucial point is that this formula is valid for **any choice** of cutoff λ (as long as it doesn't depend on N).
- Then, we have **both** formulas for $|\Psi\rangle \in \mathcal{H}_\mu$:

$$\left|S(|\Psi\rangle, M_\mu^L) + A(\mathcal{H}_\mu^L) - S(V_N |\Psi\rangle, \mathcal{B}(\mathcal{H}_N^L))\right| \underset{N\to\infty}{\longrightarrow} 0,$$

$$\left|S(|\Psi\rangle, M_\lambda^L) + A(\mathcal{H}_\lambda^L) - S(V_N |\Psi\rangle, \mathcal{B}(\mathcal{H}_N^L))\right| \underset{N\to\infty}{\longrightarrow} 0.$$

- Recall how entropy **factors out** in a CSRS:

$$S(|\Psi\rangle, M_\lambda) = S(|\Psi\rangle, M_\mu) + S(|\Psi\rangle, M_{\lambda\mu}).$$



Running the RG flow

- The crucial point is that this formula is valid for **any choice** of cutoff λ (as long as it doesn't depend on N).
- Then, we have **both** formulas for $|\Psi\rangle \in \mathcal{H}_\mu$:

$$|S(|\Psi\rangle, M_\mu^L) + A(\mathcal{H}_\mu^L) - S(V_N|\Psi\rangle, \mathcal{B}(\mathcal{H}_N^L))| \underset{N\to\infty}{\longrightarrow} 0,$$

$$|S(|\Psi\rangle, M_\lambda^L) + A(\mathcal{H}_\lambda^L) - S(V_N|\Psi\rangle, \mathcal{B}(\mathcal{H}_N^L))| \underset{N\to\infty}{\longrightarrow} 0.$$

- Recall how entropy **factors out** in a CSRS:

$$S(|\Psi\rangle, M_\lambda) = S(|\Psi\rangle, M_\mu) + S(|\Psi\rangle, M_{\lambda\mu}).$$

- We get exactly **Susskind–Uglum**!

$$|A(\mathcal{H}_\mu^L) - (S(|\Psi\rangle, M_{\lambda\mu}) + A(\mathcal{H}_\lambda^L))| \underset{N\to\infty}{\longrightarrow} 0,$$

with $M_\lambda = M_\mu \otimes M_{\lambda\mu}$.

Susskind–Uglum as ER=EPR

- This proof based on quantum error correciton provides a **reinterpretation** of Susskind–Uglum.

Susskind–Uglum as ER=EPR

- This proof based on quantum error correciton provides a **reinterpretation** of Susskind–Uglum.
- What makes it work? The **bigger** the code subspace, the **smaller** the entropy of the CJ state (i.e. the area term) will be.

Page 91 sur 104

Susskind–Uglum as ER=EPR

- This proof based on quantum error correciton provides a **reinterpretation** of Susskind–Uglum.
- What makes it work? The **bigger** the code subspace, the **smaller** the entropy of the CJ state (i.e. the area term) will be.
- This is because the missing entropy is now counted as part of the code subspace entropy!

Susskind–Uglum as ER=EPR

- This proof based on quantum error correciton provides a **reinterpretation** of Susskind–Uglum.
- What makes it work? The **bigger** the code subspace, the **smaller** the entropy of the CJ state (i.e. the area term) will be.
- This is because the missing entropy is now counted as part of the code subspace entropy!
- There is some entropy in the code that can be counted either as **bulk entropy** or as **geometry**. Whether it is one of the other amounts to making a choice of renormalization scale, which is completely **unphysical**.

Susskind–Uglum as ER=EPR

- This proof based on quantum error correciton provides a **reinterpretation** of Susskind–Uglum.
- What makes it work? The **bigger** the code subspace, the **smaller** the entropy of the CJ state (i.e. the area term) will be.
- This is because the missing entropy is now counted as part of the code subspace entropy!
- There is some entropy in the code that can be counted either as **bulk entropy** or as **geometry**. Whether it is one of the other amounts to making a choice of renormalization scale, which is completely **unphysical**.
- This is exactly **ER=EPR**: no physical distinction between **entanglement** and **geometry** in gravity.

Larger codes

- So far we assumed that the dimension of the renormalized code subspace was **fixed** in N.
- However, the Akers–Penington technology also allows to handle cases in which the dimension of the renormalized code subspace varies with N.

Larger codes

- So far we assumed that the dimension of the renormalized code subspace was **fixed** in N.
- However, the Akers–Penington technology also allows to handle cases in which the dimension of the renormalized code subspace varies with N.
- If it grows **subexponentially**, all previous assumptions can be lifted.

Larger codes

- So far we assumed that the dimension of the renormalized code subspace was **fixed** in N.
- However, the Akers–Penington technology also allows to handle cases in which the dimension of the renormalized code subspace varies with N.
- If it grows **subexponentially**, all previous assumptions can be lifted.
- If it grows **exponentially**, reconstruction becomes **state-dependent** and the map V_N can stop being (approximately) **isometric**.

Page 98 sur 104

Larger codes

- So far we assumed that the dimension of the renormalized code subspace was **fixed** in N.
- However, the Akers–Penington technology also allows to handle cases in which the dimension of the renormalized code subspace varies with N.
- If it grows **subexponentially**, all previous assumptions can be lifted.
- If it grows **exponentially**, reconstruction becomes **state-dependent** and the map V_N can stop being (approximately) **isometric**.
- Unitary reconstruction becomes too strong, only ask for **product unitaries** defined in terms of a further decomposition of the code subalgebra.

Future directions

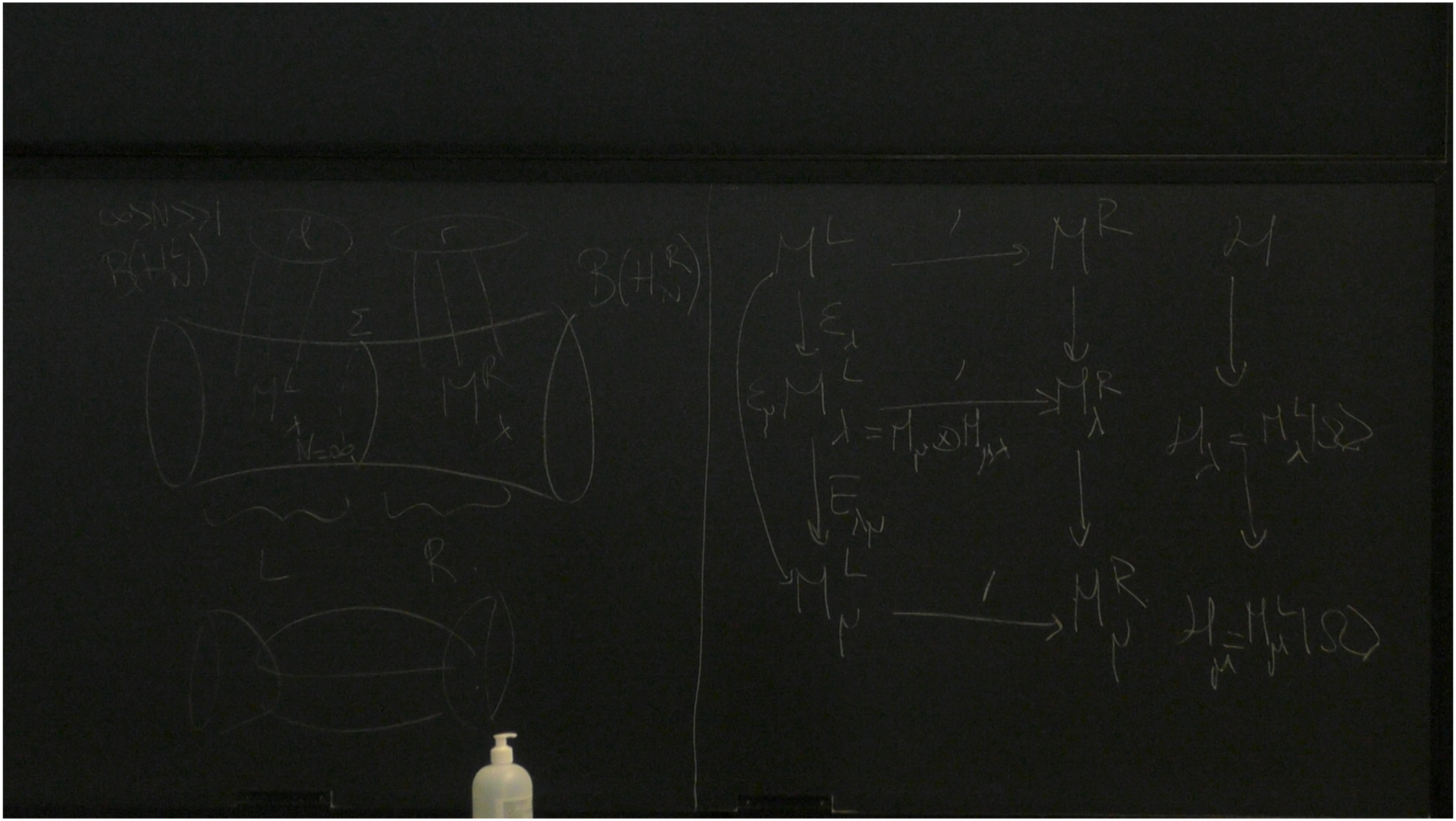
- So far this is still a proof in principle. Can one start with an actual large N theory and **construct** a CSRS? Need to embed this framework into the construction of Faulkner–Li.

Future directions

- So far this is still a proof in principle. Can one start with an actual large N theory and **construct** a CSRS? Need to embed this framework into the construction of Faulkner–Li.
- What about code subspaces that are **not invariant** under a conditional expectation? Can they be approximated by the former in some way?

Future directions

- So far this is still a proof in principle. Can one start with an actual large N theory and **construct** a CSRS? Need to embed this framework into the construction of Faulkner–Li.
- What about code subspaces that are **not invariant** under a conditional expectation? Can they be approximated by the former in some way?
- Understand the case of **large codes better**.
- QES as an instance of ER=EPR: link to the **swampland emergence proposal**?


Σ
L
R