Title: Decategorifying the singular support of coherent sheaves

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Series: Mathematical Physics

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Abstract: On smooth schemes, every coherent sheaf admits a finite resolution by vector bundles, but on singular schemes, this is no longer true. The Arinkin-Gaitsgory singular support of coherent sheaves is an invariant of coherent sheaves on certain singular spaces that measures how far a particular coherent sheaf is from having such a resolution. In this talk, I will explain how the Arinkin-Gaitsgory theory of singular support decategorifies to a notion of singular support for chains on the associated complex analytic space of our scheme, measuring the difference between cohomology and Borel-Moore homology on singular spaces. In order to do so, we take advantage of the relationship between coherent sheaves and certain categories of matrix factorizations, also know as D-branes in Landau-Ginzburg models.

Zoom link: https://pitp.zoom.us/j/95698955865?pwd=Rm9ld3FUK3hiWGUzenBuZnQyTTRYZz09

§1. Introduction

Basic idea of singular support:

- Many objects in geometry that have notions of "smoothness" and "singularity".

- Singular supp records not only location of singularities but also how they arise.

Many existing types of singular support

Ex 0: Wavefronts of distn

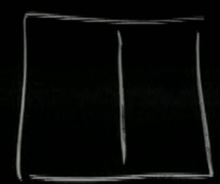
- Paley-Wiener thm:

dist u representable by smooth fn $\Longleftrightarrow$ $\mathcal{F}u$ is rapidly decreasing

$\longrightarrow$ directions of phase space space in which $\mathcal{F}u$ not rapidly decreasing resp for non-smoothness

Fu not rapidly decreasing resp. for non-smoothness

e.g. $WF(\underbrace{\delta_{y\text{-axis}}}) = N^\vee_{y\text{-axis}/\mathbb{R}^2}$

$\mathbb{R}^2$ 

Ex. 1 Sheaves on manifolds

singularity for sheaves on M = failure to be locally constant

- $\operatorname{singsupp}(\mathcal{F}) \subset T^*M$ records the directions of failure

e.g. $\operatorname{singsupp}(\mathbb{C}_{y\text{-axis}}) = N^\vee_{y\text{-axis}/\mathbb{R}^2}$

Ex 2: Arinkin-Gaitsgory sing. supp. for coherent sheaves

Deligne: if X a smooth scheme, every $\mathcal{F} \in Coh(X)$ is a perfect complex.

When X is singular, $Coh(X) \neq Perf(X)$

arise.

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derived l.c.i.

Arinkin-Gaitsgory: for Z $\overbrace{\text{quasi-smooth}}$ derived scheme

... X is singular, $\mathrm{Coh}(X) \neq \mathrm{Perf}(X)$ ... derived l.c.i.

Arinkin-Gaitsgory: for Z quasi-smooth derived scheme

associate to each $\mathcal{F} \in \mathrm{Coh}(Z)$

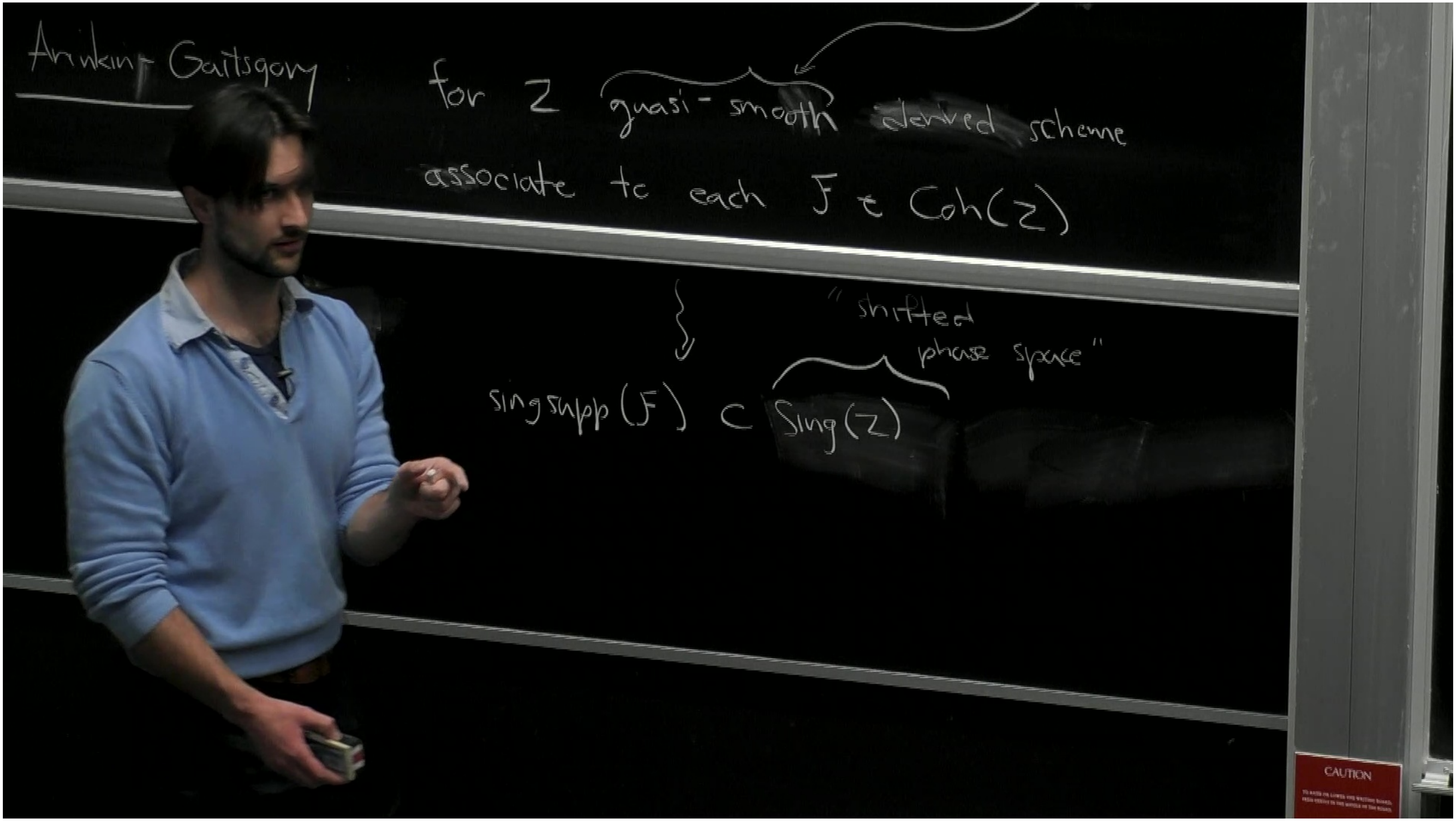
$$\mathrm{singsupp}(\mathcal{F}) \subset \underbrace{T^*[-1]Z}_{\text{"shifted phase space"}}$$

for Z $\overbrace{\text{quasi-smooth}}$ derived scheme

associate to each $\mathcal{F} \in \mathrm{Coh}(Z)$

$$\rightsquigarrow$$

$$\mathrm{singsupp}(\mathcal{F}) \subset \overbrace{T^*[-1]Z}^{\text{"shifted phase space"}} = \mathrm{Spec}_Z \mathrm{Sym}(\mathbb{T}[1])$$


Arinkin-Gaitsgory
for Z quasi-smooth derived scheme
associate to each F ∈ Coh(Z)
"shifted phase space"
singsupp(F) ⊂ Sing(Z)
CAUTION

§1.1 HKR-type theorems

Thm [Feigin–Tsygan]: $HP_\bullet(\mathrm{Perf}(Z)) \cong H^\bullet(Z)_{2\text{-per}}$

Thm [Preygel]: $HP_\bullet(\mathrm{Coh}(Z)) \cong H^{BM}_\bullet(Z)_{2\text{-per.}}$

§1.1 HKR-type theorems

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Thm [Preygel] : $HP_\bullet(\mathrm{Coh}(Z)) \cong H^{BM}_\bullet(Z)_{2\text{-per.}}$

$HP_\bullet$ = periodic cyclic homology

- gives "de Rham cohomology" of dg-category

- naturally 2-periodic.

Def: $\mathrm{Coh}_\Lambda(Z) \subset \mathrm{Coh}(Z)$ is the full subcat. on objects whose sing supp is contained in $\Lambda \subset \mathrm{Sing}(Z)$

– naturally 2-periodic.

Def: Z quasi-smooth

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Goal: decategorify $\mathrm{Coh}_\Lambda(Z)$:

find a sing supp. theory for $H_\cdot^{BM}(Z)$ such that

$$HP_0(\mathrm{Coh}_\Lambda(Z)) \cong \left\{ \begin{array}{l} \text{classes in BM hom.} \\ \text{w/ certain prescribed sing. supp.} \end{array} \right\}$$

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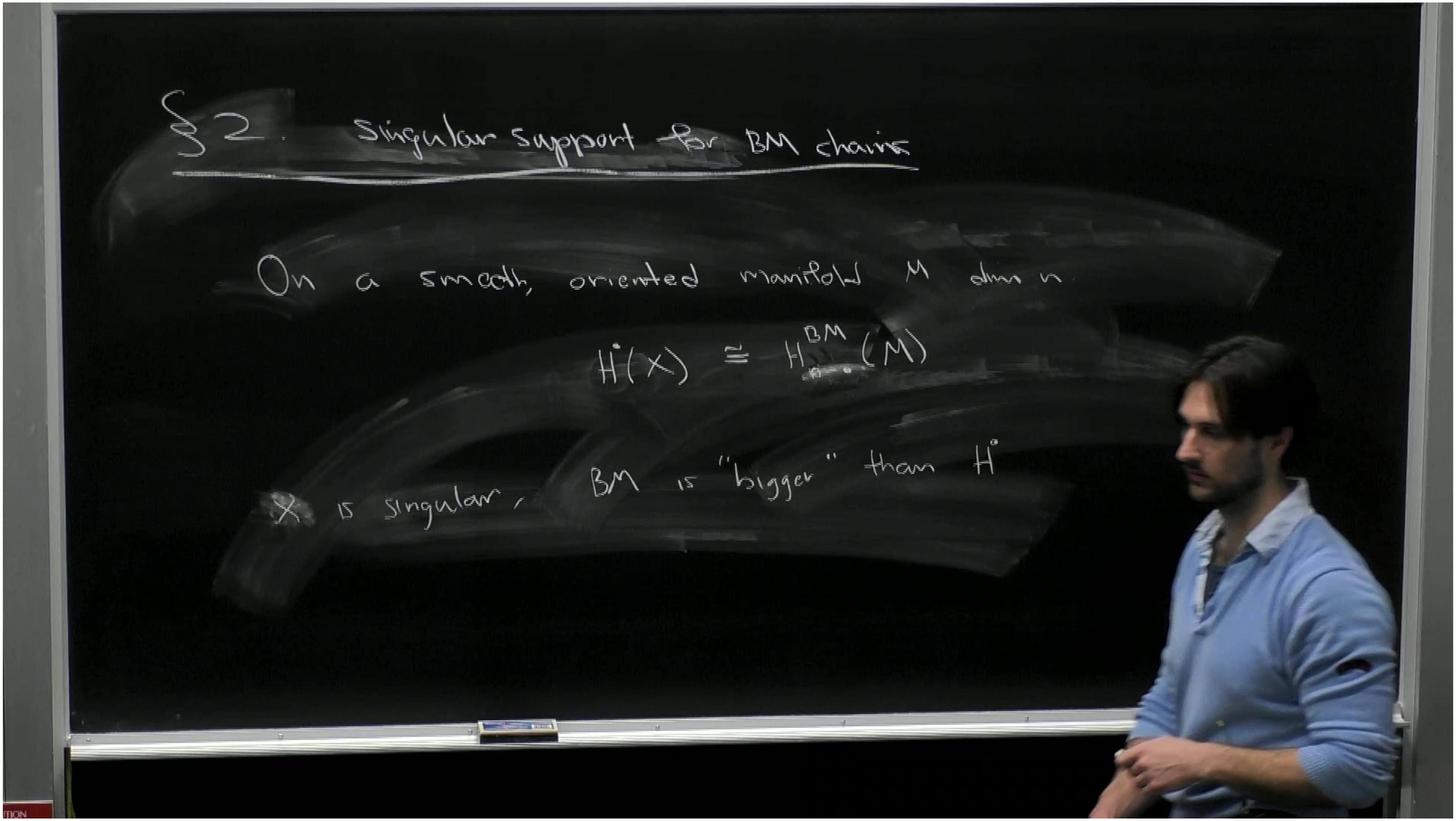
§2. Singular support for BM chains
On a smooth, oriented manifold M dim n
H°(X) ≅ H^BM (M)
X is singular, BM is "bigger" than H°

$H_{\bullet}^{BM}$ has two nontrivial groups

Failure of BM chain to have sing. cochain representative ~ kind of singularity

Goal: define sing supp for BM chains on X.

Desiderata: intrinsic to X.

Look at zero fiber of map

$$X \xrightarrow{f} V$$ map of $\mathbb{C}$-schemes

smooth complex (X)

f.d. vect. space (V)

$\rightsquigarrow$ consider $Z(f) :=$ derived zero locus of f.

Look at zero fiber of map

$$\underset{\substack{\text{smooth}\\ \text{complex}}}{X} \xrightarrow{f} \underset{\substack{\text{f.d.}\\ \text{vect. space}}}{V} \quad \text{map of } \mathbb{C}\text{-schemes}$$

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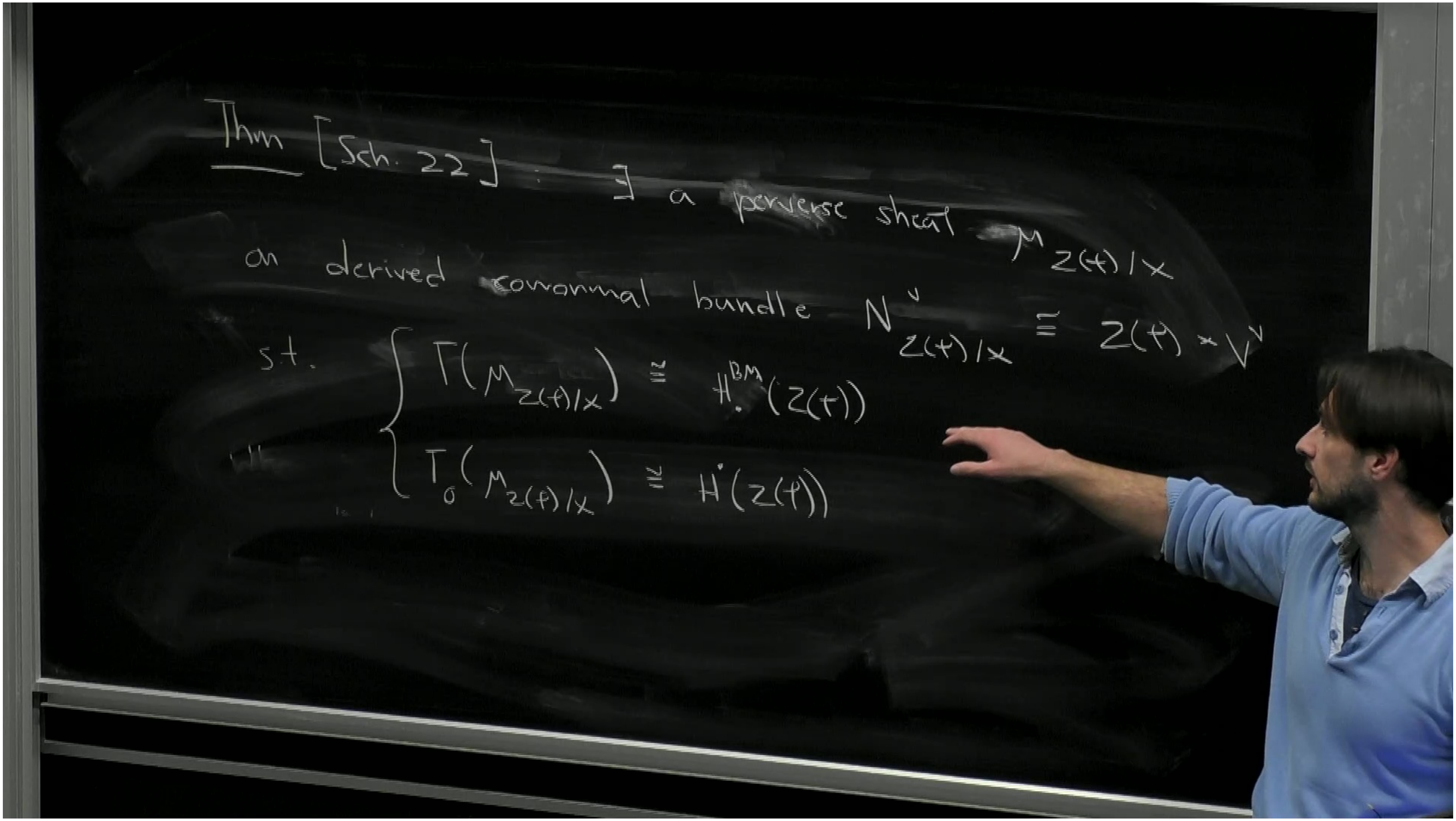
$\mathrm{Sing}(Z(f))$.


Thm [Sch. 22]: $\exists$ a perverse sheaf $\mu_{Z(t)/X}$
on derived conormal bundle $N^\vee_{Z(t)/X} \cong Z(t) \times V^\vee$
s.t. $\begin{cases} \Gamma(\mu_{Z(t)/X}) \cong H^{BM}_\bullet(Z(t)) \\ \Gamma_0(\mu_{Z(t)/X}) \cong H^\bullet(Z(t)) \end{cases}$

Idea: by varying support conditions

$$0 \subseteq \Lambda \subseteq N^\vee_{Z(f)|X},$$

get $H^\Lambda_\bullet(Z(f)) = \Gamma_\Lambda(\mathcal{M}_{Z(f)|X})$.

$\rightsquigarrow$ sing. supp of BM chain σ is the smallest Λ s.t. $\mathrm{im}(\sigma) \in H^\Lambda_\bullet(Z(f))$.

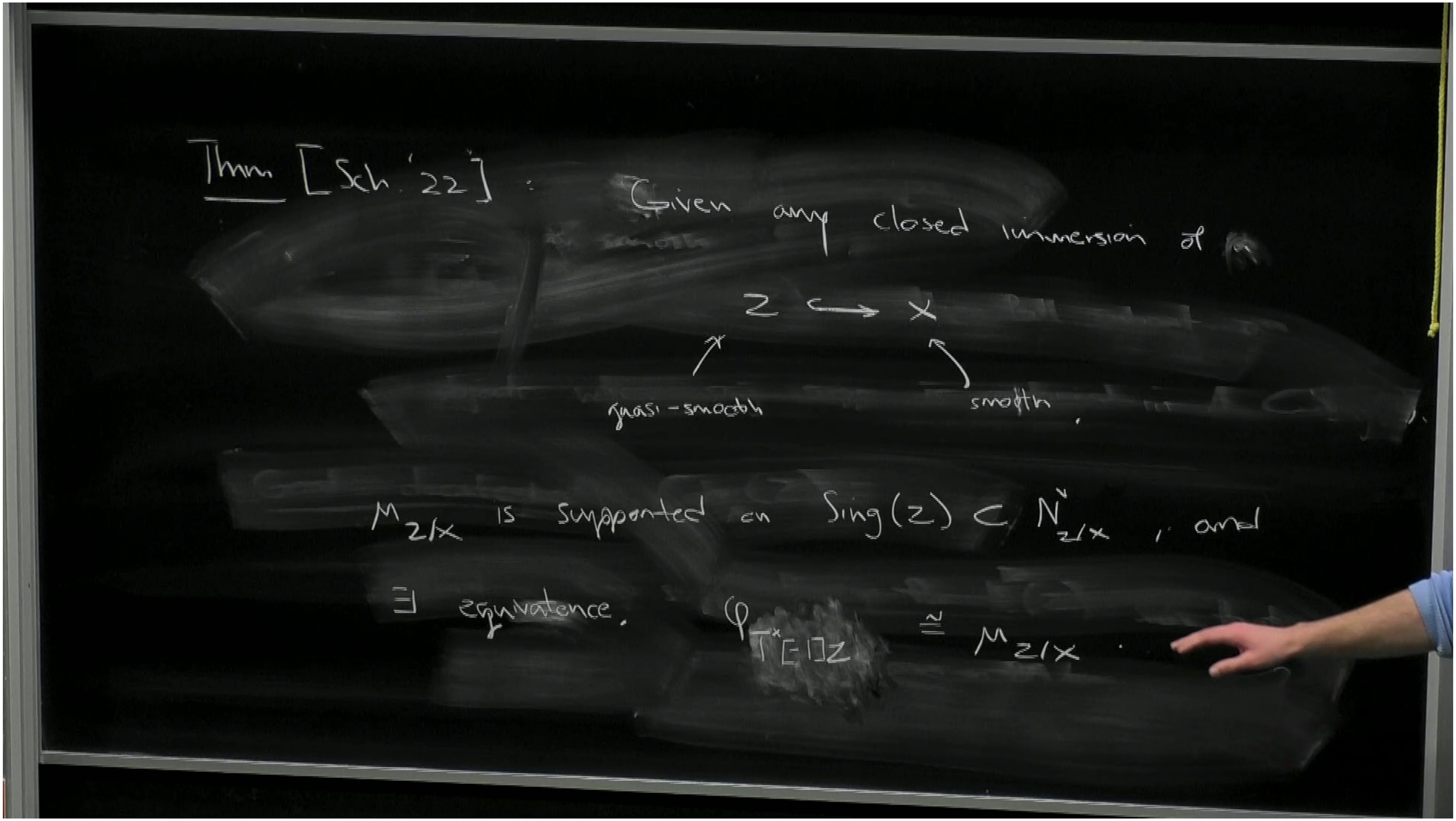
Thm [Sch '22]: Given any closed immersion of

$$Z \hookrightarrow X$$

quasi-smooth   smooth,

$M_{Z/X}$ is supported on $\mathrm{Sing}(Z) \subset N^\vee_{Z/X}$, and

$\exists$ equivalence $\varphi_{T^*[-1]Z} \cong M_{Z/X}$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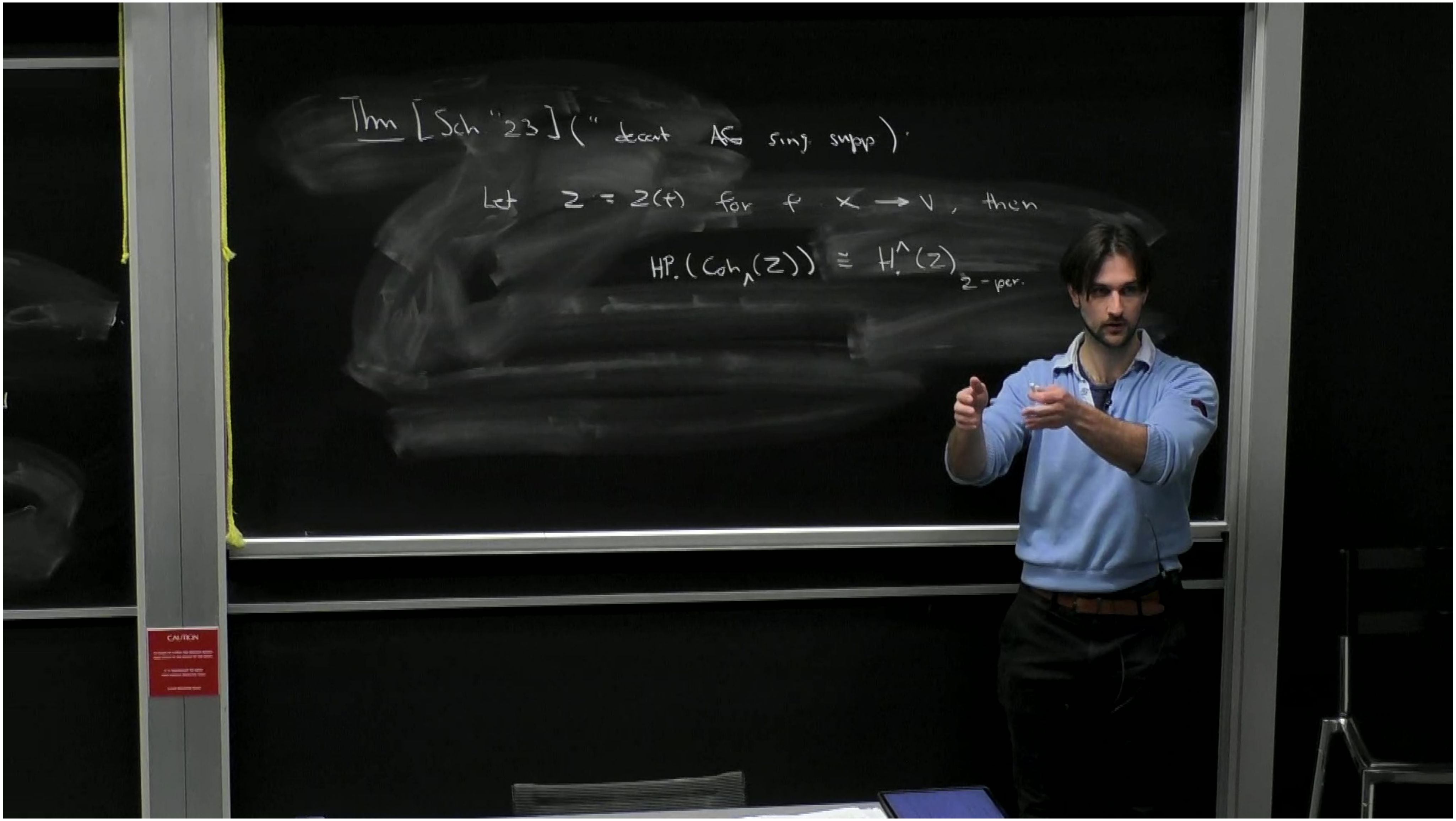
Thm [Sch '23]
decat
sing supp
Let Z = Z(f) for f: X → V, then
HP.(Coh_Λ(Z)) ≅ H^(Z) 2-per.
CAUTION