Title: On past geodesic (in)completeness, spacetime (in)extendibility, and singularities in inflationary cosmology

Speakers: Jerome Quintin

Series: Cosmology & Gravitation

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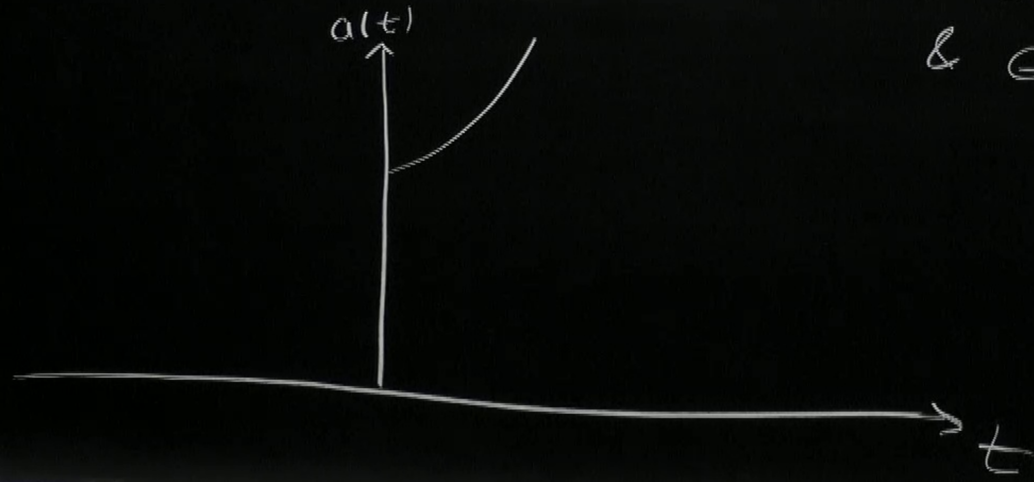
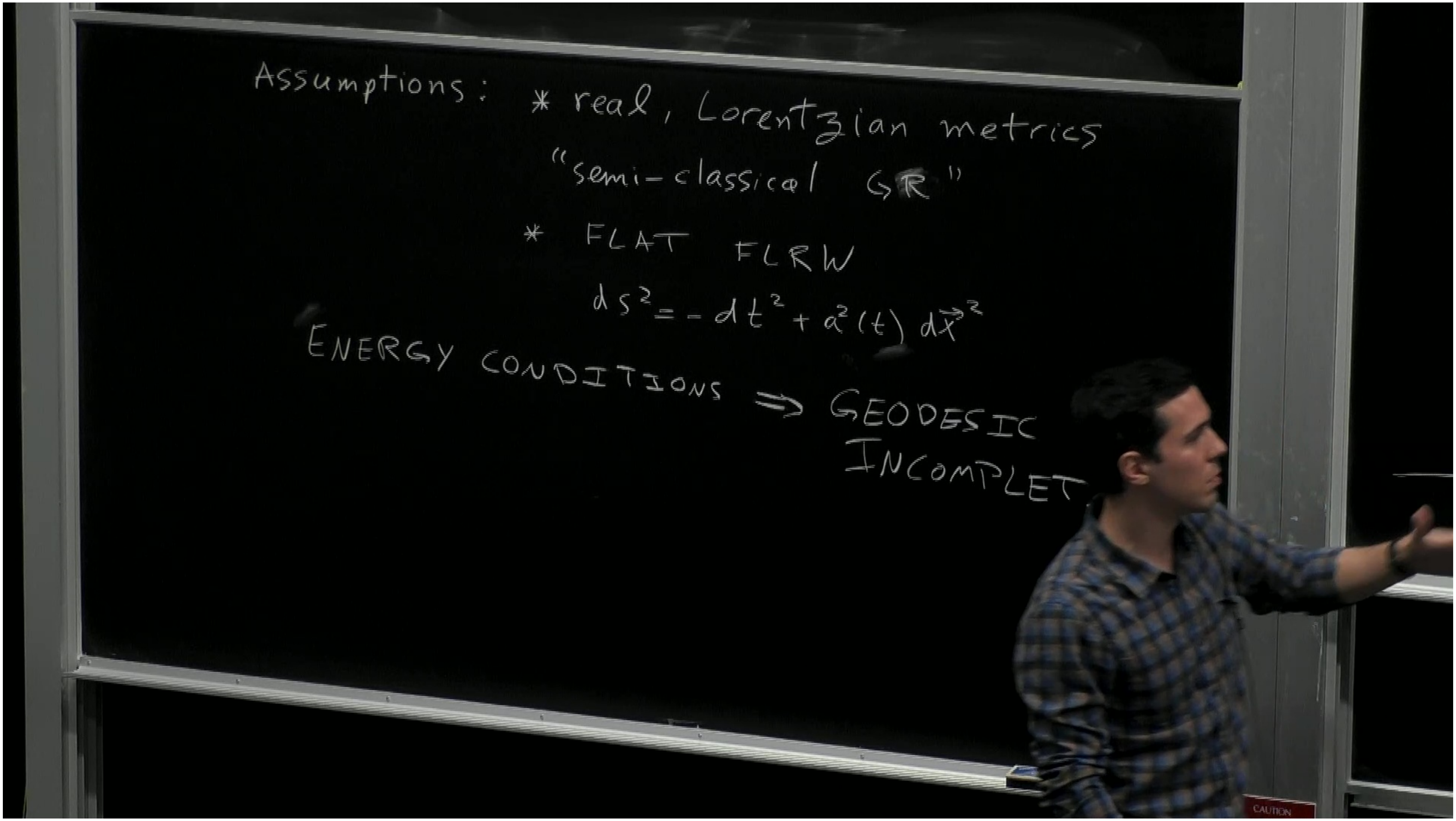
Abstract: Inflationary cosmology is notoriously past geodesically incomplete in many situations. However, it is generally unknown whether the geodesic incompleteness implies the existence of an initial spacetime curvature singularity or whether the spacetime may be extended beyond its null past boundary. In homogeneous and isotropic cosmology with flat spatial sections, we classify which past inflationary histories have a scalar curvature singularity and which might be extendible/non-singular. We derive rigorous extendibility criteria of various regularity classes for quasi-de Sitter spacetimes that evolve from infinite proper time in the past. Beyond homogeneity and isotropy, we show that continuous extensions respecting the Einstein field equations with a perfect fluid must have the equation of state of a de Sitter universe asymptotically. An interpretation of our results is that past-eternal inflationary scenarios are most likely physically singular, except in very special situations.

Zoom link: https://pitp.zoom.us/j/98334550627?pwd=UnR3eUxZRFZpeEZoOGEwNkJuc0M0UT09

ON PAST GEODESIC (IN)COMPLETENESS, SPACETIME (IN) EXTENDIBILITY, AND SINGULARITIES IN INFLATION

Based ON WORK TO APPEAR WITH ERIC LING & GHAZAL G.




Assumptions: * real, Lorentzian metrics
"semi-classical GR"
* FLAT FLRW
$ds^2 = -dt^2 + a^2(t)\, d\vec{x}^2$
ENERGY CONDITIONS ⇒ GEODESIC
INCOMPLET

ON PAST GEODESIC (IN)COMPLETENESS,
SPACETIME (IN)EXTENDIBILITY,
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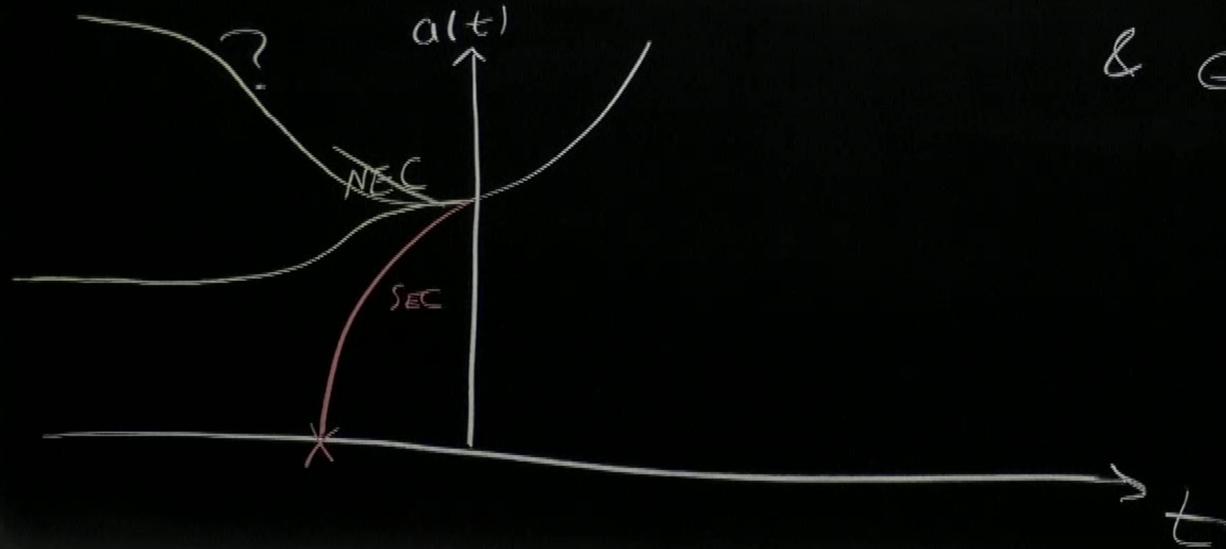

GEODESIC COMPLETENESS:

$$\lambda = \int dt\, a(t) \to -\infty \quad \text{as } t \to -\infty$$

$$\text{if } \begin{cases} a \to \infty \\ a \to \text{const.} \end{cases}$$

GEO. INCOMPLETE:

$$a \searrow 0 \quad \text{as } t \to t_i > -\infty \quad \Rightarrow \quad |R_{\mu\nu} R^{\mu\nu}| \to \infty$$

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$$a \searrow 0 \quad \text{as } t \to -\infty \quad \Rightarrow \quad ?$$

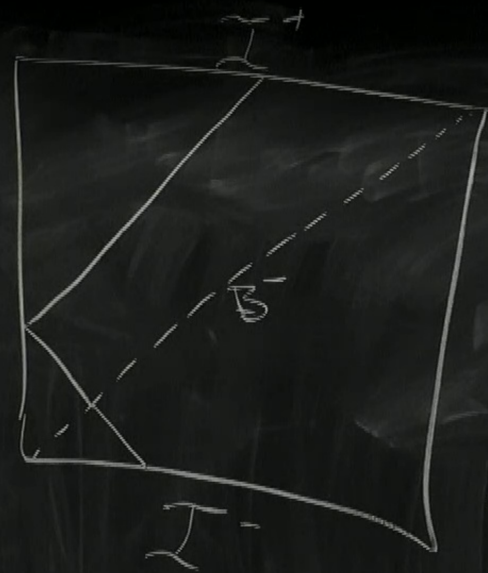
EXTENDIBILITY

de Sitter: $ds^2 = -dt^2 + e^{2t} d\vec{x}^2$

$(\Lambda = 3)$ $\quad \lambda = \int dt\, a(t) = e^{t} \Rightarrow$ finite as $t \to -\infty$

$\Rightarrow$ extend. to $ds^2 = -dt^2 + \cosh^2 t \, d\Omega^2_{(3)}$

$\Rightarrow$ non-sing.



cood. :

$$\begin{cases} d\lambda \equiv a\, dt \\ \qquad = a^2\, d\eta \\ dv \equiv d\eta + dr \end{cases}$$

$$ds^2 = a^2(\eta)\left(-d\eta^2 + dr^2 + r^2 d\Omega^2_{(2)}\right)$$

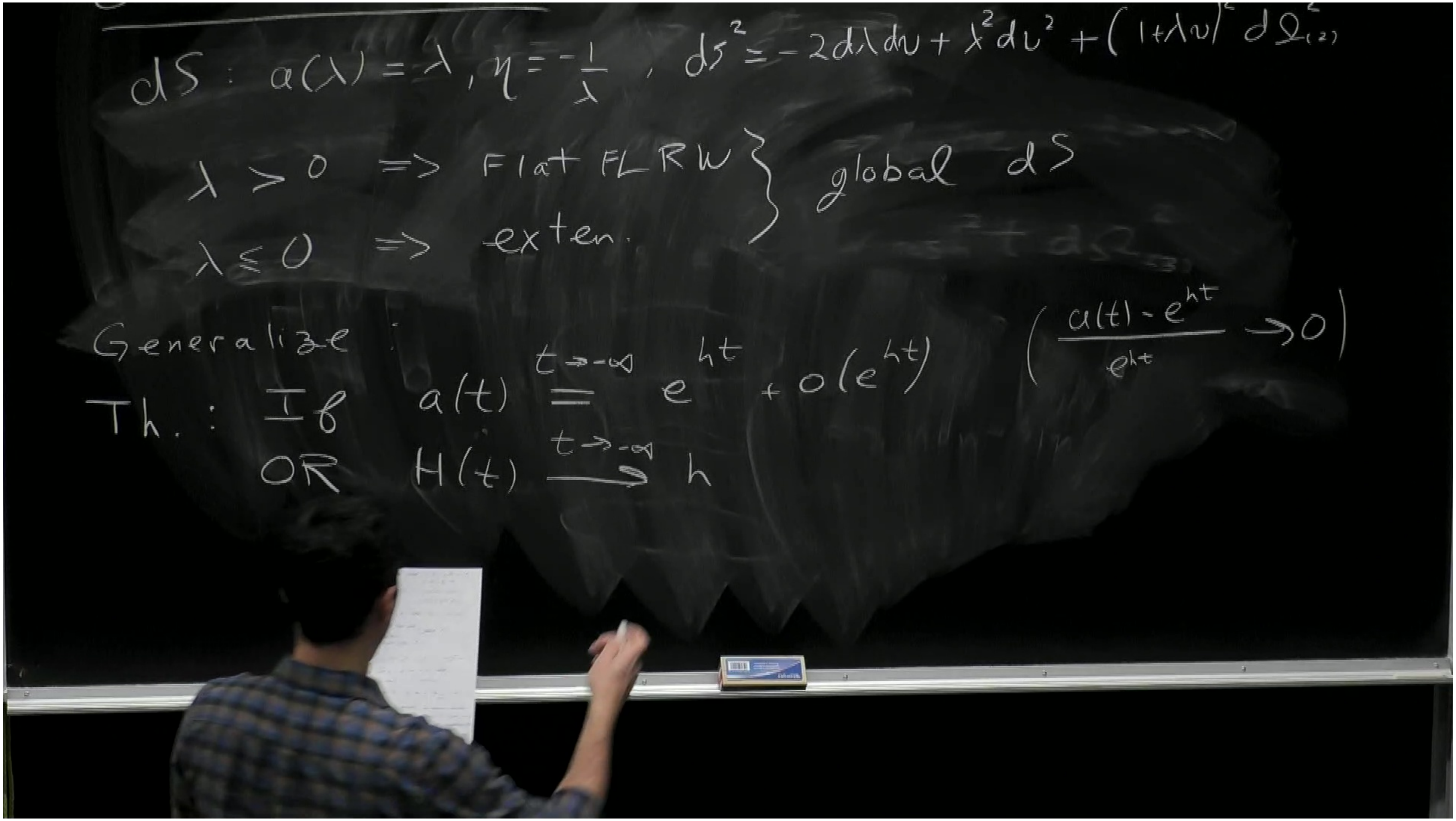
$$= -2\,d\lambda\, dv + a^2(\lambda)\, dv^2$$

$$+ a^2(\lambda)\left(v - \eta(\lambda)\right)^2 d\Omega^2_{(2)}$$

EXTENDIBILITY

$dS: a(\lambda) = \lambda, \eta = -\frac{1}{\lambda}, ds^2 = -2d\lambda dv + \lambda^2 dv^2 + (1+\lambda v)^2 d\Omega^2_{(2)}$

$\lambda > 0 \Rightarrow$ Flat FLRW



dS : $a(\lambda) = \lambda$, $\eta = -\frac{1}{\lambda}$, $ds^2 = -2d\lambda dv + \lambda^2 dv^2 + (1+\lambda v)^2 d\Omega^2_{(2)}$
$\lambda > 0 \Rightarrow$ Flat FLRW } global dS
$\lambda \leq 0 \Rightarrow$ exten.
Generalize :
Th. : If $a(t) \overset{t\to-\infty}{=} e^{ht} + o(e^{ht})$ $\left(\frac{a(t) - e^{ht}}{e^{ht}} \to 0 \right)$
OR $H(t) \xrightarrow{t\to-\infty} h$

EXTENDIBILITY

dS : $a(\lambda) = \lambda$, $\eta = -\frac{1}{\lambda}$, $ds^2 = -2\,d\lambda\,dv + \lambda^2 dv^2 + (1+\lambda v)^2 d\Omega^2_{(2)}$

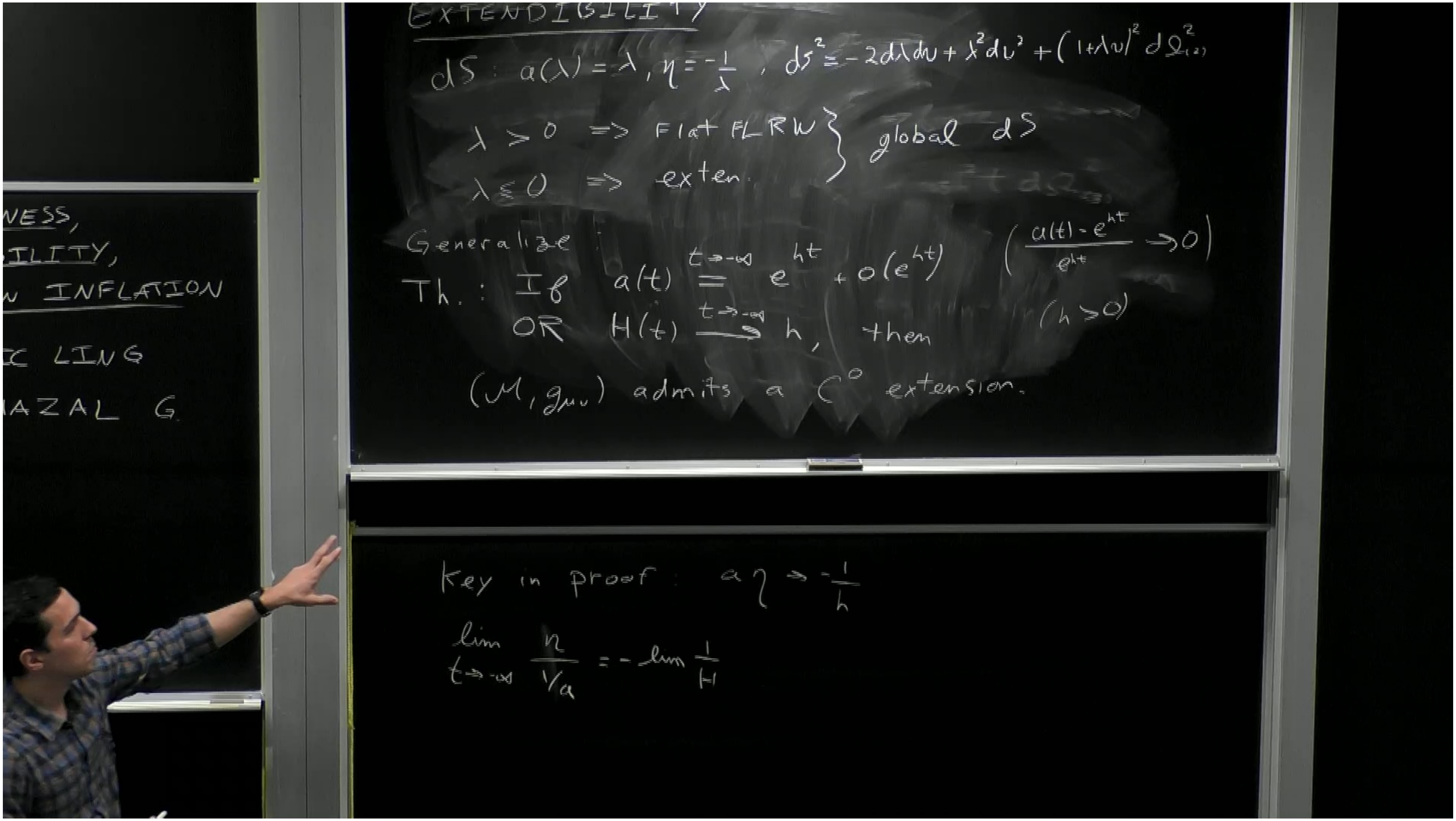
$$\left.\begin{array}{l} \lambda > 0 \Rightarrow \text{Flat FLRW} \\ \lambda \leq 0 \Rightarrow \text{exten.} \end{array}\right\} \text{global } dS$$

Generalize :

Th. : If $a(t) \overset{t \to -\infty}{=} e^{ht} + o(e^{ht})$ $\left(\frac{a(t) - e^{ht}}{e^{ht}} \to 0 \right)$

OR $H(t) \xrightarrow{t \to -\infty} h$, then

$(\mathcal{M}, g_{\mu\nu})$ admits a C^0 extension.


EXTENDIBILITY
dS : a(λ) = λ, η = -1/λ , ds² = -2dλdu + λ²du² + (1+λu)² dΩ²(2)
λ > 0 => Flat FLRW
λ ≤ 0 => exten.
global dS
Generalize :
Th. : If a(t) = e^{ht} + o(e^{ht}) (t→-∞)
((a(t) - e^{ht}) / e^{ht} → 0)
OR H(t) → h (t→-∞), then
(h > 0)
(M, g_μν) admits a C⁰ extension.
Key in proof : aη → -1/h
lim (t→-∞) η / (1/a) = - lim 1/H
NESS,
ILITY,
INFLATION
LING
AZAL G.

Key in proof: $a\eta \to -\frac{1}{H}$

$$\lim_{t\to-\infty} \frac{\eta}{1/a} = -\lim \frac{1}{H}$$

Th.: If $\displaystyle\lim_{t\to-\infty} \frac{\dot H(t)}{a^2(t)}$ is finite, then $(\mathcal{M}, g_{\mu\nu})$ admits a C^2 extension.

Key in proof $a\eta \to -\frac{1}{h}$

$$\lim_{t\to-\infty} \frac{\eta}{1/a} = -\lim \frac{1}{H}$$

Th.: If $\lim\limits_{t\to-\infty} \dfrac{\dot{H}(t)}{a^2(t)}$ is finite, then $(\mathcal{M}, g_{\mu\nu})$

admits a C^2 extension.

$\Rightarrow$ more than "asympt. dS"

$(H \to h, \ \dot{H} \to 0)$

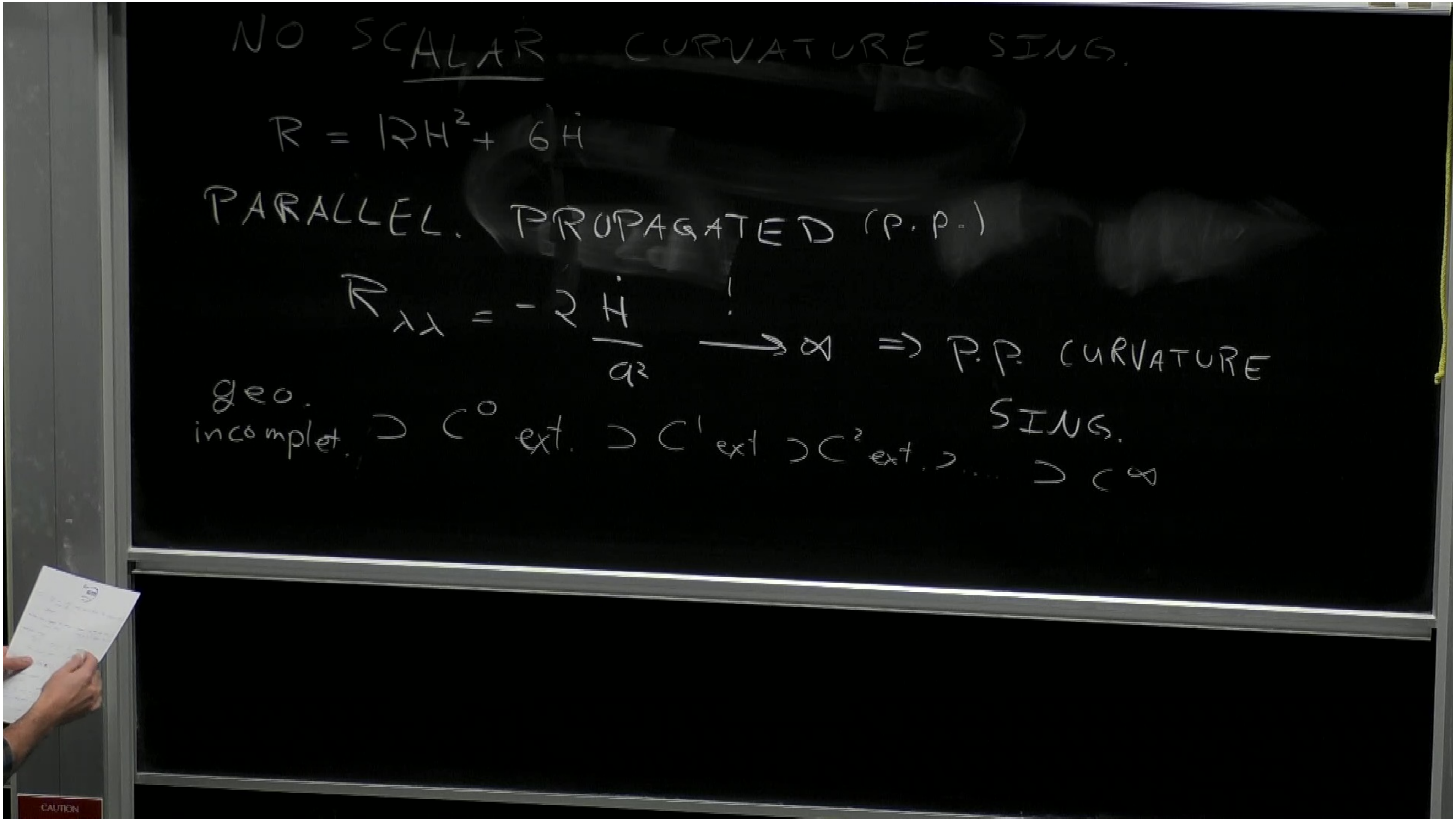
E.g: $a(t) = e^{ht} + e^{2ht}$

NO SCALAR CURVATURE SING.

$R = 12H^2 + 6\dot{H}$

PARALLEL PROPAGATED (P.P.)

$R_{\lambda\lambda} = -2\frac{\dot{H}}{a^2} \longrightarrow \infty \Rightarrow$ P.P. CURVATURE SING.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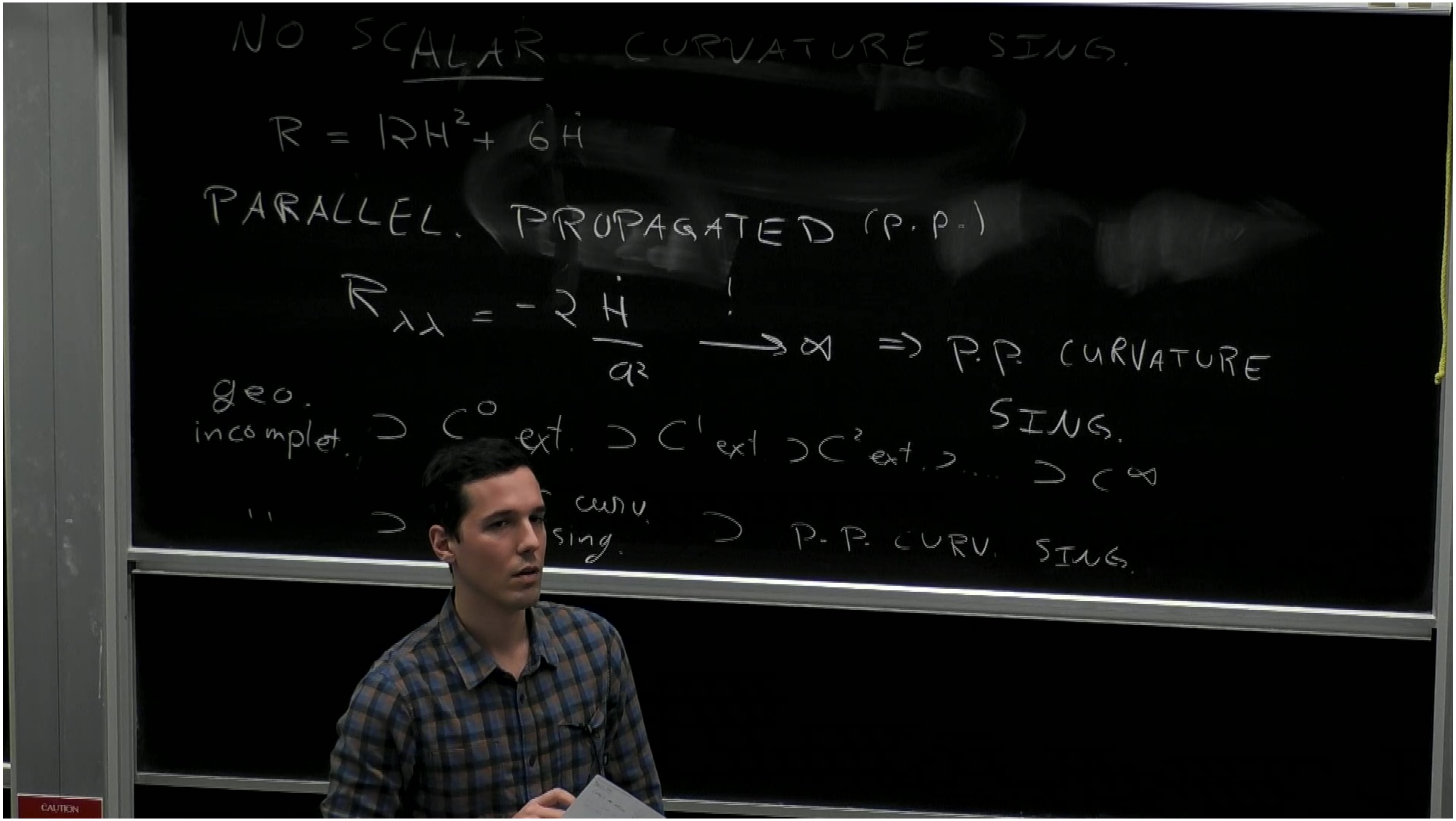
geo. incomplet. $\supset C^0$ ext. $\supset C^1$ ext $\supset C^2$ ext. $\supset \dots \supset C^\infty$


NO SCALAR CURVATURE SING.
R = 12H² + 6Ḣ
PARALLEL PROPAGATED (P.P.)
R_λλ = -2 Ḣ/a² → ∞ ⇒ P.P. CURVATURE SING.
geo. incomplet. ⊃ C⁰ ext. ⊃ C¹ ext ⊃ C² ext. ⊃ ... ⊃ C^∞
curv. sing. ⊃ P.P. CURV. SING.

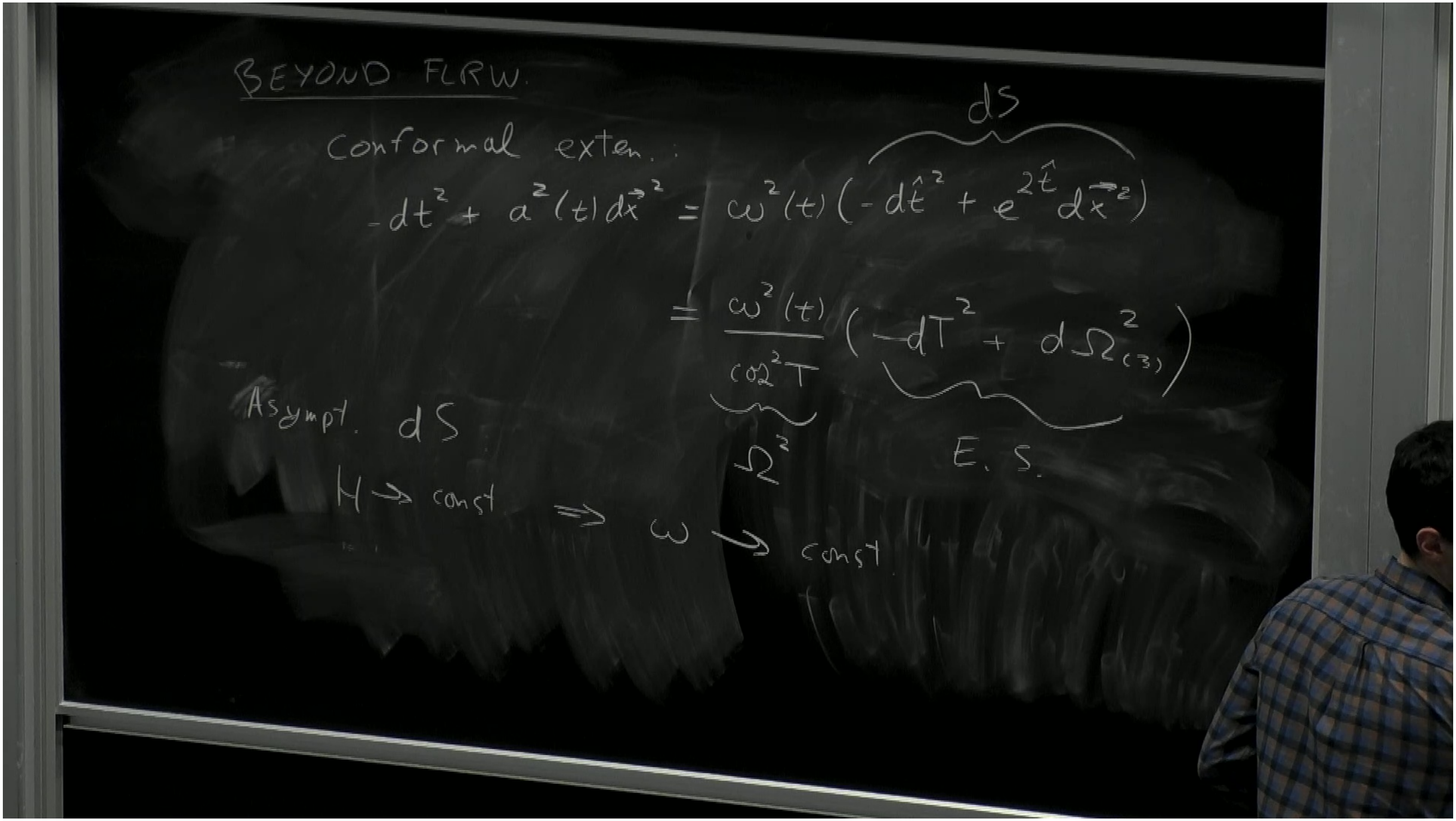
$$\lim_{t\to-\infty} \frac{n}{1/a} = -\lim \frac{1}{H}$$

Th.: If $\lim\limits_{t\to-\infty} \frac{\dot{H}(t)}{a^2(t)}$ is finite, then $(\mathcal{M}, g_{\mu\nu})$ admits a C^2 extension.

$\Rightarrow$ more than "asympt. dS"

$(H \to h,\ \dot{H} \to 0)$

Th.: $\dot{H}/a^2$ is analytic in a as $a \searrow 0 \Rightarrow C^\infty$


BEYOND FLRW.
dS
conformal exten.:
$-dt^2 + a^2(t)\,d\vec{x}^2 = \omega^2(t)\left(-d\hat{t}^2 + e^{2\hat{t}}\,d\vec{x}^2\right)$
$= \frac{\omega^2(t)}{\cos^2 T}\left(-dT^2 + d\Omega^2_{(3)}\right)$
Ω^2
E. S.
Asympt. dS
$H \to$ const $\Rightarrow \omega \to$ const

Key in proof $a\eta \to -\frac{1}{h}$

$$\lim_{t\to-\infty} \frac{\eta}{1/a} = -\lim \frac{1}{H}$$

Th.: If $\displaystyle\lim_{t\to-\infty} \frac{\dot H(t)}{a^2(t)}$ is finite, then $(\mathcal{M}, g_{\mu\nu})$ admits a C^2 extension.

$\to$ more than "asympt. dS"

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E.g: $a(t) = e^{ht} + e^{2ht}$

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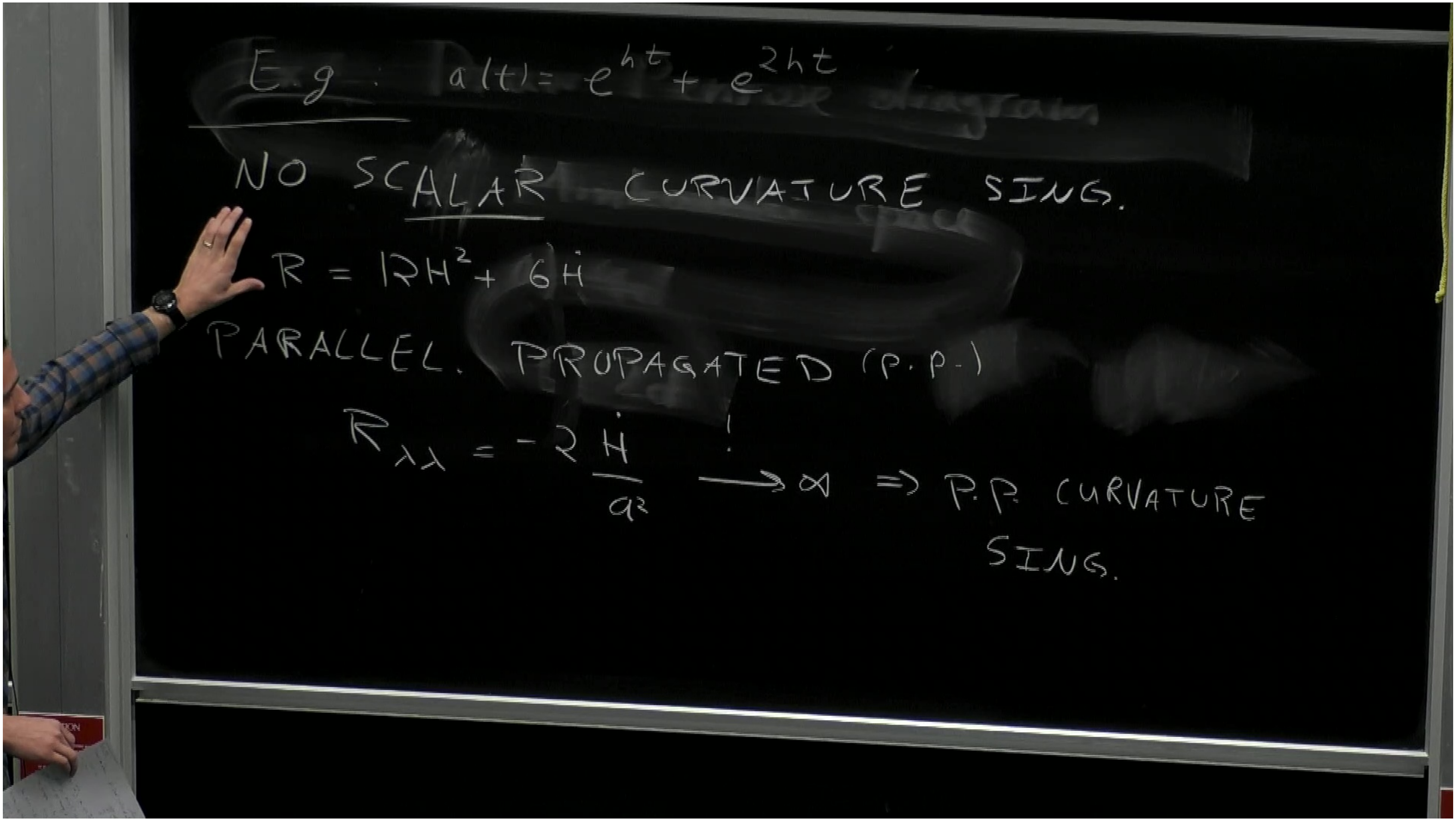
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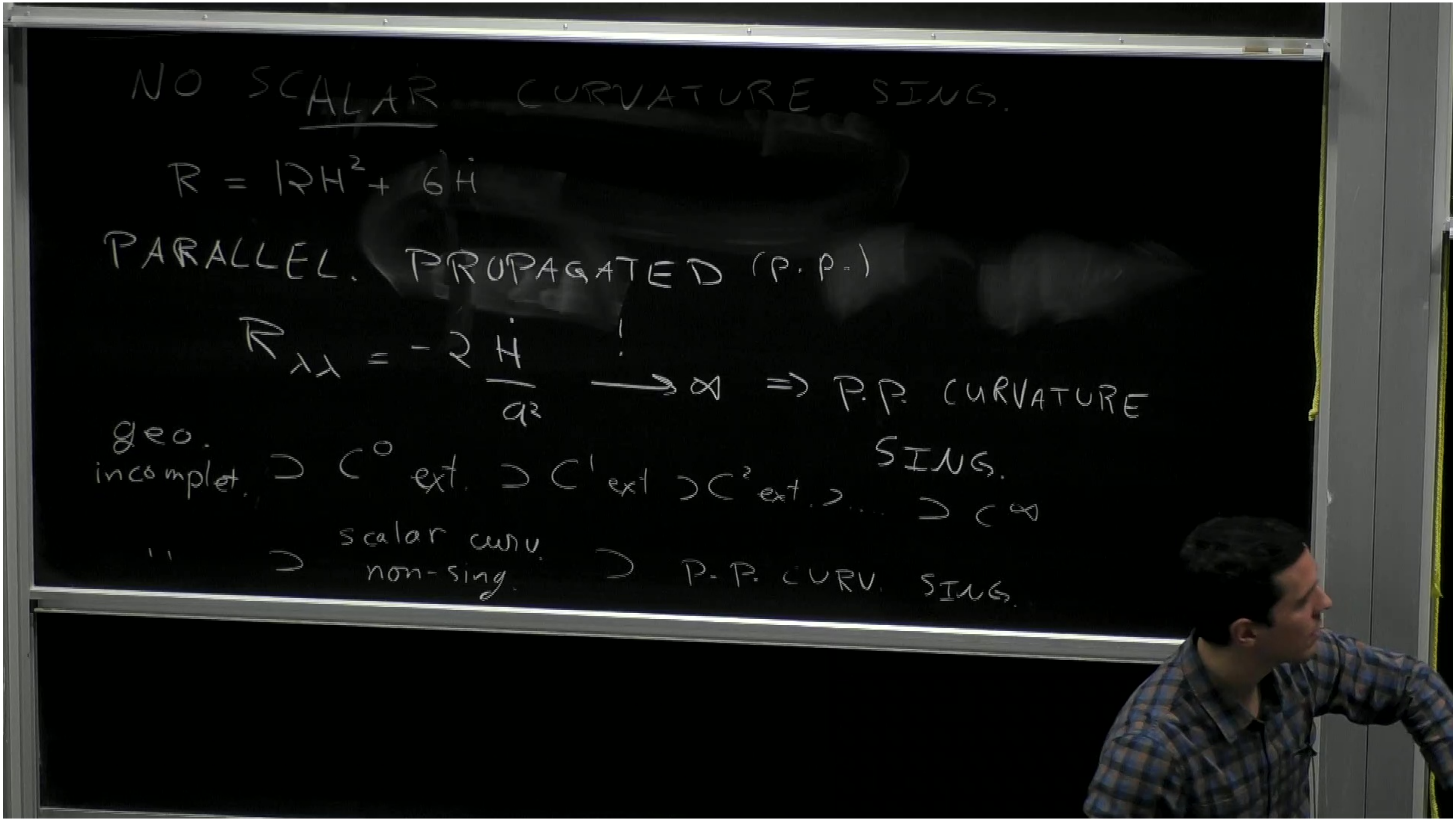
$$R = 12H^2 + 6\dot{H}$$

PARALLEL PROPAGATED (P.P.)

$$R_{\lambda\lambda} = -2\frac{\dot{H}}{a^2} \; ! \longrightarrow \infty \Rightarrow \text{P.P. CURVATURE SING.}$$

geo. incomplet. $\supset C^0$ ext. $\supset C^1$ ext $\supset C^2$ ext. $\supset \dots \supset C^\infty$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geo. incomplet. $\supset C^0$ ext. $\supset C^1$ ext $\supset C^2$ ext. $\supset \dots \supset C^\infty$
" $\supset$ scalar curv. non-sing. $\supset$ P.P. CURV. SING.

$$\lim_{t\to-\infty} \frac{\eta}{1/a} = -\lim \frac{1}{H}$$

Th.: If $\displaystyle\lim_{t\to-\infty} \frac{\dot H(t)}{a^2(t)}$ is finite, then $(\mathcal{M}, g_{\mu\nu})$

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BEYOND FLRW.

Conformal exten.:

$$-dt^2 + a^2(t)\, d\vec{x}^{\,2} = \omega^2(t) \overbrace{\left(-d\hat{t}^{\,2} + e^{2\hat{t}}\, d\vec{x}^{\,2}\right)}^{dS}$$

$$= \frac{\omega^2(t)}{\underbrace{\cos^2 T}_{\Omega^2}} \underbrace{\left(-dT^2 + d\Omega^2_{(3)}\right)}_{E.\,S.}$$

Asympt. dS

$H \rightarrow$ const $\Rightarrow$ $\omega \rightarrow$ const

Th. Let $(\mathcal{M}, g_{\mu\nu})$ has a C^0 conformal extension

with conformal factor Ω and past boundary $\mathcal{B}^-$.

Main assumptions: (a) $G_{\mu\nu} = M_{Pl}^{-2}\left((\rho+p)u_\mu u_\nu + p\, g_{\mu\nu}\right)$

(b) ρ, p, Ric conf ext. (C^0)

Then $p = -\rho$ on $\mathcal{B}^-$

If $p \neq -\rho$

with conformal factor Ω and past boundary $\mathcal{B}^-$.

Main assumptions: (a) $G_{\mu\nu} = M_{Pl}^{-2}\left((\rho+p)u_\mu u_\nu + p\, g_{\mu\nu}\right)$

(b) ρ, p, Ric conf. ext. (C^0)

Then $p = -\rho$ on $\mathcal{B}^-$

If $p \neq -\rho \Rightarrow$ inextendible?

$$2\dot{H} = -(\rho+p) \longrightarrow 0 \Rightarrow p \to -\rho$$

$$3H^2 = \rho = \Lambda - \frac{k}{a^2} + \frac{\rho_m^0}{a^3} + \frac{\rho_r^0}{a^4} + \frac{\sigma_0^2}{a^6}$$