Title: 4-dimensional covariant Loop Quantum Gravity and complex Chern-Simons theory

Speakers: Muxin Han

Series: Quantum Gravity

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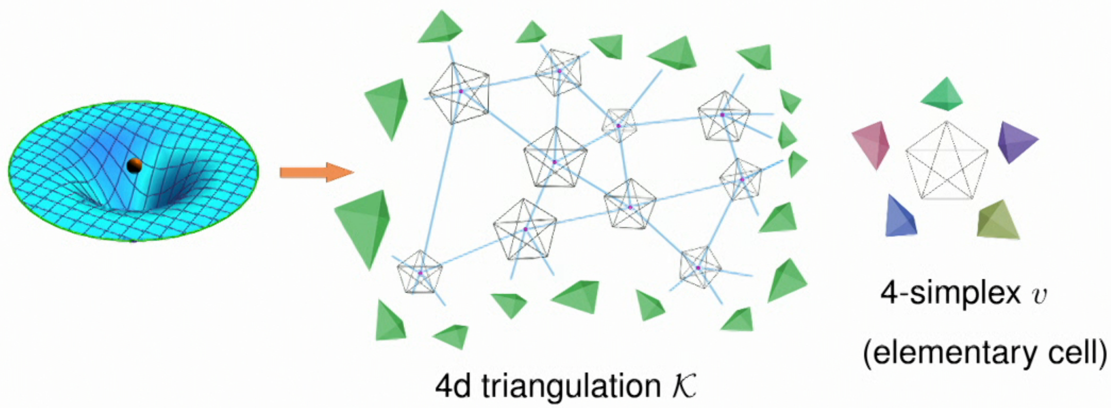
Abstract: We present an improved formulation of 4-dimensional Lorentzian spinfoam quantum gravity with cosmological constant. The construction of spinfoam amplitudes uses the state-integral model of PSL(2,C) Chern-Simons theory and the implementation of simplicity constraint. The formulation has 2 key features: (1) spinfoam amplitudes are all finite, and (2) With suitable boundary data, the semiclassical asymptotics of the vertex amplitude has two oscillatory terms, with phase plus or minus the 4-dimensional Lorentzian Regge action with cosmological constant for the constant curvature 4-simplex.

Zoom link: https://pitp.zoom.us/j/92219187641?pwd=RUsvcWo2SHFmVTE3NmxDMUZlVEV2UT09

Overview of Loop Quantum Gravity and Spinfoams

- Loop Quantum Gravity (LQG) is a *background-independent* and *non-perturbative* approach to quantum gravity in 3+1 dimensions.

- Background independence: Quantum Gravity = Quantum Spacetime Geometry



- Non-perturbative quantum gravity: We construct the full transition amplitude of quantum gravity instead of perturbative expansion

Spinfoam Amplitude: $$A(\mathcal{K}) = \sum_{\{j,i\}} \prod_f A_f(j) \prod_e A_e(j,i) \prod_v A_v(j,i)$$

A_f: amplitude associated to each triangle f in $\mathcal{K}$. A_e: amplitude associated to each tetrahedron e. A_v: amplitude associated to each 4-simplex v. $\{j,i\}$: intermediate states (spin-network state in LQG)

Overview of Loop Quantum Gravity and Spinfoams

Plebanski-Holst theory of GR: $$S=-\frac{1}{2}\int_{\mathcal{B}_4}\left(\mathrm{Tr}\left[\left(\star B+\frac{1}{\gamma}B\right)\wedge\mathcal{F}(\mathcal{A})\right]+\varphi_{IJKL}B^{IJ}\wedge B^{KL}-\frac{|\Lambda|}{6}\mathrm{Tr}\left[\left(\star B+\frac{1}{\gamma}B\right)\wedge B\right]\right)$$

B is $\mathfrak{sl}_2\mathbb{C}$-valued 2-form. $\mathcal{F}(\mathcal{A})$ is the curvature of $\mathrm{SL}(2,\mathbb{C})$ connection $\mathcal{A}$. γ is the Barbero-Immirzi parameter.

φ is Lagrangian multiplier saptisfying $\epsilon^{IJKL}\varphi_{IJKL}=0$. $\star$ is the dual for the internal $\mathrm{SL}(2,\mathbb{C})$.

$$\frac{\delta S}{\delta\varphi}=0,\qquad\Rightarrow\qquad\epsilon^{\mu\nu\rho\sigma}B^{IJ}_{\mu\nu}B^{KL}_{\rho\sigma}=V\epsilon^{IJKL}\qquad\Rightarrow\qquad B^{IJ}=\pm e^I\wedge e^J,$$ (Simplicity constraint)

The simplicity constraint reduces the Plebanski-Holst theory to GR in the tetrad-connection formulation.

Quantization of the Plebanski-Holst theory is a constrained BF theory

$$\int\mathscr{D}\mathcal{A}\mathscr{D}B\mathscr{D}\varphi\; e^{iS}=\int\mathscr{D}\mathcal{A}\mathscr{D}B\;\delta(\text{simplicty})\; e^{-\frac{i}{2}\int_{\mathcal{B}_4}\mathrm{Tr}\left[\left(\star B+\frac{1}{\gamma}B\right)\wedge\mathcal{F}\right]}$$

Overview of Loop Quantum Gravity and Spinfoams

$$\int \mathcal{D}\mathcal{A}\mathcal{D}B\mathcal{D}\varphi\, e^{iS} = \int \mathcal{D}\mathcal{A}\mathcal{D}B\ \delta(\text{simplicty})\ e^{-\frac{i}{2}\int_{\mathcal{B}_4} \mathrm{Tr}\left[\left(\star B + \frac{1}{\gamma}B\right)\wedge\mathcal{F}\right]}$$

Spinfoam quantization:

- Let $\mathcal{B}_4$ be a 4-simplex. The above partition function should give the spinfoam vertex amplitude A_v (local dynamics of LQG).
- Firstly quantize the BF theory,

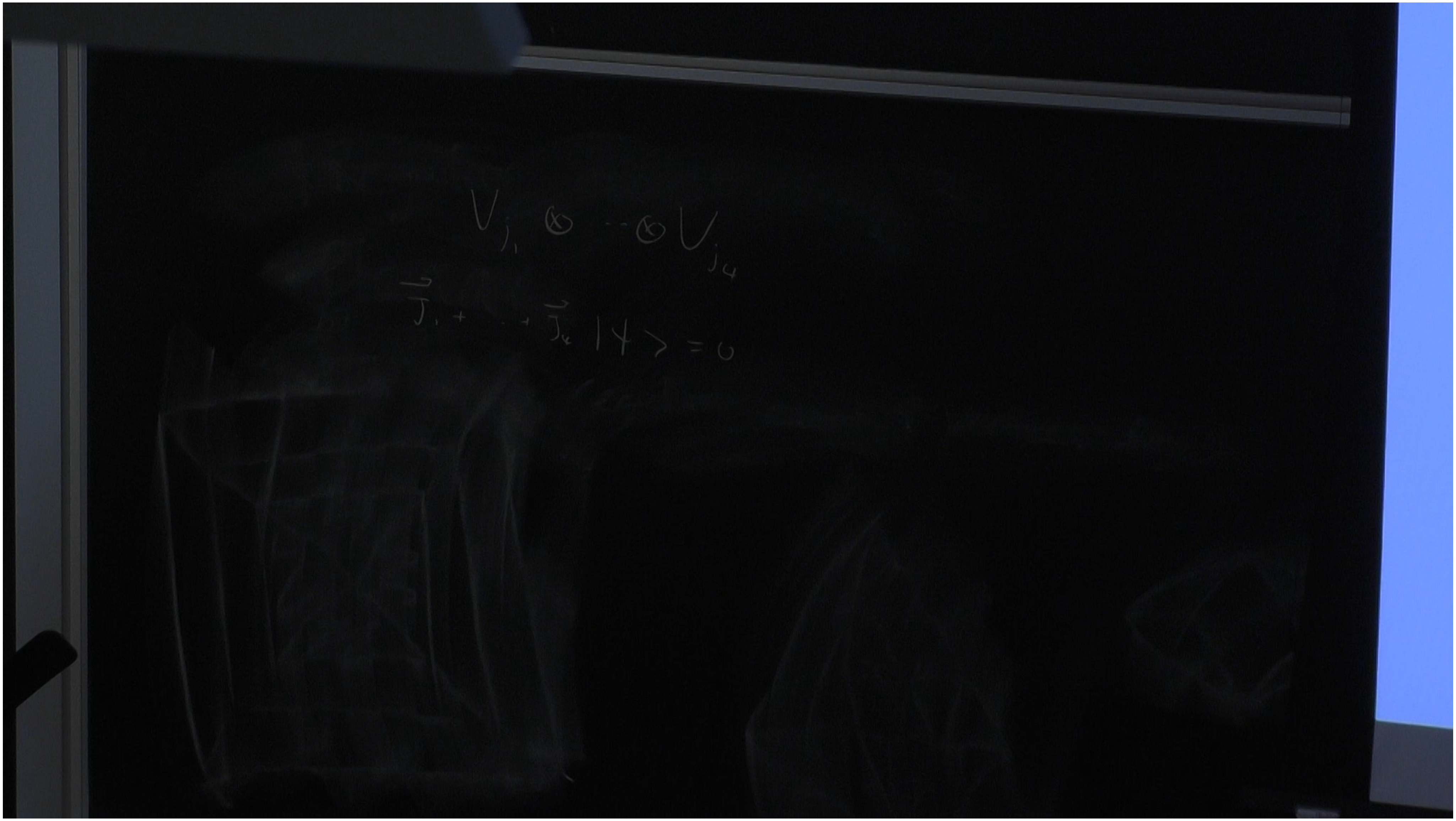
$$\int \mathcal{D}\mathcal{A}\mathcal{D}B\ e^{-\frac{i}{2}\int_{\mathcal{B}_4} \mathrm{Tr}\left[\left(\star B + \frac{1}{\gamma}B\right)\wedge\mathcal{F}\right]} = \int \mathcal{D}\mathcal{A}\ \delta(\mathcal{F}(A))\ .$$

- Then quantize the simplicity constraint and impose the operator constraint to the BF theory.
- The Engle-Pereira-Rovelli-Livine-Freidel-Krasnov (EPRL-FK) model: The simplicity constraint results in the Y-map

$$Y:\quad \text{4-valent SU(2) invariant tensor } i_e \quad \hookrightarrow \quad \text{4-valent SL(2,C) invariant tensor } Y(i_e)$$

(Quantum tetrahedon in spin-network states) (Boosted quantum tetrahedon)

- The vertex amplitude $A_v = \mathrm{Tr}\Big[Y(i_1)\otimes\cdots\otimes Y(i_5)\Big]$. Spinfoam Amplitude: $A(\mathcal{K}) = \sum_{\{j,i\}}\prod_f \dim(j)\prod_v A_v(j,i)$


V_{j1} ⊗ ··· ⊗ V_{j4}
J1 + ··· + J4 |ψ> = 0

Overview of Loop Quantum Gravity and Spinfoams

Semiclassical consistency: recovering discrete gravity dynamics in the semiclassical large-j regime

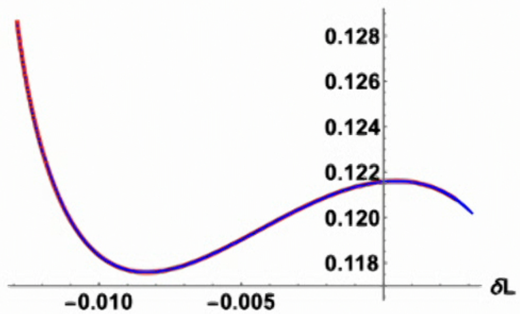
- Large-j asymptotics of vertex amplitude [Barrett et al 2009, Freidel and Conrady 2008]:

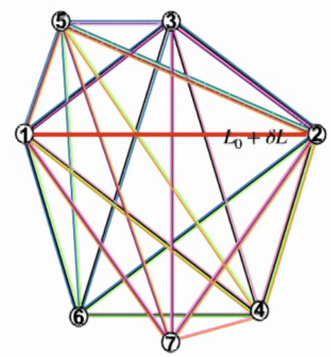
$$A_v = \left(\mathcal{N}_+ e^{iS_{Regge}} + \mathcal{N}_- e^{-iS_{Regge}}\right)\left[1 + O\left(1/j\right)\right]$$

S_{Regge} is the Regge action (discrete Einstein-Hilbert action) of the 4-simplex.

- Recovering semiclassical discrete gravity dynamics on larger 4d simplicial complex [MH, Liu, Qu 2021-2023]

$S_{Spinfoam}$ v.s. S_{Regge}





Spinfoam LQG with cosmological constant

- The cosmological observation confirms the positive cosmological constant Λ (dark energy) in our universe.

Einstein equation: $$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- The formulation of LQG needs to include the consmological constant.
- In LQG, we treat the (bare) cosmological constant as a fundamental parameter of the theory.

Spinfoam Amplitude with Λ: $$A^{(\Lambda)}(\mathcal{K}) = \sum_{\{j,i\}}^{\Lambda} \prod_f A_f^{(\Lambda)}(j) \prod_e A_e^{(\Lambda)}(j,i) \prod_v A_v^{(\Lambda)}(j,i)$$

- The EPRL-FK model ($\Lambda = 0$) has infra-red divergence due to the unbounded sum over spins. The cosmological constant Λ should provide a natural (infra-red) cut-off to the state-sum and makes the amplitude finite.

A brief history of the research on spinfoam with Λ

- Spinfoam model of 3d gravity with quantum SU(2) group (Tureav-Viro model):
  - Finiteness,
  - Semiclassical consistency: Recovering 3d Regge calculus (discrete GR) with positive Λ in the large-j asymptotics.

  Lesson: Cosmological constant should regularize the QG model finite.

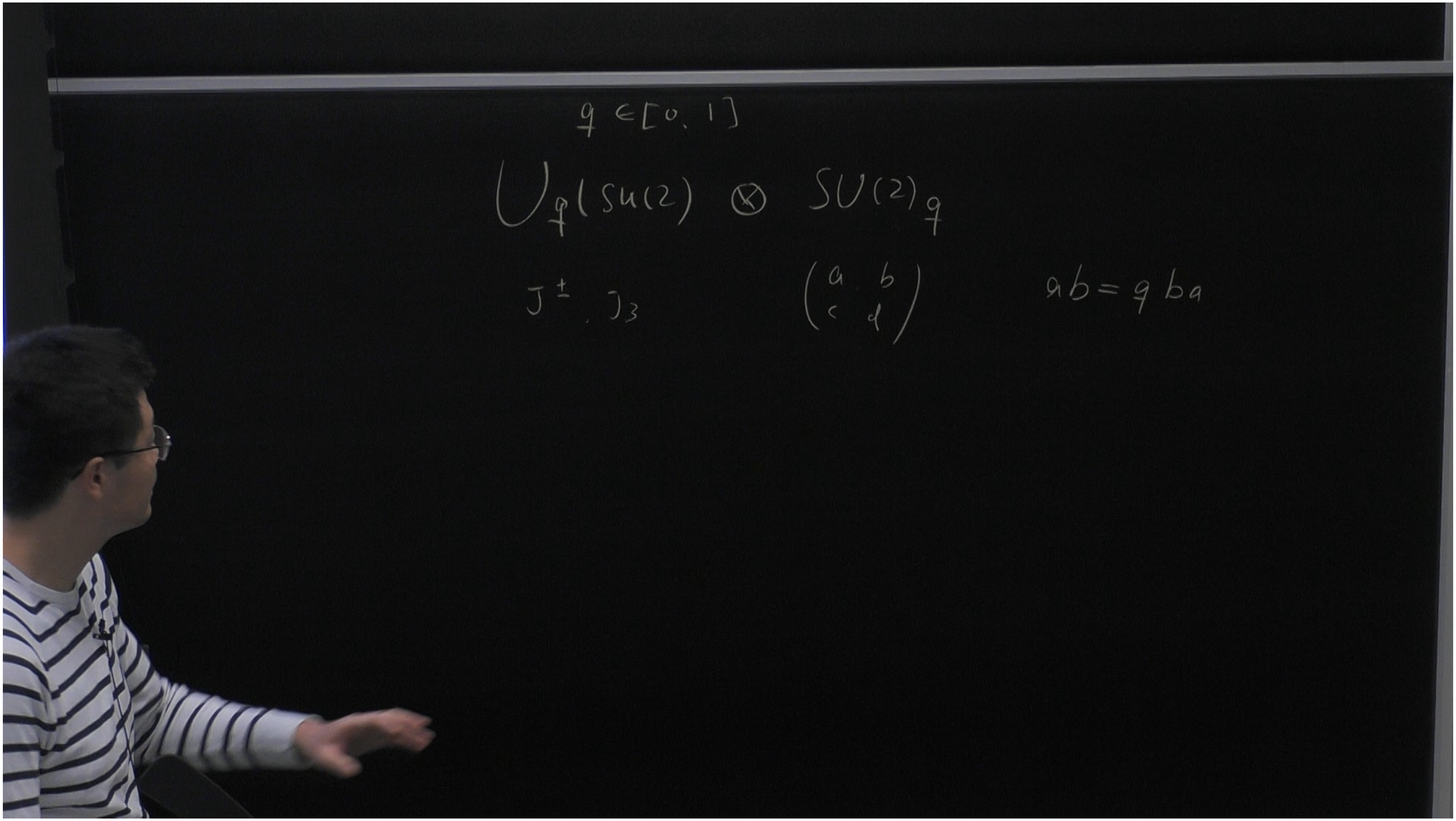
- Quantum Lorentz group spinfoam model of 4d gravity: [Noui, Roche 2002, MH 2010, Fairbairn, Meusburger 2010]
  - The models are all finite.
  - But the semiclassical limit is difficult to extract.

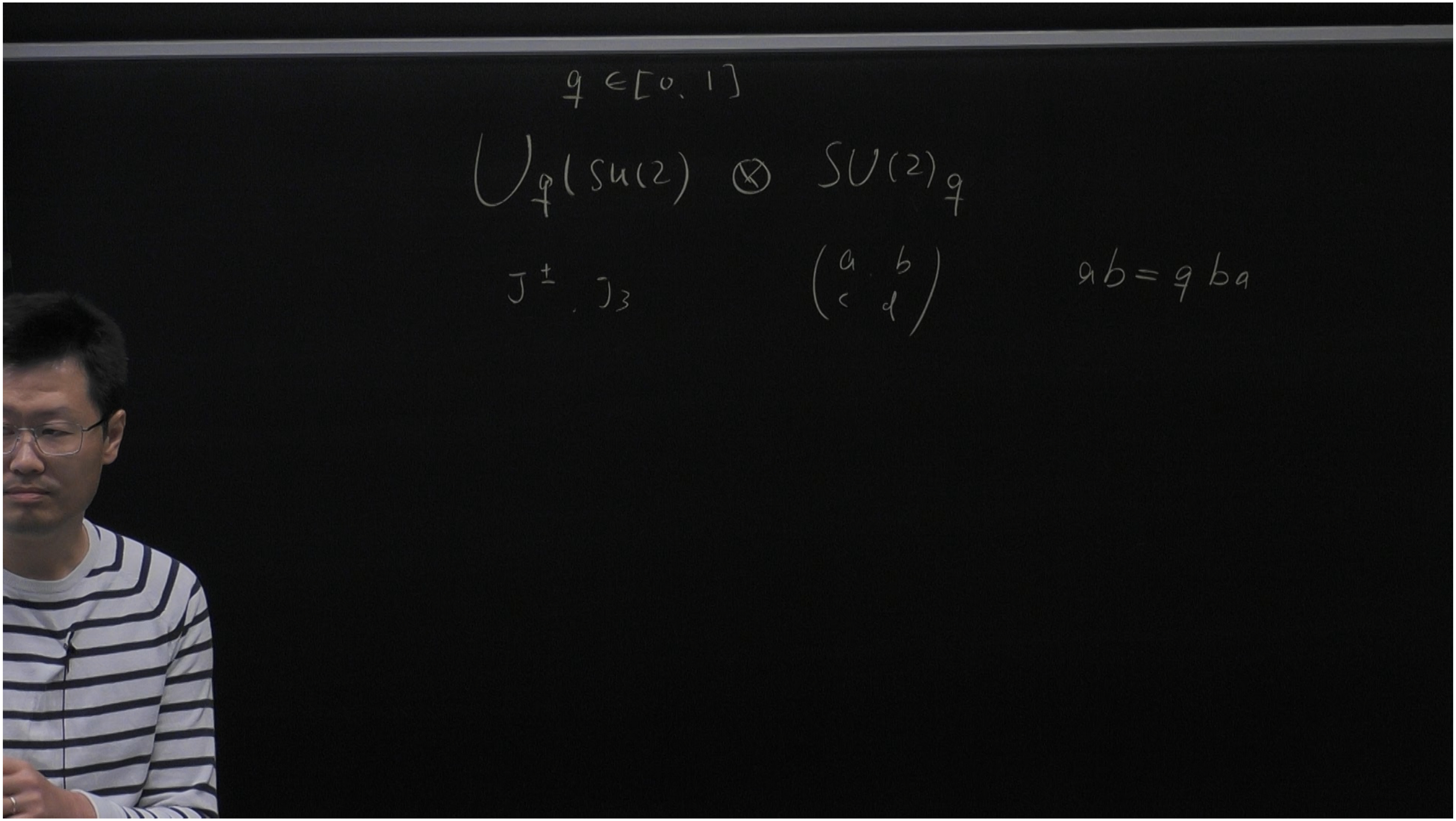
- Spinfoam model of 4d gravity with cosmological constant: [Haggard, MH, Kaminski, Riello 2014-2015]

  The definition of the model is based on SL(2,C) Chern-Simons theory (on the boundary).
  - The semiclassical limit reproduces correctly 4d Regge calculus with Λ (positive or negative)
  - But the finiteness of the model is not manifest.

- This talk: Improved model of 4d gravity with cosmological constant: [MH 2021]

  We have both
  - Finiteness,
  - The semiclassical limit reproduces 4d Regge calculus with Λ (positive or negative)


q ∈ [0, 1]
U_q(su(2)) ⊗ SU(2)_q
J^± , J_3
(a b
c d)
ab = q ba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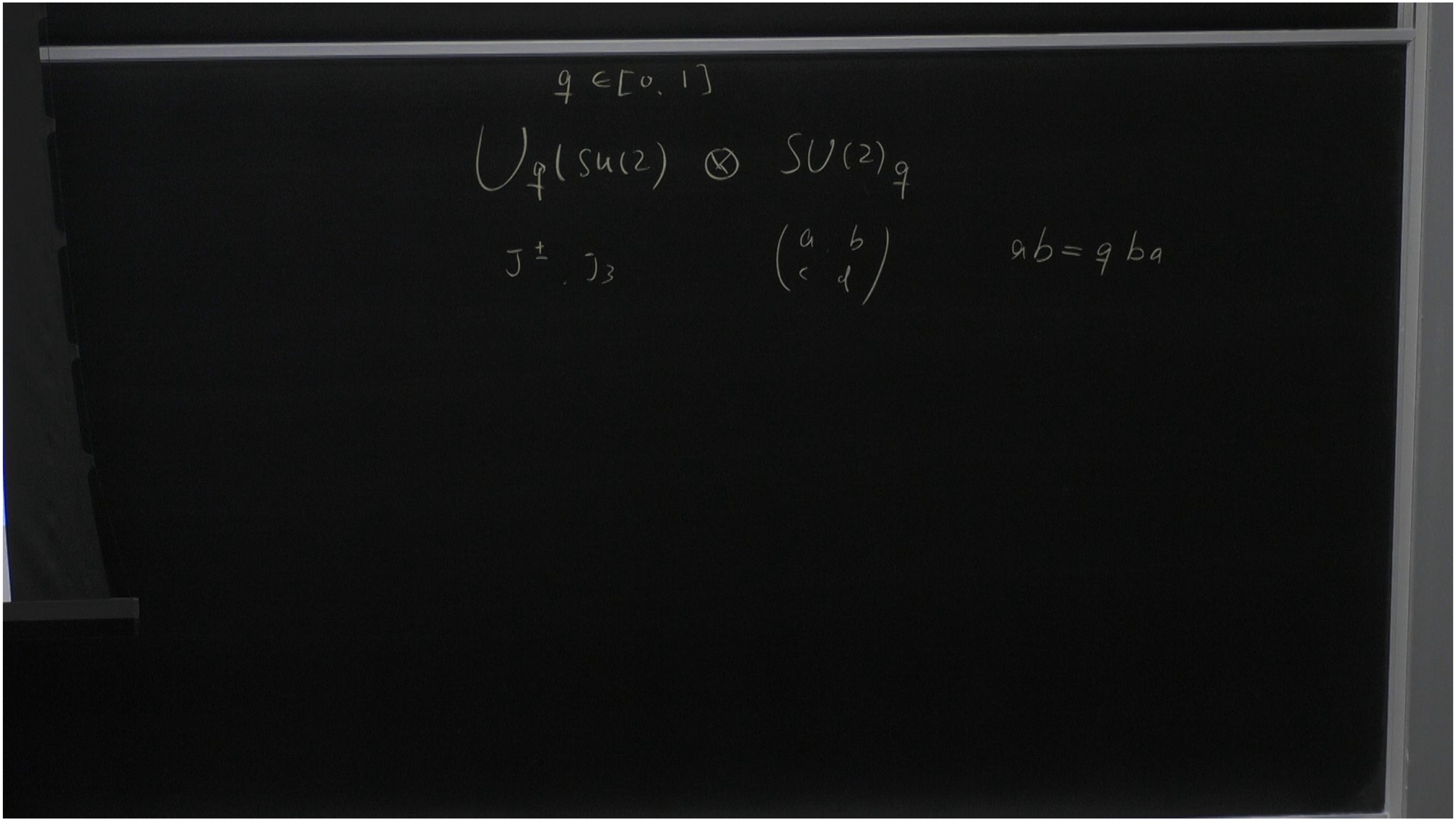

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Plebanski, BF, and boundary Chern-Simons

Plebanski formulation of GR: 4d gravity = BF theory + simplicity constraint

BF theory: $$S_{BF} = -\frac{1}{2}\int_{\mathcal{B}_4} \mathrm{Tr}\left[\left(\star B + \frac{1}{\gamma}B\right)\wedge \mathcal{F}(\mathcal{A})\right] - \frac{|\Lambda|}{12}\int_{\mathcal{B}_4} \mathrm{Tr}\left[\left(\star B + \frac{1}{\gamma}B\right)\wedge B\right]$$

B is $\mathfrak{sl}_2\mathbb{C}$-valued 2-form. $\mathcal{F}(\mathcal{A})$ is the curvature of $\mathrm{SL}(2,\mathbb{C})$ connection $\mathcal{A}$. γ is the Barbero-Immirzi parameter.
$\star$ is the dual for the internal $\mathrm{SL}(2,\mathbb{C})$.

Simplicity constraint: $B = \pm e \wedge e$ relates B to cotetrad e.

Inserting the simplicity constraint to the BF action, we obtain the Holst action of gravity

$$S_{\mathrm{Holst}} = -\frac{1}{2}\int_{\mathcal{B}_4} \mathrm{Tr}\left[\left(\star e\wedge e + \frac{1}{\gamma}e\wedge e\right)\wedge \mathcal{F}(\mathcal{A})\right] - \frac{\Lambda}{12}\int_{\mathcal{B}_4} \mathrm{Tr}\left[(\star e\wedge e)\wedge e\wedge e\right]$$

The Holst action of gravity is on-shell equivalent to the Einstein-Hilbert action

$$\delta S_{\mathrm{Holst}} \quad \Longrightarrow \quad \text{Einstein equation with } \Lambda$$

Plebanski, BF, and boundary Chern-Simons

The general procedure to construct the spinfoam amplitude:

1. Quantize the BF theory,

BF theory: $$S_{BF} = -\frac{1}{2}\int_{\mathcal{B}_4} \mathrm{Tr}\left[\left(\star B + \frac{1}{\gamma}B\right)\wedge \mathcal{F}(\mathcal{A})\right] - \frac{|\Lambda|}{12}\int_{\mathcal{B}_4} \mathrm{Tr}\left[\left(\star B + \frac{1}{\gamma}B\right)\wedge B\right]$$

2. Impose the simplicity constraint at the quantum level

Zero cosmological constant: The partition function of BF + simplicity constraint give the EPRL-FK spinfoam amplitude in LQG

Nonzero cosmological constant: Integrating out B field in $\int dBd\mathcal{A}\, e^{iS_{BF}}$ (Gaussian integral) gives the constraint

$$\mathcal{F}(\mathcal{A}) = \frac{|\Lambda| B}{3}$$ relating B to the curvature of the connection

Integrating out B field gives the boundary $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory

$$\frac{3i}{4|\Lambda|}\int_{\mathcal{B}_4} \mathrm{Tr}\left[\left(\star + \frac{1}{\gamma}\right)\mathcal{F}\wedge\mathcal{F}\right] = \frac{t}{8\pi}\int_{S^3}\mathrm{Tr}(\mathcal{A}\wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) + \frac{\bar{t}}{8\pi}\int_{S^3}\mathrm{Tr}(\bar{\mathcal{A}}\wedge \mathrm{d}\bar{\mathcal{A}} + \frac{2}{3}\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\wedge\bar{\mathcal{A}})$$

complex CS coupling constant: $$t = k(1+i\gamma), \quad k = \frac{12\pi}{|\Lambda|\ell_P^2\gamma} \in \mathbb{Z}_+$$

Implementation of simplicity constraint

We need to impose the simplicity constraint $B = \pm e \wedge e$ to the quantum Chern-Simons theory

$\mathcal{F}(\mathcal{A}) = \frac{|\Lambda| B}{3}$ translate the constraint on B to the constraint on the Chern-Simons connection

The simplicity constraint has the 1st-class and 2nd-class components (according to the CS symplectic form)

1st-class constraint: $\{C_i, C_j\} = {f_{ij}}^k C_k$

2nd-class constraint: $\{C_i, C_j\} =$ non-vanishing on the constraint surface

- Quantizing the 1st-class constraints and imposing them strongly

$$\hat{C}_j |\Psi\rangle = 0$$

- Imposing the 2nd-class constraints weakly with coherent states

$$|\Psi_{(p,q)}\rangle$$

The 2nd-class constraints are imposed on the label (p, q) (classical phase space point), where the coherent state is peaked.

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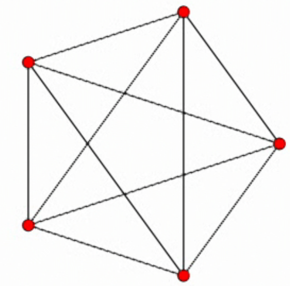
Boundary of 4-simplex and defects



Consider the 4-manifold $\mathcal{B}_4$ (where BF theory lives) is a 4-simplex.

The boundary of 4-simplex is a 3-sphere triangulated by 5 tetrahedra

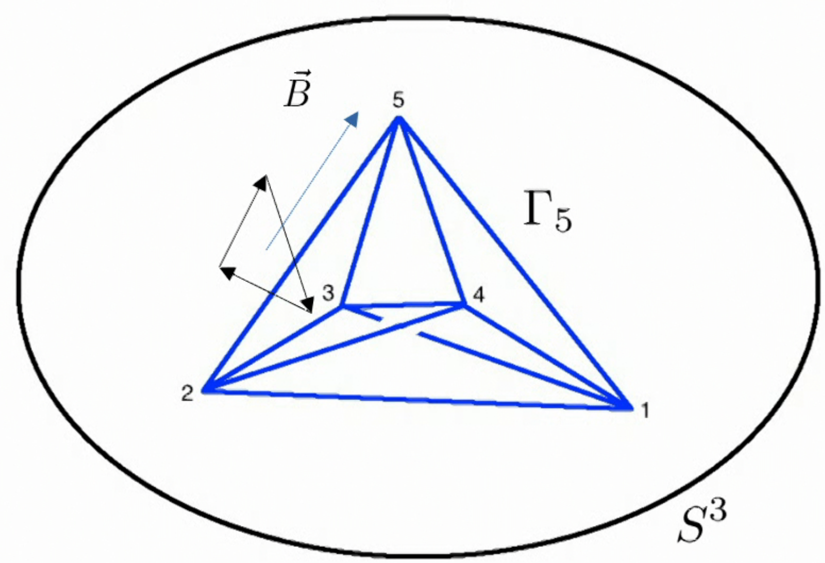
The SL(2,C) Chern-Simons theory is on the 3-sphere

The dual graph Γ_5 on the 3-sphere: 5 nodes dual to tetrahedra, 10 links dual to triangles

$$\mathcal{F}(\mathcal{A}) = \frac{|\Lambda| B}{3}$$

The flux of B (relating to cotetrad) equals to the magnetic flux of Chern-Simons connection on the triangle

Dual graph of the triangulated sphere

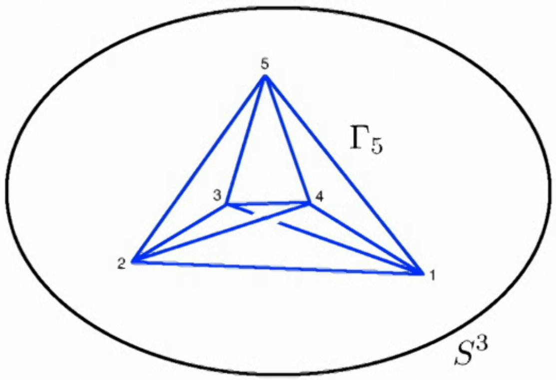


This motivates us to remove the dual graph (and the tubular neighborhood) from the 3-sphere, to create the flux

$S^3 \setminus \Gamma_5$ is the graph-complement 3-manifold

The $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory is defined on $S^3 \setminus \Gamma_5$

Chern-Simons theory and spinfoam amplitude



Spinfoam 4-simplex amplitude [Haggard, HM, Kaminski, Riello 2014]

$$A_v^0 := \int D\mathcal{A}D\bar{\mathcal{A}}\, e^{-iS_{CS}(\mathcal{A},\bar{\mathcal{A}})}\Psi_{\Gamma_5}(\mathcal{A},\bar{\mathcal{A}})$$

$S_{CS}(\mathcal{A},\bar{\mathcal{A}})$: $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons action on S^3

$\Psi_{\Gamma_5}(\mathcal{A},\bar{\mathcal{A}})$: Wilson graph on Γ_5 with representation labels j

The semiclassical consistency: We take j large and Λ small with $j\Lambda$ fixed. By the stationary phase approximation of the path integral,

$$A_v^0 = \left(\mathcal{N}_+ e^{iS_{\mathrm{Regge},\Lambda}} + \mathcal{N}_- e^{-iS_{\mathrm{Regge},\Lambda}}\right)\left[1 + O\left(1/j\right)\right].$$

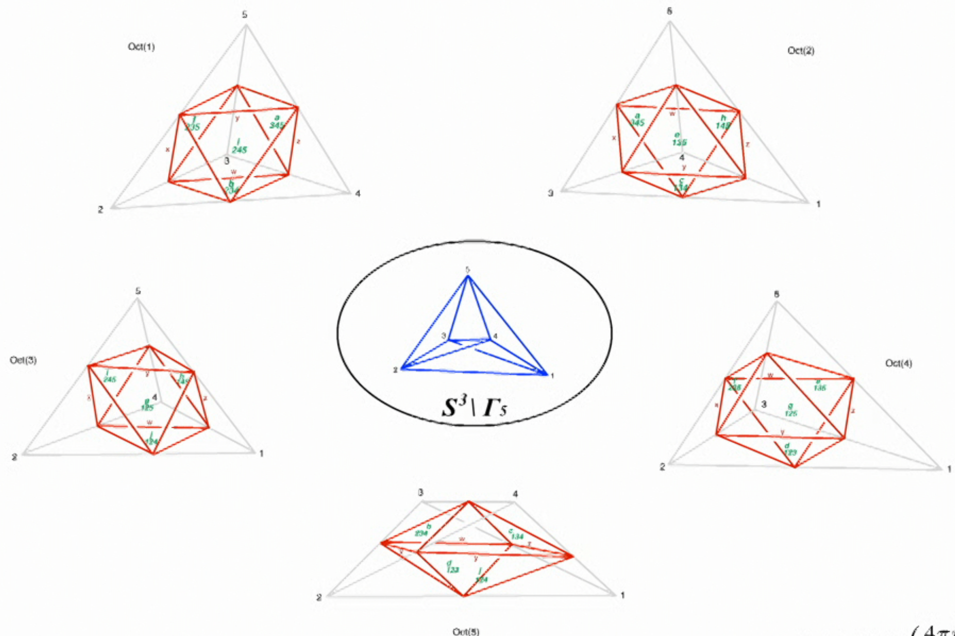
It correctly reproduces the classical Regge action (discrete Einstein-Hilbert action) with Λ in 3+1 dimensions.

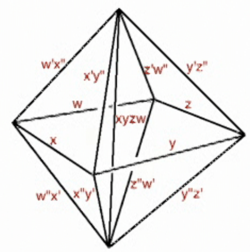
Drawback: The model is an infinite dimensional formal functional integral. The finiteness is obscure.

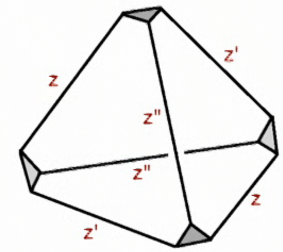
State integral model of SL(2,C) Chern-Simons theory

The quantum Chern-Simons theory is defined on an ideal triangulation of the 3-manifold.

The 3-manifold $S^3 \setminus \Gamma_5$ can be decomposed by 5 idea octahedra, each of which is 4 ideal tetrahedra (tetrahedra with vertices truncated)







Chern-Simons partition function on an ideal tetrahedron: quantum dilogarithm

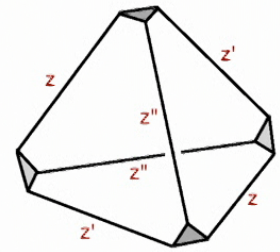
$$\Psi_\Delta(\mu \mid m) = \begin{cases} \prod_{j=0}^{\infty} \frac{1 - q^{j+1} z^{-1}}{1 - \widetilde{q}^{-j} \widetilde{z}^{-1}} & |q| < 1, \\ \prod_{j=0}^{\infty} \frac{1 - \widetilde{q}^{j+1} \widetilde{z}^{-1}}{1 - q^{-j} z^{-1}} & |q| > 1. \end{cases}$$

$$q = \exp\left(\frac{4\pi i}{t}\right), \widetilde{q} = \exp\left(\frac{4\pi i}{\bar{t}}\right), z = \exp\left[\frac{2\pi i}{k}(-ib\mu - m)\right], \widetilde{z} = \exp\left[\frac{2\pi i}{k}\left(-ib^{-1}\mu + m\right)\right]$$

$$b^2 = \frac{1 - i\gamma}{1 + i\gamma} \qquad \mathrm{Re}(b) > 0$$

Faddeev 1995, Kashaev 1996,
Dimofte, Gaiotto, Gukov, 2011-2015

State integral model of SL(2,C) Chern-Simons theory



$\Psi_\Delta(\mu|m)$ is analytic in the upper-half plane $\mathrm{Im}(\mu) > 0$

Considering $\alpha, \beta > 0$, we have the following property (useful for the finiteness)

$$\left| e^{-\frac{2\pi}{k}\beta\mu} \Psi_\Delta(\mu + i\alpha|m) \right| \sim \begin{cases} \exp\left[-\frac{2\pi}{k}\beta\mu\right] & \mu \to \infty \\ \exp\left[-\frac{2\pi}{k}\mu(\alpha + \beta - Q/2)\right] & \mu \to -\infty \end{cases} . \qquad Q = b + \bar{b} > 0$$

Therefore $e^{-\frac{2\pi}{k}\beta\mu}\Psi_\Delta(\mu + i\alpha|m)$ is a Schwartz function of μ if α, β is inside the open triangle $\mathfrak{P}(\Delta)$:

$$\mathfrak{P}(\Delta) = \{(\alpha, \beta) \in \mathbb{R}^2 | \alpha, \beta > 0,\ \alpha + \beta < Q/2\}.$$

$\alpha, \beta \in \mathfrak{P}(\Delta)$ is called a *positive angle structure*.

Dimofte 2014
Andersen, Kashaev 2014

The Fourier transform $\int \mathrm{d}\mu\, e^{\frac{2\pi i}{k}\nu\mu}\Psi_\Delta(\mu|m)$ is convergent if the integration contour is shifted away from the real axis while $\alpha = \mathrm{Im}(\mu)$, $\beta = \mathrm{Im}(\nu)$ belong to $\mathfrak{P}(\Delta)$

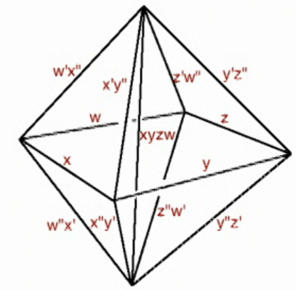
State integral model of SL(2,C) Chern-Simons theory

Chern-Simons partition function on ideal octahedron

$$Z_{\rm oct}(\mu_X,\mu_Y,\mu_Z|m_X,m_Y,m_Z)$$

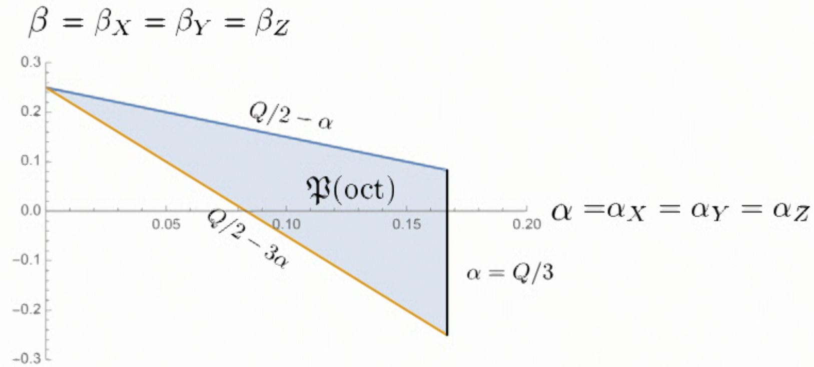
$$= \Psi_\Delta(\mu_X|m_X)\,\Psi_\Delta(\mu_Y|m_Y)\,\Psi_\Delta(\mu_Z|m_Z)\,\Psi_\Delta(\mu_W|m_W)$$

$$\mu_W = iQ-\mu_X-\mu_Y-\mu_Z, \quad m_W=-m_X-m_Y-m_Z$$



$e^{-\frac{2\pi}{k}\sum_i \beta_i\mu_i} Z_{\rm oct}(\{\mu_i+i\alpha_i\}\mid\{m_i\})$ is a Schwartz function of μ_X,μ_Y,μ_Z, if $(\alpha_X,\beta_X,\alpha_Y,\beta_Y,\alpha_Z,\beta_Z)\in\mathbb{R}^6$ is contained by a non-empty open polytope $\mathfrak{P}(\rm oct)$

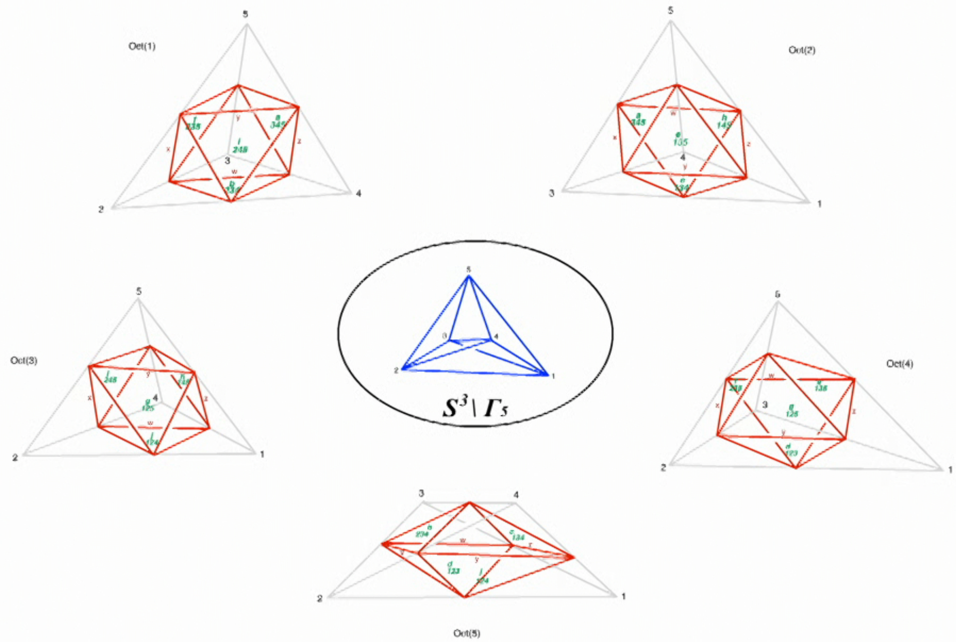
$(\vec\alpha,\vec\beta)\in\mathfrak{P}(\rm oct)$ is a positive angle structure of the ideal octahedron.



Theorem: *The partition function of SL(2,C) Chern-Simons theory converges absolutely if the 3-manifold admits a positive angle structure*

Dimofte 2014
Andersen, Kashaev 2014

State integral model of SL(2,C) Chern-Simons theory



CS partition function on $S^3 \setminus \Gamma_5$ [MH 2021]

$$\mathcal{Z}_{S^3\setminus\Gamma_5} = \text{Fourier transform} \cdot \prod_{a=1}^{5} Z_{\mathrm{oct(a)}}$$

which is a finite sum of 15-dimensional integrals (state inegral model).

The space of postive angle structures is not empty

$$\mathfrak{P}(S^3 \setminus \Gamma_5) = \mathfrak{P}(\mathrm{oct})^{\times 5}$$

$\mathcal{Z}_{S^3\setminus\Gamma_5}$ converges absolutely.

Quantum simplicity constraint

$\mathcal{F}(\mathcal{A}) = \frac{|\Lambda| B}{3}$ The flux of B equals to the magnetic flux of Chern-Simons connection on the triangle

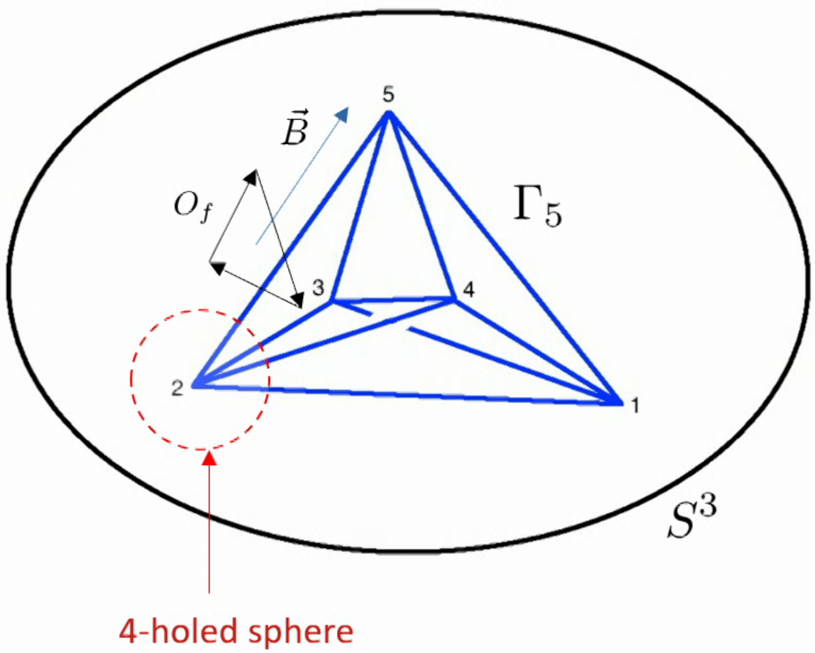
$O_f = e^{|\Lambda| B_f / 3}$ holonomy around the triangle

Discrete simplicity constraint on each tetrahedron:

$\exists$ timelike normal N^J, $B_f^{IJ} N_J = 0$ for all tetrahedron faces f.

Near each node, all four O_f of the tetrahedron can be conjugated to SU(2) simultaneously.

Dual graph of the triangulated sphere



Definition: Semiclassically the simplicity constraint restricts the moduli spaces of $\mathrm{SL}(2,\mathbb{C})$ flat connections on 4-holed spheres to the ones that can be gauge-transformed to $\mathrm{SU}(2)$ flat connections.

Quantum simplicity constraint

The 1st-class component of the simplicity constraint:

The eigenvalue of O_f is the exponental area

$$\lambda_f = \exp\left(\frac{4\pi i}{k} j_f\right) \in \mathrm{U}(1), \qquad j_f = 0, \frac{1}{2}, \cdots, \frac{k-1}{2}$$

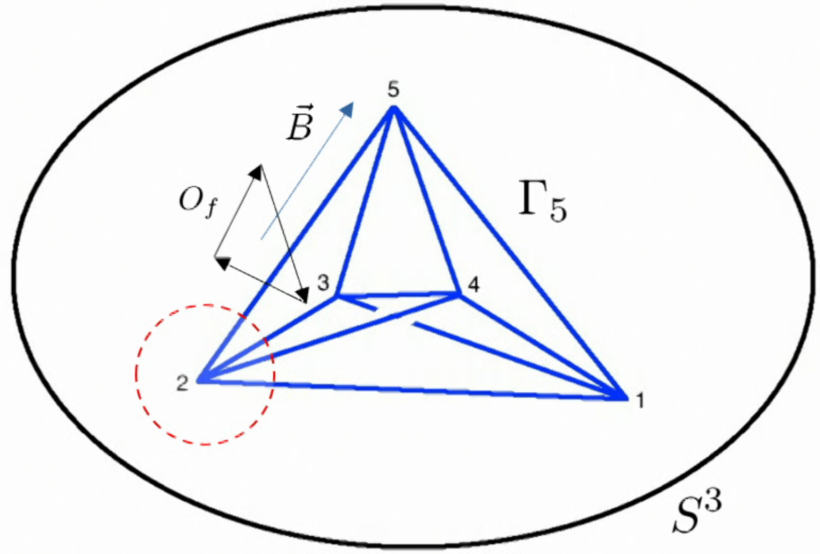
The CS level $k \sim \Lambda^{-1}$ gives a cut-off of the area.

All λ_f are mutually commutative in quantum Chern-Simons theory.

The 1st-class simplicity constraint is imposed strongly to the CS partition function

$$\mathcal{Z}_{S^3\setminus\Gamma_5}\left(\{\lambda_f\}, \{\mathcal{X}_{a=1,\cdots 5}\}\right)$$

Dual graph of the triangulated sphere



"4d area = 3d area" gives a restriction to the positive angle structure, as an analog of the similar consistency condition in the EPRL model [Ding, MH, Rovelli 2010].

Second-class simplicity constraint

Fixing all λ_f,

The moduli space of $\mathrm{SL}(2,\mathbb{C})$ flat connection on 4-holed sphere is 4 real dimensional (non-compact space)

Semiclassical 2nd-class constraint →

The moduli space of $\mathrm{SU}(2)$ flat connection on 4-holed sphere is 2 real dimensional (compact space)

Weakly impose the 2nd-class constraint to the CS partition function:

1. Express the CS partition function in the coherent state basis

$$A_v(j,\rho) = \langle \Psi_\rho \mid \mathcal{Z}_{S^3\setminus\Gamma_5} \rangle$$

   where ρ labels the point in the moduli space where the coherent state is peaked

2. Imposing the semiclassical second-class constraint to the coherent state label ρ

The resulting spinfoam vertex amplitude A_v is a finite sum of absolutely convergent finite-dimensional integrals.

Gluing 3-manifold through 4-holed sphere corresponds to taking inner product $L^2(\mathbb{R}) \otimes \mathbb{C}^k$

$$\begin{aligned}
\sum_{m \in \mathbb{Z}/k\mathbb{Z}} \int \mathrm{d}\mu \, f(\mu \mid m) f'(\mu \mid m) &= \sum_{m,n \in \mathbb{Z}/k\mathbb{Z}} \int \mathrm{d}\mu \mathrm{d}\nu \, f(\mu \mid m) \left[\delta(\mu, \nu)\delta_{m,n}\right] f'(\nu \mid n) \\
&= \sum_{m,n \in \mathbb{Z}/k\mathbb{Z}} \int \mathrm{d}\mu \mathrm{d}\nu \, f(\mu \mid m) \left[\int \mathrm{d}\rho \, \Psi^*_\rho(\mu \mid m) \Psi_\rho(\nu \mid n)\right] f'(\nu \mid n)
\end{aligned}$$

Restrict ρ integral to the compact moduli space of SU(2) flat connection.

The finiteness of spinfoam amplitude

The full amplitude on a 4d simplicial complex

$$A(\mathcal{K}) = \sum_{j_f=1/2}^{(k-1)/2} \prod_f A_f(j_f) \int \mathrm{d}\mu(\rho) \prod_v A_v(j, \rho)$$

Theorem: *$A(\mathcal{K})$ is finite, provided that $\mathrm{d}\mu(\rho)$ is regular on the compact moduli space.* MH 2021

Large-j asymptotics of the 4-simplex amplitude

- The semiclassical limit is $j \to \infty$ and $\Lambda \to 0$ $(k \to \infty)$ with $j\Lambda$ fixed
- The stationary phase analysis can be applied to the finite-dimensional integral $A_v(j,\rho) = \langle \Psi_\rho \mid \mathcal{Z}_{S^3\setminus\Gamma_5} \rangle$
- With the boundary condition corresponding to a non-degenerage 4-simplex, the integral has exactly 2 critical points corresponding to the constant curvature 4-simplex geometry with opposite orientations. Haggard, MH, Kaminski, Riello 2014-2015
- The asymptotics of the amplitude

$$A_v = \left(\mathcal{N}_+ e^{iS_{Regge,\Lambda}+C} + \mathcal{N}_- e^{-iS_{Regge,\Lambda}-C}\right)\left[1 + O\left(1/j\right)\right]$$

MH 2021

$S_{Regge,\Lambda}$ is the 4d Regge action with cosmological constant, C is a geometry-independent constant.

Conclusion & Outlook

- The 4d spinfoam amplitude with cosmological constant is constructed with the state-integral model of CS theory
- The spinfoam amplitudes are all finite.
- The semiclassical behavior of 4-simplex amplitude reproduces 4d Regge calculus with cosmological constant
- This spinfoam model is so far the best 4-dimensional analog of the Tureav-Viro model.

Some interesting future perspectives:

- Simplicity constraint and SL(2,C) | SU(2) interface: Analog of Y-map
- Degrees of divergence and radiative corrections in the small Λ limit
- Numerical computation, critical points and Lefschetz thimbles
- ……