Title: Topological orders and topological quantum computing - Lecture 20230314

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Collection: Mini introductory course on topological orders and topological quantum computing

Date: March 14, 2023 - 10:00 AM

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Abstract: In this mini course, I shall introduce the basic concepts in 2D topological orders by studying simple models of topological orders and then introduce topological quantum computing based on Fibonacci anyons. Here is the (not perfectly ordered) syllabus.

Overview of topological phases of matter
Z2 toric code model: the simplest model of 2D topological orders
Quick generalization to the quantum double model
Anyons, topological entanglement entropy, S and T matrices
Fusion and braiding of anyons: quantum dimensions, pentagon and hexagon identities
Fibonacci anyons
Topological quantum computing.

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Non-Abelian Anyons and TQC

Sunday, November 20, 2022 3:26 PM

Braiding & Fusion of Anyons

We have seen basic fusion & braiding of anyons in the $\mathbb{Z}_2$ toric code model. Now we study the general rules of fusion & braiding

- Fusion algebra (actually, fusion category)

  - A finite set of (anyon) types

    $\{a, b, c, \cdots\}$

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- A finite set of fusion rules

$|a-b|$ $\min(a+b,\ 2k-(a+b))$

$2\times 2=0$

$1\times 1=0+2$

$$a\times b=\sum_c N_{ab}^{c}\, c,\quad N_{ab}^{c}\in \mathbb{N}$$

E.g. Think about spins: $\frac{1}{2}\otimes\frac{1}{2}=0\oplus 1$

each has an internal space

$SU(2)_k$ quantum group

{First k irreducible reps of SU(2): $0, 1, \cdots, k-1$}

Non-Abelian: If any $N_{ab}^{c}>1$ and/or more than 1 term on the RHS.

Abelian: If only 1 term on the RHS & $N_{ab}^{c}=1$ for that term.

E.g. $e\times e=m\times m=\epsilon\times\epsilon=1$, $N_{ee}^{1}=N_{mm}^{1}=N_{\epsilon\epsilon}^{1}=1$;

$e\times m=\epsilon$, $N_{em}^{\epsilon}=1$; $e\times\epsilon=m$, $N_{e\epsilon}^{m}=1$; $m\times\epsilon=e$, $N_{m\epsilon}^{e}=1$

- Quantum dimension d_a

▷ The asymptotic dimension of the internal Hilbert space of an anyon

Caution: d_a may be irrational

$d_a \geq 1, \quad d_1 \equiv 1$ $\qquad$ $\mathcal{D} = \sqrt{\sum_a d_a^2}$ total quantum dimension

The anyons in any 2-dimensional topological phase form a fusion algebra.

▷ $\exists$ more fundamental ways of computing quantum dimensions; however, if we know the fusion rules, we can compute d_a easily.

$d_a d_b = \sum_c N_{ab}^c d_c$ $\qquad$ (Have you seen this Eq. before?)

E.g. $\mathbb{Z}_2$ toric code

$d_e^2 = d_m^2 = d_\epsilon^2 = 1 \Rightarrow d_e = d_m = d_\epsilon = 1$ $\qquad$ Abelian $d_a \equiv 1$

2 terms $\Rightarrow$ σ is non-abelian

$$\Rightarrow d_\psi^2 = 1 \Rightarrow d_\psi = 1, \text{ then } d_\sigma \times d_\sigma = d_1 + d_\psi = 2 \Rightarrow d_\sigma = \sqrt{2} \quad \text{irrational}$$

- Fusion space / Hilbert space of anyons

How to discribe a Hilbert space of multiple anyons?

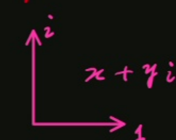
- Fusion space & punctured sphere

▷ $a \times b = \sum_c N_{ab}^c \, c$ e.g. $a \times b = C_1 \oplus 2C_2 \oplus C_3$

each is regarded as a linear space

For given a, b, c, the space of a & b fusing to c is V_{ab}^c, then

Think of



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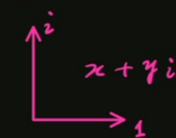
a Fusion space

Think of



$$\dim V_{ab}^c = N_{ab}^c$$

▷ The fusion space V_{ab}^c can be topologically interpreted as a 3-punctured

For given a, b, c, the space of a & b fusing to c is V_{ab}^{c}, then

a Fusion space

Think of

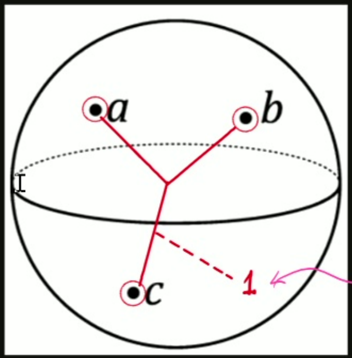
i $x + yi$ 1

$$\dim V_{ab}^{c} = N_{ab}^{c}$$

▷ The fusion space V_{ab}^{c} can be topologically interpreted as a 3-punctured sphere

- $GSD_{S^2} \equiv 1$ Why?
- V_{ab}^{c} is the Hilbert space of three anyons a, b, c on the sphere
- $\because$ No other anyons on the sphere

  $\therefore$ a, b, c must fuse to 1, trivial total topological charge.

$V_{ab}^{c} =$  $= V_{abc}^{1}$

total topological charge is trivial

- Fusion space / Hilbert space of anyons

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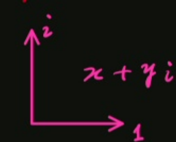
a Fusion space

$$\dim V_{ab}^c = N_{ab}^c$$

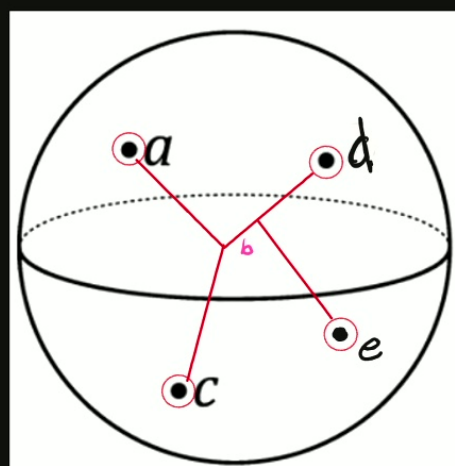
Think of



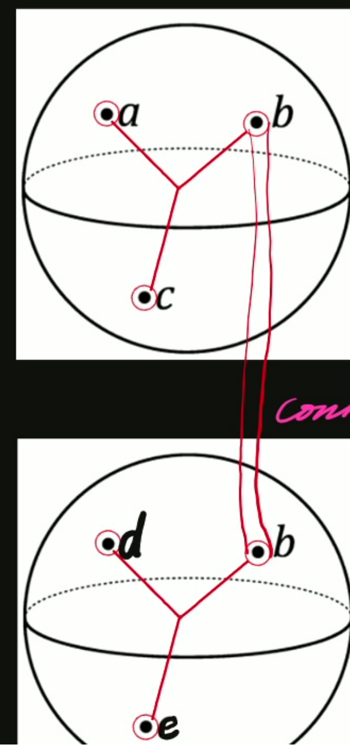
▷ The fusion space of n anyons is an n-punctured sphere, which can be obtained by glueing 3-punctured spheres.

E.g. Glueing 2 3-punctured spheres ⟶ a 4-punctured sphere.

$$V^{e}_{acd} =$$

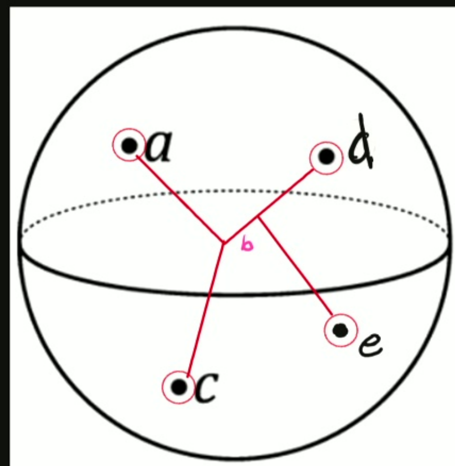


$$= \bigoplus_{b} V^{b}_{ac} V^{e}_{bd} =$$

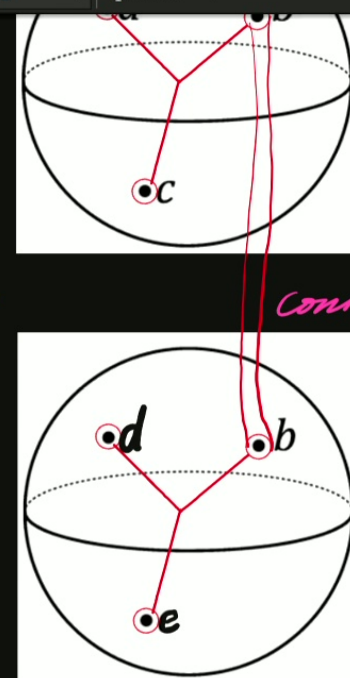


Connected sum

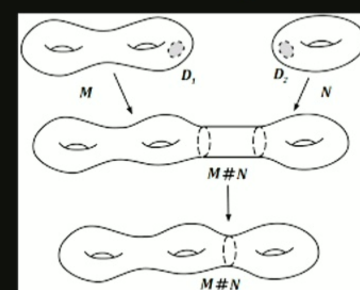
$V^{e}_{acd} =$



$= \bigoplus_{b} V^{b}_{ac} V^{e}_{bd} =$

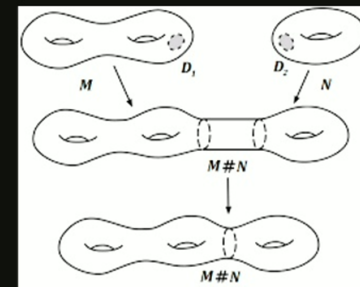


connected sum



- Given any 2-dimensional topological phase $\mathcal{C}$,

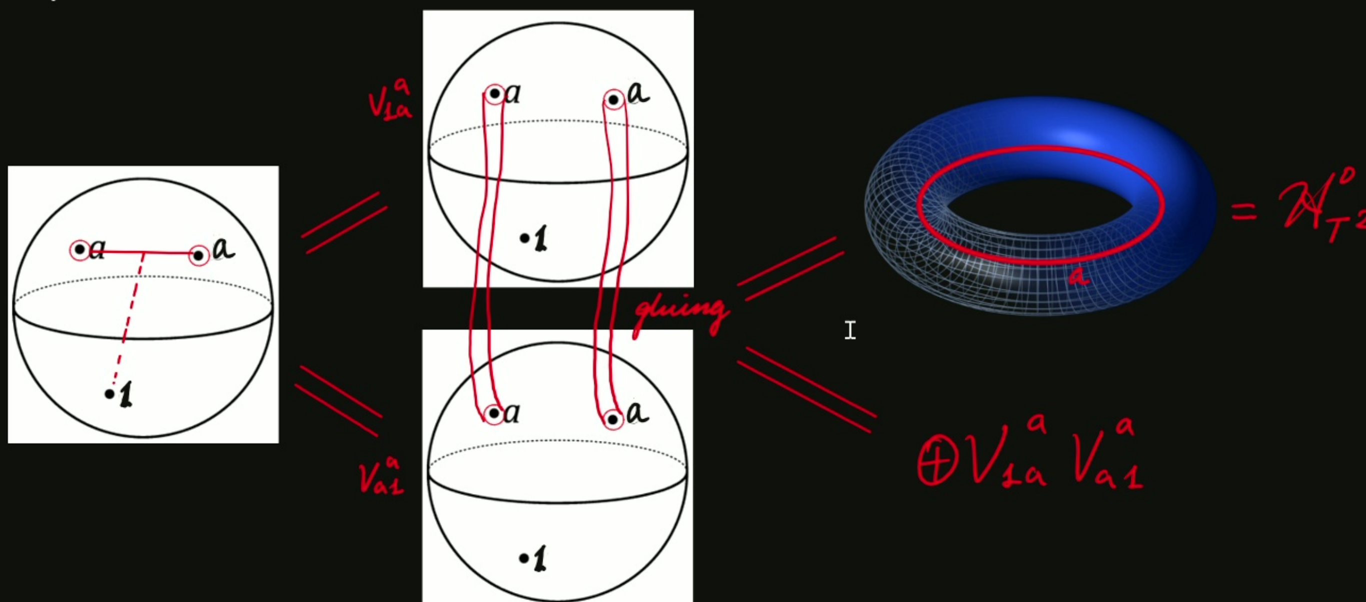
$GSD^{\mathcal{C}}_{T^2}$ = # anyon types in $\mathcal{C}$.



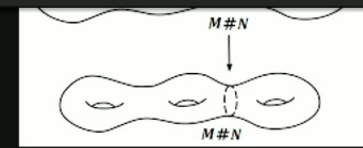
- Given any 2-dimensional topological phase $\mathcal{C}$,

$$GSD^{\mathcal{C}}_{T^2} = \#\text{ anyon types in } \mathcal{C}.$$

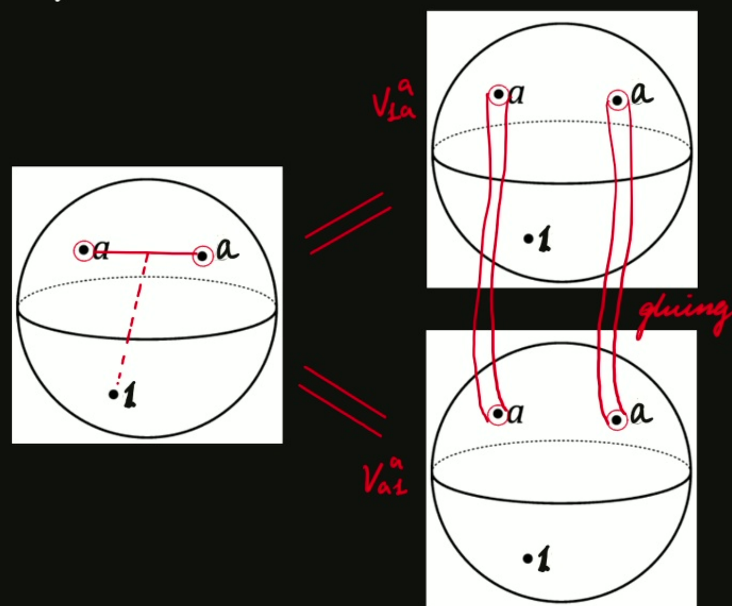
Proof.

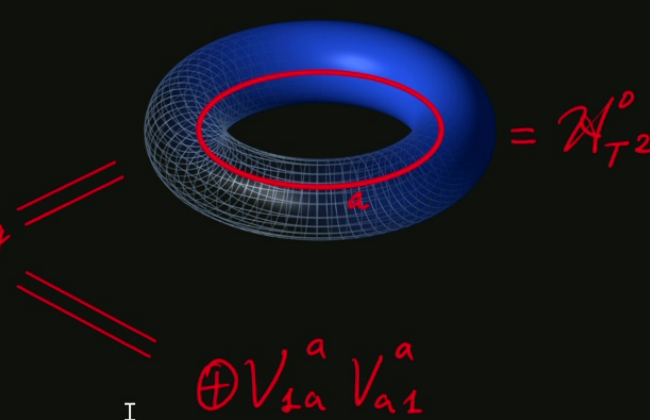


$$GSD^{C}_{T^2} = \#\ \text{anyon types in } C.$$



Proof.



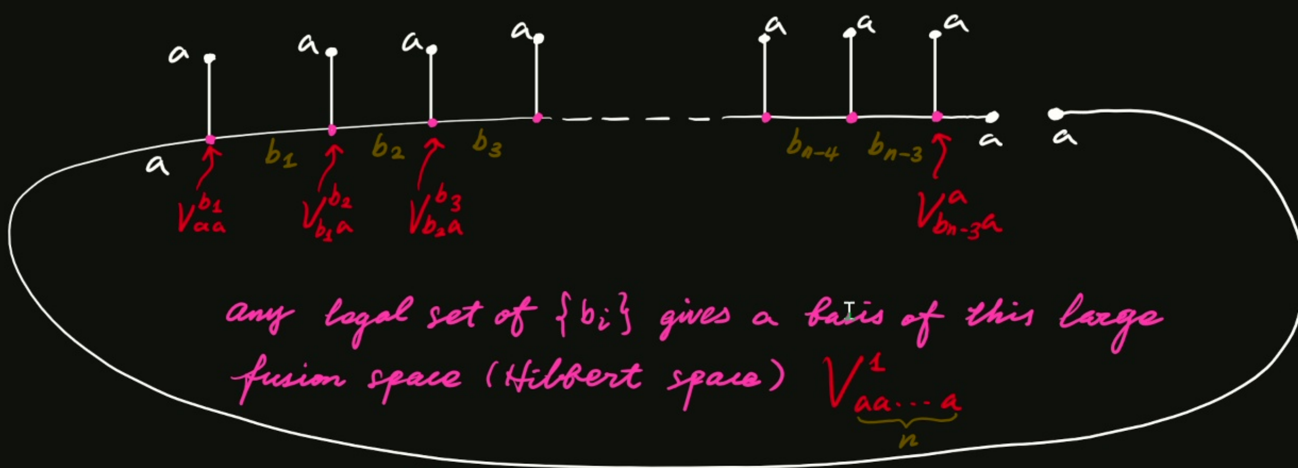


$$GSD^{C}_{T^2} = \dim \mathcal{H}^0_{T^2} = \dim \bigoplus_a V^a_{1a} V^a_{a1} = \sum_a \underbrace{N^a_{1a}}_{\equiv 1} \underbrace{N^a_{a1}}_{\equiv 1} = \sum_a 1 = \#\ \text{anyon types}$$

Consider an n-punctured sphere with anyon a at each puncture.

$\sim$ $(n-1)$ anyons a on the plane, & the n-th anyon a at ∞

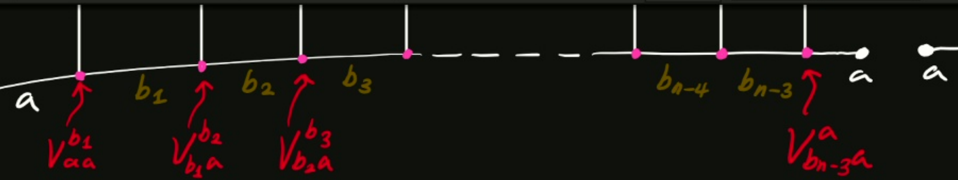
$\sim$ $(n-1)$ anyons a on the disk, & the n-th anyon a on the boundary.



any legal set of $\{b_i\}$ gives a basis of this large fusion space (Hilbert space) $V^1_{\underbrace{aa\cdots a}_{n}}$

$$V^1_{a\cdots a} = \bigoplus_{b_i} V^{b_1}_{aa} V^{b_2}_{ab_1} \cdots V^{a}_{ab_{n-3}}$$

$\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = \frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}$
$= (2)\left(\frac{1}{2}\right)\oplus\frac{3}{2}$



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$$= (2)\left(\tfrac{1}{2}\right)\oplus\tfrac{3}{2}$$

If this is hard to understand, think about first 2 spin-$\frac{1}{2}$'s

$$\tfrac{1}{2}\otimes\tfrac{1}{2} = 0+1 \qquad \dim V^{0}_{\frac{1}{2}\frac{1}{2}} = 1.$$

Then what about $V^{0}_{\underbrace{\frac{1}{2}\frac{1}{2}\cdots\frac{1}{2}}_{n}}$, the singlet space (intertwiner space) of n spin-$\frac{1}{2}$'s ? What is $\dim V^{0}_{\underbrace{\frac{1}{2}\frac{1}{2}\cdots\frac{1}{2}}_{n}}$?

$V_{a\cdots a} = \bigoplus_{b_i} V_{aa} V_{ab_1} \cdots V_{ab_{n-3}}$

$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$

$= 2\left(\frac{1}{2}\right) \oplus \frac{3}{2}$

If this is hard to understand, think about first 2 spin-$\frac{1}{2}$'s

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$$\dim V^1_{\underbrace{aa\cdots a}_{n}} = \operatorname{Tr} \sum_{b_i} (N_a)^{b_1}_{a} (N_a)^{b_2}_{b_1} \cdots (N_a)^{a}_{b_{n-3}}$$

$\Downarrow$ $n \to \infty$

let d_a be the largest eigenvalue of (N_a)

$$= d_a^{n-2} \sim d_a^n$$

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$$\dim V^{1}_{\underbrace{aa\cdots a}_{n}} = \mathrm{Tr}\sum_{b_i} (N_a)_a^{b_1} (N_a)_{b_1}^{b_2} \cdots (N_a)_{b_{n-3}}^{a}$$

$$\Big\Downarrow \begin{array}{l} n \to \infty \\ \text{let } d_a \text{ be the largest eigenvalue of } (N_a) \end{array}$$

$$= d_a^{n-2} \sim d_a^{n}$$

Clearly, as said, the total dimension can certainly not be evenly distributed to each anyon as an integer.

$\Rightarrow$ The Hilbert space is nonlocal

For Abelian anyons, $d_a \equiv 1 \;\therefore\; N_{ab}^{c} \equiv \delta_{ab}^{c}$ no degeneracy in a fusion space.

• The pentagon identity

- Fusion is associative

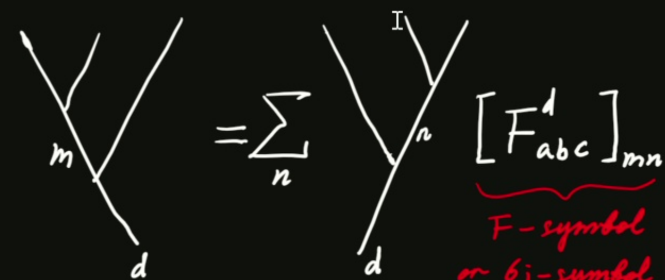
$$\underbrace{(a\times b)\times c}_{\text{a fusion space}} = \underbrace{a\times(b\times c)}_{\text{a fusion space}}$$

isomorphism between 2 fusion spaces

more precisely: $(a\times b)\times c \cong a\times(b\times c)$

$\Downarrow$

basis transformation:



F-symbol or 6j-symbol

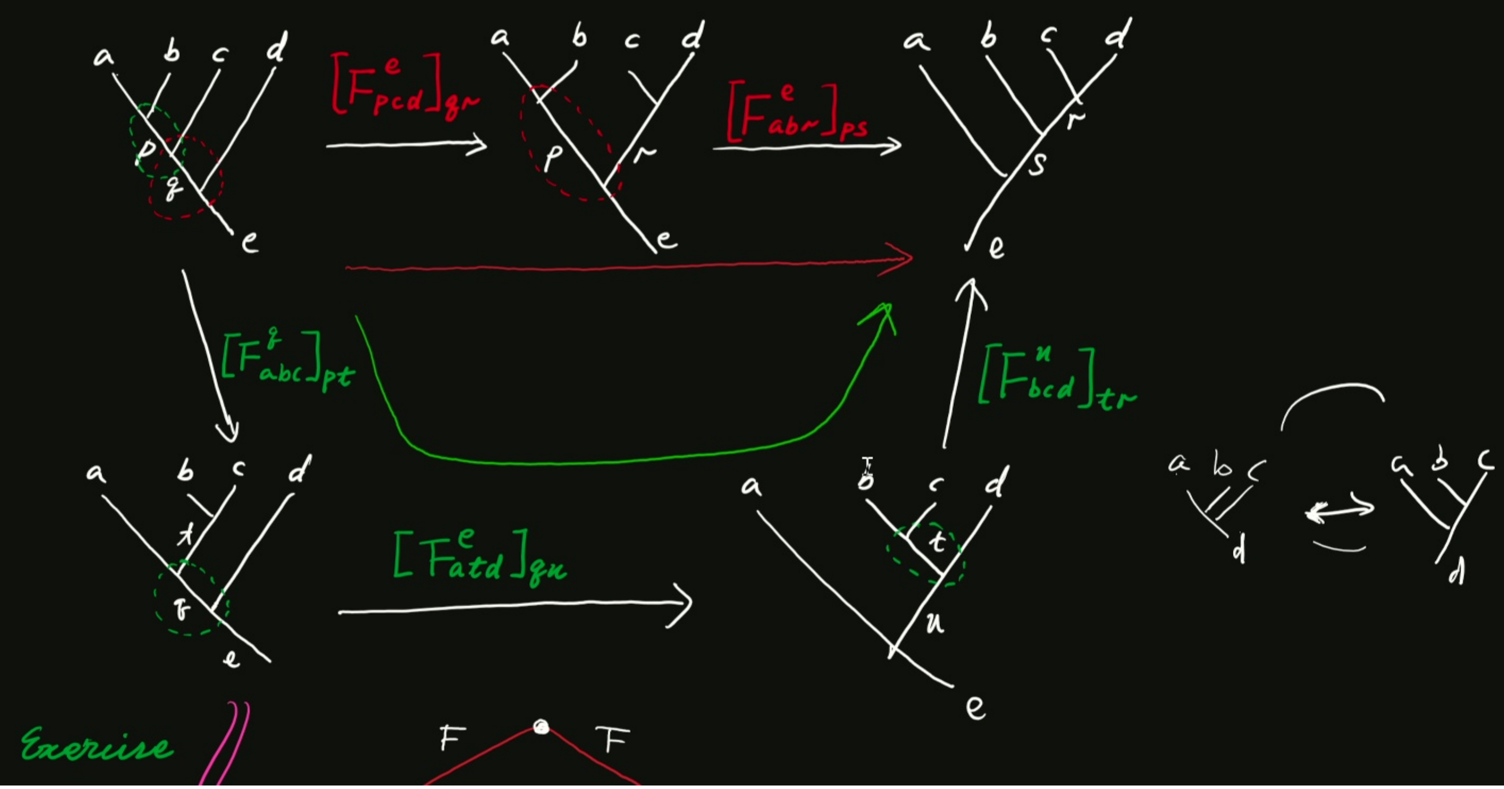
fusion (tree) basis

SU(2) irreps

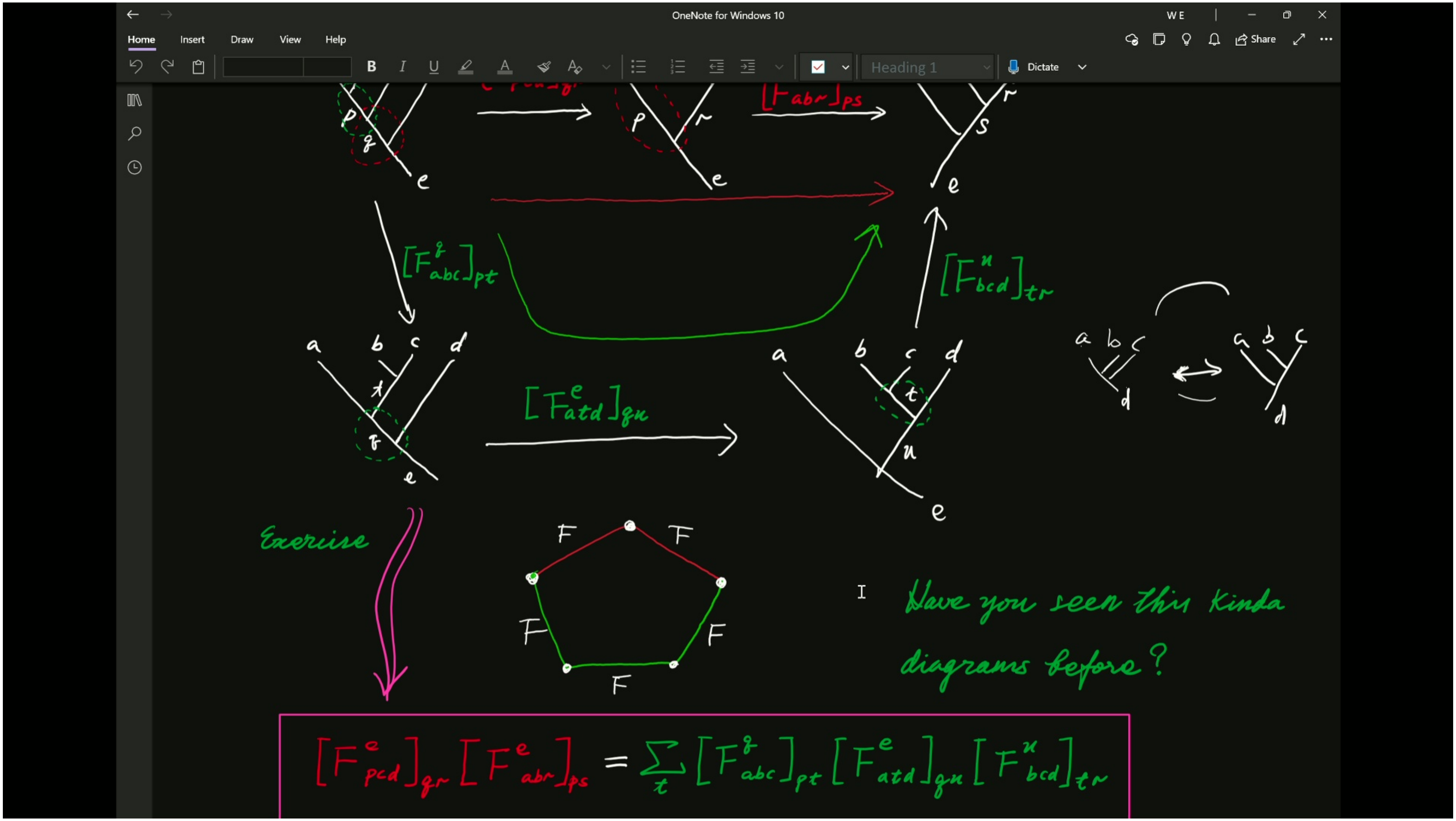
$(i\otimes j)\otimes k \cong i\otimes(j\otimes k)$

What are the coefficients in the basis transformation

No. One needs to care about this up to fusing 4 anyons because of the pentagon identity. (The coherence theorem)



Exercise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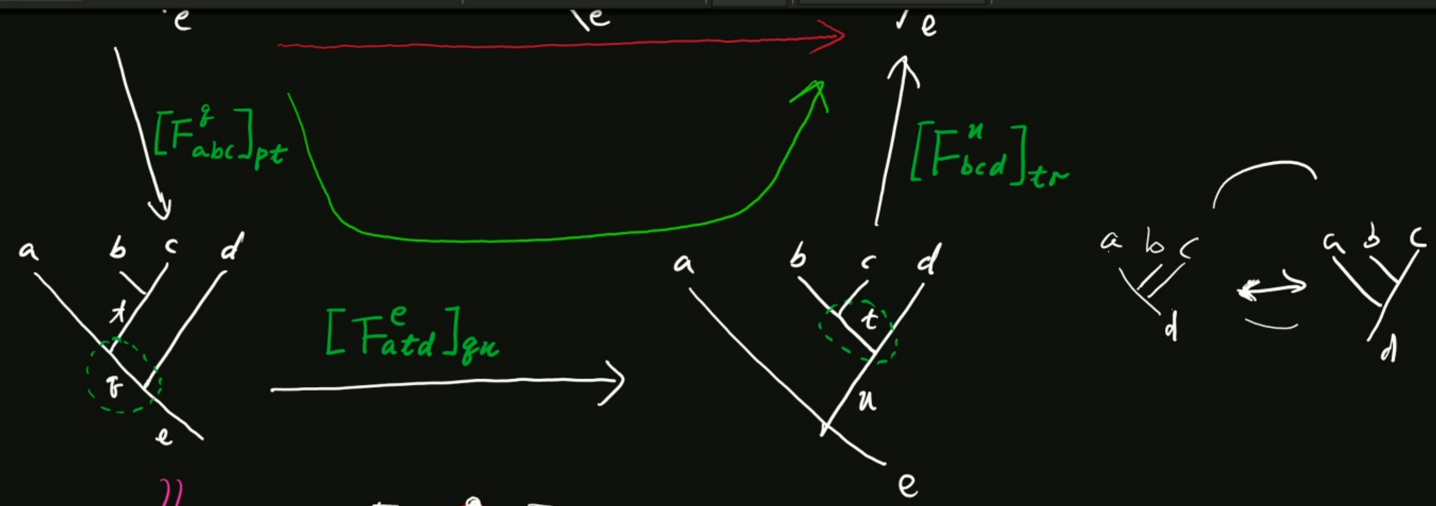
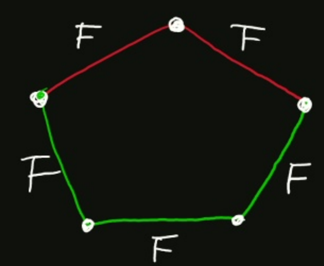
View
Help
Share
Heading 1
Dictate
$[F_{abr}]_{ps}$
$[F^{q}_{abc}]_{pt}$
$[F^{u}_{bcd}]_{tr}$
$[F^{e}_{atd}]_{qu}$
Exercise
F
F
F
F
F
Have you seen this kinda diagrams before?
$$[F^{e}_{pcd}]_{qr}[F^{e}_{abr}]_{ps} = \sum_{t}[F^{q}_{abc}]_{pt}[F^{e}_{atd}]_{qu}[F^{u}_{bcd}]_{tr}$$



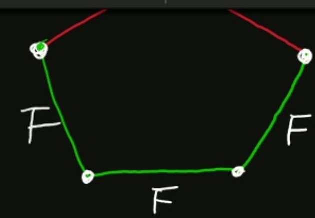
Exercise



Have you seen this kinda diagrams before?

$$\boxed{[F^{e}_{pcd}]_{qr}\,[F^{e}_{abr}]_{ps} = \sum_{t} [F^{q}_{abc}]_{pt}\,[F^{e}_{atd}]_{qu}\,[F^{u}_{bcd}]_{tr}}$$

Not a single equation! Nonlinear!



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Not a single equation! Nonlinear!

Warning: Given a set of labels, not an arbitrary set of fusion rules is consistent! Need to solve the pentagon identity to find the F-symbols. No solution ⇒ illegal fusion rules.

a set of labels

} = a fusion category

rules is consistent! Need to solve the pentagon identity to find the F-symbols. No solution $\Rightarrow$ illegal fusion rules.

$$\left.\begin{array}{l} \text{a set of labels} \\ + \text{ a set of consistent fusion rules} \\ + \text{ a solution to the pentagon identity} \end{array}\right\} = \text{a fusion category}$$

Exercise: Given the set $\{0, 1\}$

(1) Try solve the pentagon identity for the following fusion rules

$$\begin{Bmatrix} 1 \times 1 = 0 \\ 0 \times 1 = 1 \end{Bmatrix} \xrightarrow[\text{pentagon}]{\text{solve}} 2 \text{ solutions} \begin{cases} F \equiv 1 & \mathbb{Z}_2 \text{ fusion category} \\ \text{some } F = -1 & \text{semion fusion category} \end{cases}$$

(1) Try solve the pentagon identity for the following fusion rules

$$\left\{\begin{array}{l} 1\times 1 = 0 \\ 0\times 1 = 1 \end{array}\right\} \xrightarrow[\text{pentagon}]{\text{solve}} 2 \text{ solutions} \left\{\begin{array}{ll} F\equiv 1 & \mathbb{Z}_2 \text{ fusion category} \\ \text{some } F=-1 & \text{semion fusion category} \end{array}\right.$$

(2) Try solve the pentagon identity for the following fusion rules

$$\left\{\begin{array}{l} 1\times 1 = 0+1 \\ 0\times 1 = 1 \end{array}\right\} \xrightarrow[\text{pentagon}]{\text{solve}} 1 \text{ solution: Fibonacci fusion category}$$

Find the non-zero F-symbols

(3) Show that the pentagon identity has no solution for fusion rules

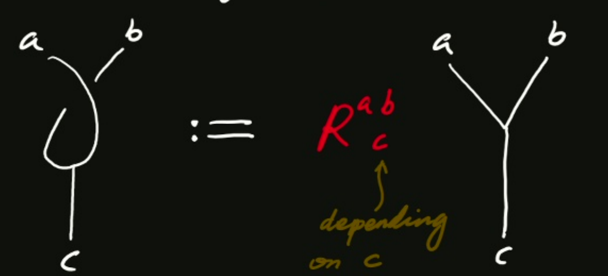
$$\left\{\begin{array}{l} 1\times 1 = 0+1+1 \\ 0\times 1 = 1 \end{array}\right\}$$

$0 \times 1 = 1$

• Braiding (Braided fusion category)

On top of fusing anyons, if we allow exchanging anyons, more basis transformations in a fusion space are possible.

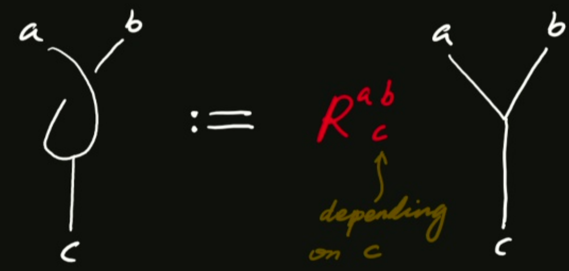
- R matrix: Exchanging anyons is nontrivial



R is a matrix acting on the fusion basis, as you'll see.

basis transformations in a fusion space are possible.

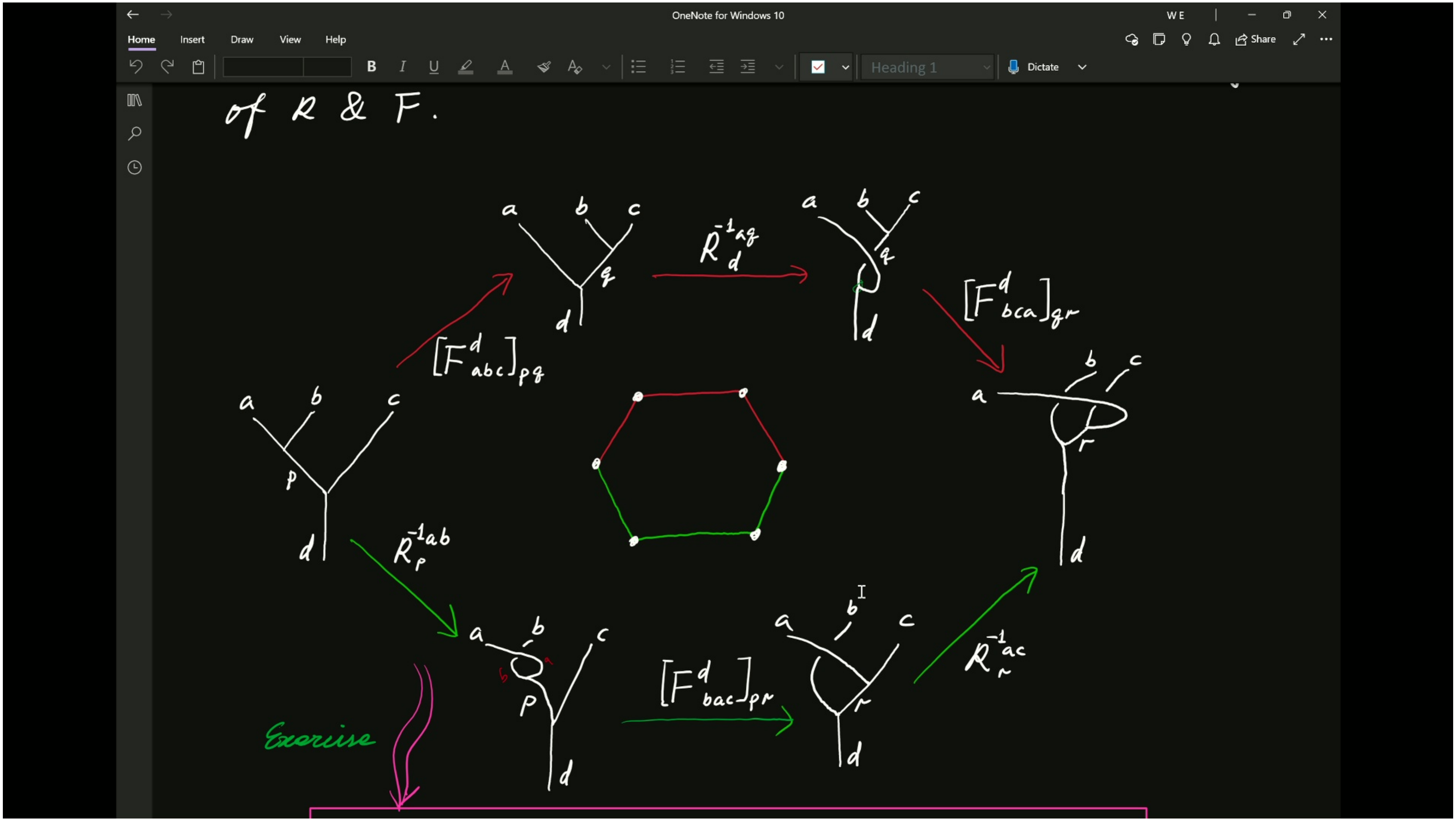
- R matrix: Exchanging anyons is nontrivial



R is a matrix acting on the fusion basis, as you'll see.

- The Hexagon identity

One cannot introduce arbitrary R on top of a fusion category but has to choose the R compatible with the fusion. When R is introduced, more basis transformations are allowed apparently;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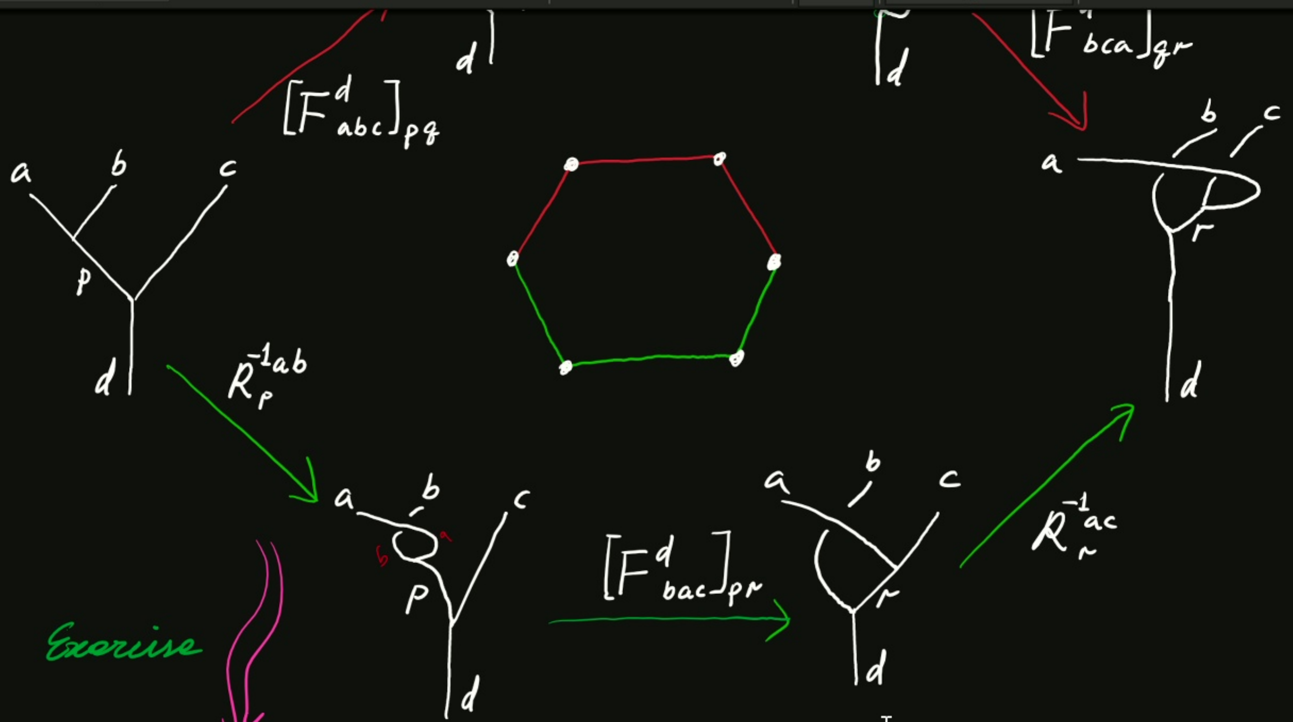
of R & F.
$[F^{d}_{abc}]_{pq}$
$R^{-1\,aq}_{d}$
$[F^{d}_{bca}]_{qr}$
$R^{-1\,ab}_{p}$
$[F^{d}_{bac}]_{pr}$
$R^{-1\,ac}_{r}$
Exercise



$$\boxed{R^{-1\,ab}_{p}\,[F^{d}_{bac}]_{pr}\,R^{-1\,ac}_{r} = \sum_{q} [F^{d}_{abc}]_{pq}\,R^{-1\,aq}_{d}\,[F^{d}_{bca}]_{qr}}$$

Not a single equation! Nonlinear!

and then use the pentagon & hexagon identities to fix some phase factors for consistency.

Now we can study an interesting braided fusion category describing the simplest non-abelian anyons, which support TQC.

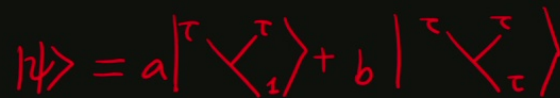
Fibonacci anyons & TQC

Let's consider a example of non-Abelian anyons of particular importance.

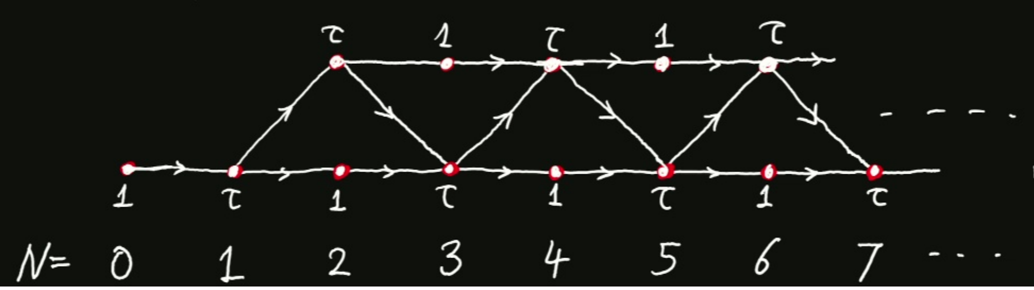
- Fibonacci anyons
  - Fusion algebra

- Hilbert space dimension

τ	$\dim \mathcal{H}_\tau = 1$
$\tau \times \tau = 1 + \tau$	$\dim \mathcal{H}_{2\tau} = 2$
$\tau \times \tau \times \tau = 1 + \tau + \tau$	$\dim \mathcal{H}_{3\tau} = 3$
$\tau \times \tau \times \tau \times \tau = 1 + 1 + \tau + \tau + \tau$	$\dim \mathcal{H}_{4\tau} = 5$
$\vdots$	$\vdots$



▷ Bratelli diagram/fusion path method

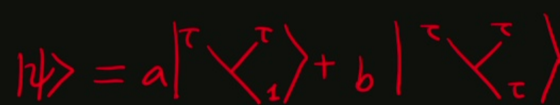


- **Fibonacci anyons**
  - Fusion algebra
    - ▷ Anyons : $\{1, \tau\}$
    - ▷ Fusion rules : $1 \times \tau = \tau, \quad \tau \times \tau = \underbrace{1 + \tau}_{\text{2 terms: non-Abelian}}$
  - Topological spins

$$\theta_1 = 0, \qquad \theta_\tau = \frac{2}{5}$$

  - Hilbert space dimension

τ	$\dim \mathcal{H}_\tau = 1$
$\tau \times \tau = 1 + \tau$	$\dim \mathcal{H}_{2\tau} = 2$
$\tau \times \tau \times \tau = 1 + \tau + \tau$	$\dim \mathcal{H}_{3\tau} = 3$

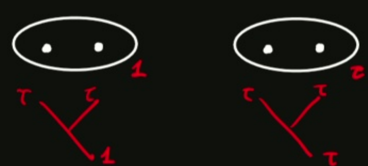


- F-Symbol, R matrix & Braiding

A fusion path = a state in the Hilbert space

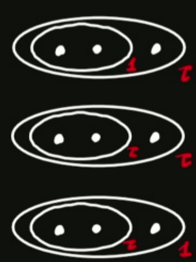
- A logical qubit

$N=2$



$\longleftrightarrow \; |(\cdot\ \cdot)_1\rangle, \quad |(\cdot\ \cdot)_\tau\rangle$

$N=3$



$\longleftrightarrow \; |((\cdot\ \cdot)_1, \cdot)_\tau\rangle =: |0\rangle$

$\longleftrightarrow \; |((\cdot\ \cdot)_\tau, \cdot)_\tau\rangle =: |1\rangle$

$\longleftrightarrow \; |((\cdot\ \cdot)_\tau, \cdot)_1\rangle =: |N\rangle$

} a logical qubit

Will see why.

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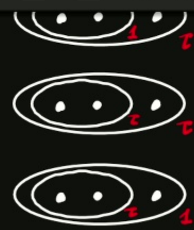
$N=3$



$$|((\cdot\ \cdot)_1,\cdot)_\tau\rangle =: |0\rangle$$

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$$|((\cdot\ \cdot)_\tau,\cdot)_1\rangle =: |N\rangle$$

} a logical qubit

Will see why.

- The F-symbols & basis change

$$|(\cdot,(\cdot\ \cdot)_i)_k\rangle = \sum_j [F^k_{\tau\tau\tau}]_{ij}\, |((\cdot\ \cdot)_j,\cdot)_k\rangle$$

$i,j,k\in\{1,\tau\}$

$$\Big\Downarrow$$

Exercise:

Solving the pentagon identity

Hint: Note that $\dim \mathcal{H}^\tau_{\tau\tau\tau} = 2$

non-zero F-symbols:

$\Downarrow$

non-zero F-symbols:

$k=1 \qquad F^{1}_{\tau\tau\tau} = 1$

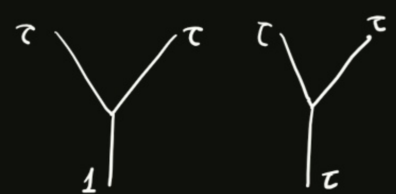
$$k=\tau \qquad [F^{\tau}_{\tau\tau\tau}] = \begin{pmatrix} F_{11} & F_{1\tau} \\ F_{\tau 1} & F_{\tau\tau} \end{pmatrix} = \begin{pmatrix} \phi^{-1} & \sqrt{\phi^{-1}} \\ \sqrt{\phi^{-1}} & -\phi^{-1} \end{pmatrix}$$

$\phi = d_\tau$

$\phi^{-1} = \frac{\sqrt{5}-1}{2}$

- The R-matrix

First note that in this case the R-matrix is diagonal because we have only



$|(\cdot\ \cdot)_1\rangle \qquad |(\cdot\ \cdot)_\tau\rangle \qquad \Rightarrow$

$$R|(\cdot\ \cdot)_1\rangle = R^{\tau\tau}_1 |(\cdot\ \cdot)_1\rangle$$

$$R|(\cdot\ \cdot)_\tau\rangle = R^{\tau\tau}_\tau |(\cdot\ \cdot)_\tau\rangle$$

- The R-matrix

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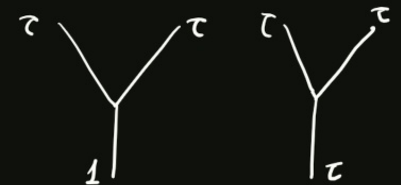


$|(\cdot\ \cdot)_1\rangle \qquad |(\cdot\ \cdot)_\tau\rangle$

$$\Rightarrow \quad \underline{R\,|(\cdot\ \cdot)_1\rangle} = R_1^{\tau\tau}\,|(\cdot\ \cdot)_1\rangle$$

$$\underline{R\,|(\cdot\ \cdot)_\tau\rangle} = R_\tau^{\tau\tau}\,|(\cdot\ \cdot)_\tau\rangle$$

Then the hexagon identity reduces to

$$\underline{R_p [F]_{pr} R_r = [F]_{p1}[F]_{1r} + [F]_{p\tau} R_\tau [F]_{\tau r}}$$

$\Downarrow$ Exercise: check & solve this

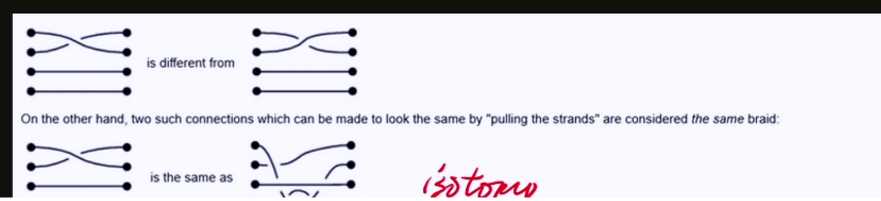
$$\begin{pmatrix} e^{-i\frac{4\pi}{5}} & \end{pmatrix} \qquad \begin{pmatrix} |(\cdot\ \cdot)_1\rangle \end{pmatrix}$$

$(z-w)^{-\frac{4}{5}} \longrightarrow$ phase factor $e^{-i\frac{4\pi}{5}}$ full round $\Rightarrow e^{-i\frac{2\pi}{5}}$ for exchange only

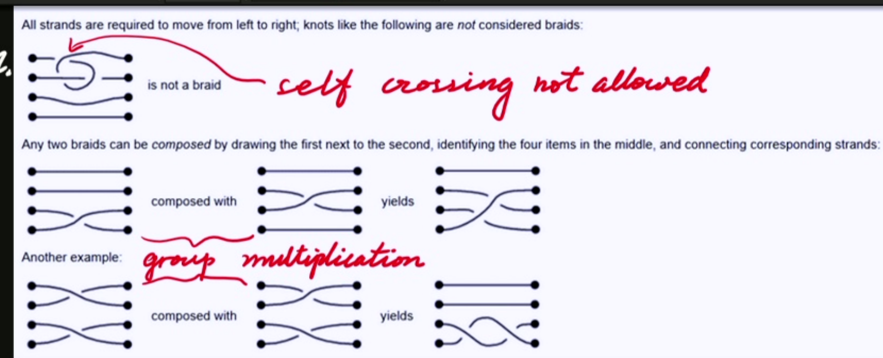
- Braid group

You have seen braids very often since a child, but do you know that all possible braids of n strands form a group? This group is the n-strand braid group B_n.

To study the braid group mathematically, we need to



set up the rules & abstraction,

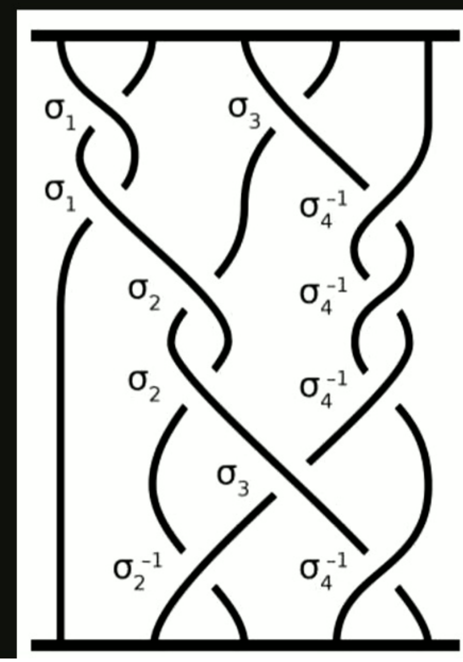


Multiplication is concatenation :

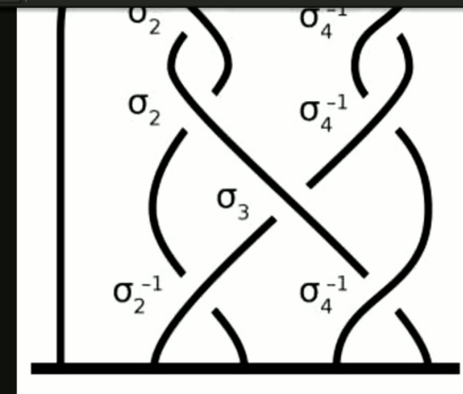
- braid generators

We can break up any braid into the multiplication of a number of generators :

An n-strand braid has (n-1) generators, each being a single crossing of two adjacent strands:



multiplication of a number of generators:



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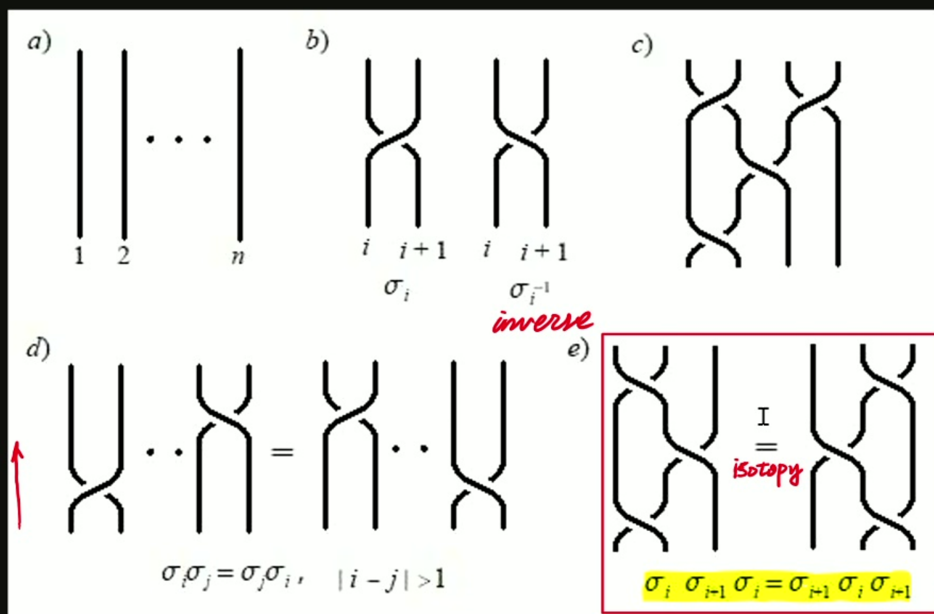
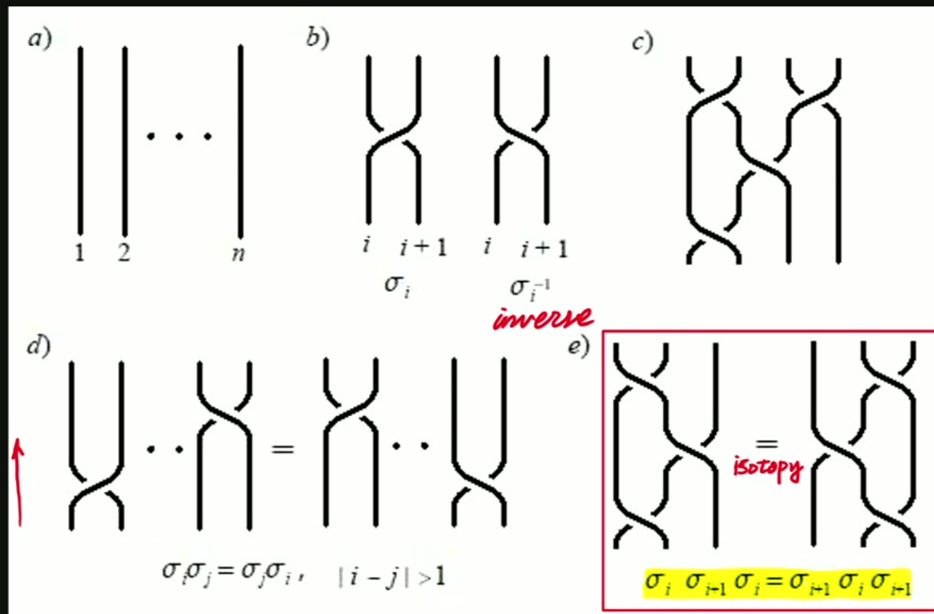


The generators are not all free but subject to a few conditions

The Yang-Baxter Eq.



The generators are not all free but subject to a few conditions

The Yang-Baxter Eq.

- The presentation

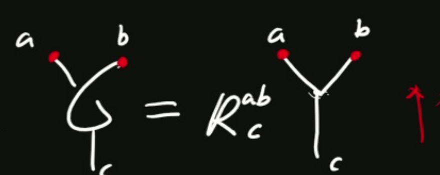
$$B_n = \left(\sigma_1, \sigma_2, \cdots, \sigma_n \mid \sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1}, \forall |i-j|>1 ; \; \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}, \forall i \right)$$

representations of the group over certain relevant representation spaces (vector / Hilbert spaces).

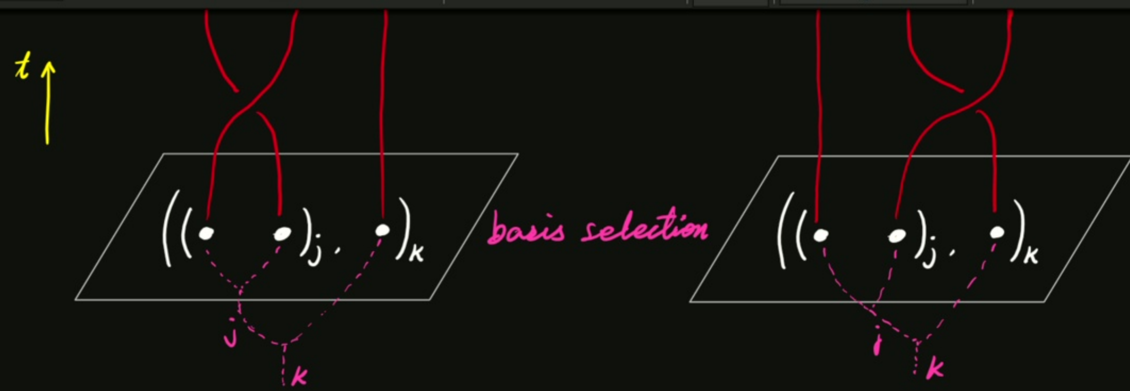
For our purposes, we shall focus on B_3 and construct its representatio[n] over the Hilbert space of 3 Fibonacci anyons.

- Braiding 3 Fibonacci anyons

The R matrix doesn't really swap two anyons.



We need the braiding operators, i.e., the braid group generators to swap two anyons physically.

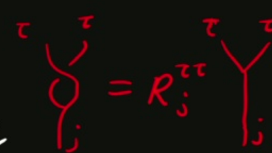


Remember?

The question is how to represent σ_1 & σ_2 on $\mathcal{H}_{3\tau} = \text{span}\{|0\rangle, |1\rangle, |N\rangle\}$?
That is, we need to find the 3×3 unitary matrices $\rho(\sigma_1)$ & $\rho(\sigma_2)$
acting on $\mathcal{H}_{3\tau}$.

R links 2 basis states with the same fusion product j.

▷ $\rho(\sigma_1)$

This one is easy because 

, so in the basis chosen, $\rho(\sigma_1)$

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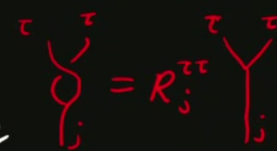


, so in the basis chosen, $\rho(\sigma_1)$ is automatically block-diagonalized, with a 2×2 block identified with the 2×2 R matrix we obtained before:

$$\rho(\sigma_1)\begin{pmatrix} |0\rangle \\ |1\rangle \\ |N\rangle \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} e^{-i\frac{4\pi}{5}} & 0 \\ 0 & -e^{-i\frac{2\pi}{5}} \end{matrix}}^{\,R} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \underbrace{-e^{-i\frac{2\pi}{5}}}_{R^{\tau\tau}_{\tau}} \end{pmatrix} \begin{pmatrix} |0\rangle := |((\cdot\ \cdot)_1, \cdot)_\tau\rangle \\ |1\rangle := |((\cdot\ \cdot)_\tau, \cdot)_\tau\rangle \\ |N\rangle := |((\cdot\ \cdot)_\tau, \cdot)_1\rangle \end{pmatrix}$$

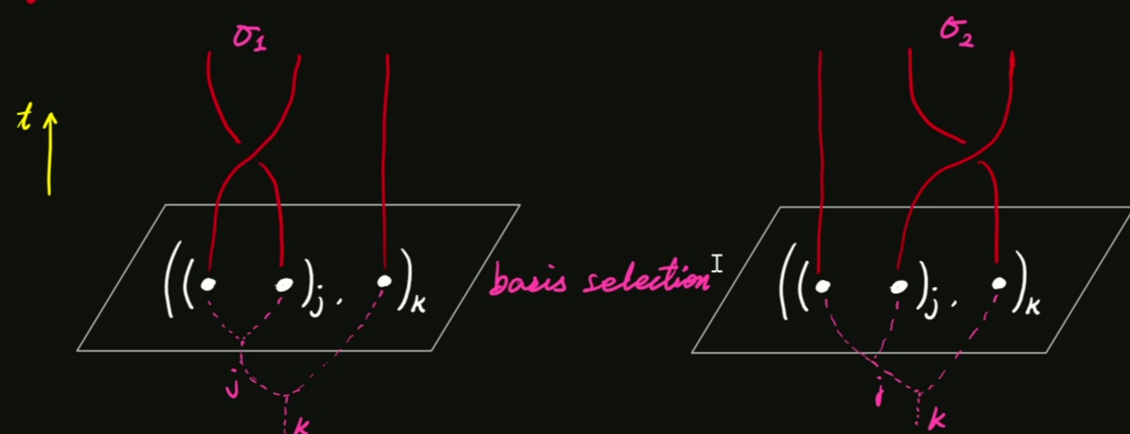
Justifies this logical qubit? Not yet.

We need the braiding operators, i.e., the braid group generators to swap two anyons physically.

- Braiding Matrices

Consider 3 τ's. We have to choose a basis of $\mathcal{H}_{3\tau}$

Only 2 generators:



Remember?

are not fused first in the basis chosen. What can we do about this?

We need to first change the basis using the F-matrix, then apply the R-matrix, and in the end change the basis back to the basis chosen again by the F-matrix. Formally,

$$\rho(\sigma_2) = F^{-1} \underset{\substack{\uparrow \\ \text{exchange the right 2 } \tau\text{'s}}}{R} F$$

Let's do this:

$$|0\rangle = |((\cdot\ \cdot)_1,\ \cdot)_\tau\rangle = F_{11}|(\cdot\ (\cdot\ \cdot)_1)_\tau\rangle + F_{1\tau}|(\cdot,\ (\cdot\ \cdot)_\tau)_\tau\rangle$$

$$= \sum_{p=1}^{\tau} \ [F^{\tau}_{\tau\tau\tau}]_{1p}$$

$$\Big\Downarrow F^{-1}$$

$$= \left([F^{-1}]_{11} e^{-i\frac{4\pi}{5}} F_{11} - [F^{-1}]_{1\tau} e^{-i\frac{2\pi}{5}} F_{\tau 1}\right)|0\rangle$$

$$+ \left([F^{-1}]_{\tau 1} e^{-i\frac{4\pi}{5}} F_{11} - [F^{-1}]_{\tau\tau} e^{-i\frac{2\pi}{5}} F_{\tau 1}\right)|1\rangle$$

$$= e^{-i\frac{\pi}{5}} \phi^{-1} |0\rangle - i e^{-i\frac{\pi}{10}} \sqrt{\phi^{-1}}\, |1\rangle = \rho(\sigma_2)|0\rangle$$

Exercise: Find $\rho(\sigma_2)|1\rangle$ & $\rho(\sigma_2)|N\rangle$ similarly.

Therefore,

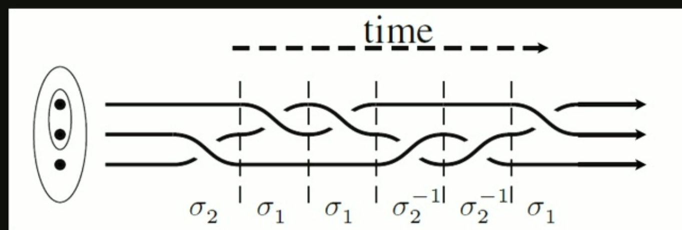
$$\rho(\sigma_2)\begin{pmatrix}|0\rangle \\ |1\rangle \\ |N\rangle\end{pmatrix} = \begin{pmatrix} -e^{-i\frac{2\pi}{5}}\phi^{-1} & -i e^{-i\frac{\pi}{10}}\sqrt{\phi^{-1}} & 0 \\ -i e^{-i\frac{\pi}{10}}\sqrt{\phi^{-1}} & -\phi^{-1} & 0 \\ 0 & 0 & -e^{-i\frac{2\pi}{5}} \end{pmatrix}\begin{pmatrix}|0\rangle \\ |1\rangle \\ |N\rangle\end{pmatrix}$$

Justifies the logical qubit

TQC with Fibonacci anyons

- Combining σ_1 & σ_2 $\Rightarrow B_3$

E.g.



Recall

$$\sigma_2 \, \sigma_1 \, \sigma_2 = \sigma_1 \, \sigma_2 \, \sigma_1$$

- In fact, $\rho(\sigma_1)$ & $\rho(\sigma_2)$ generate a dense subset of $SU(2)$

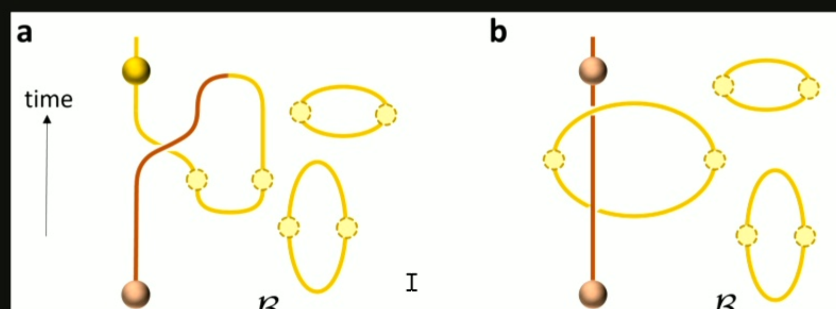
essentially $SU(2)$

- Thus, Fibonacci anyons ⇒ universal TQC

TQC is robust against errors due to local perturbations.
Keep anyons far enough, it's hard to have nonlocal errors.

Two possible error sources:

But actually fine!



- TQC is an approximation of the quantum circuit model!

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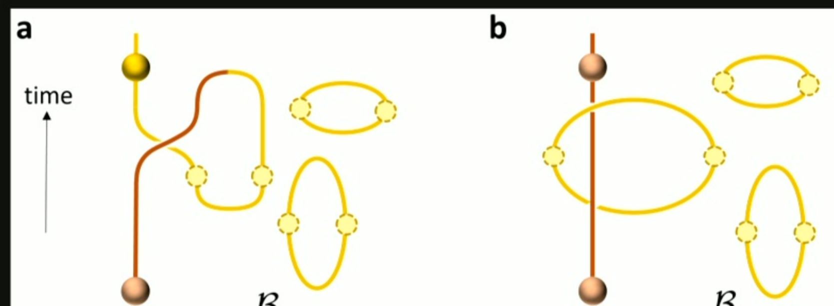

But actually fine!

- TQC is an approximation of the quantum circuit model!

- Challenge: leakage in 2-qubit & multi-qubit gates.

Exercises: