Title: Topological orders and topological quantum computing - Lecture 20230309

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Collection: Mini introductory course on topological orders and topological quantum computing

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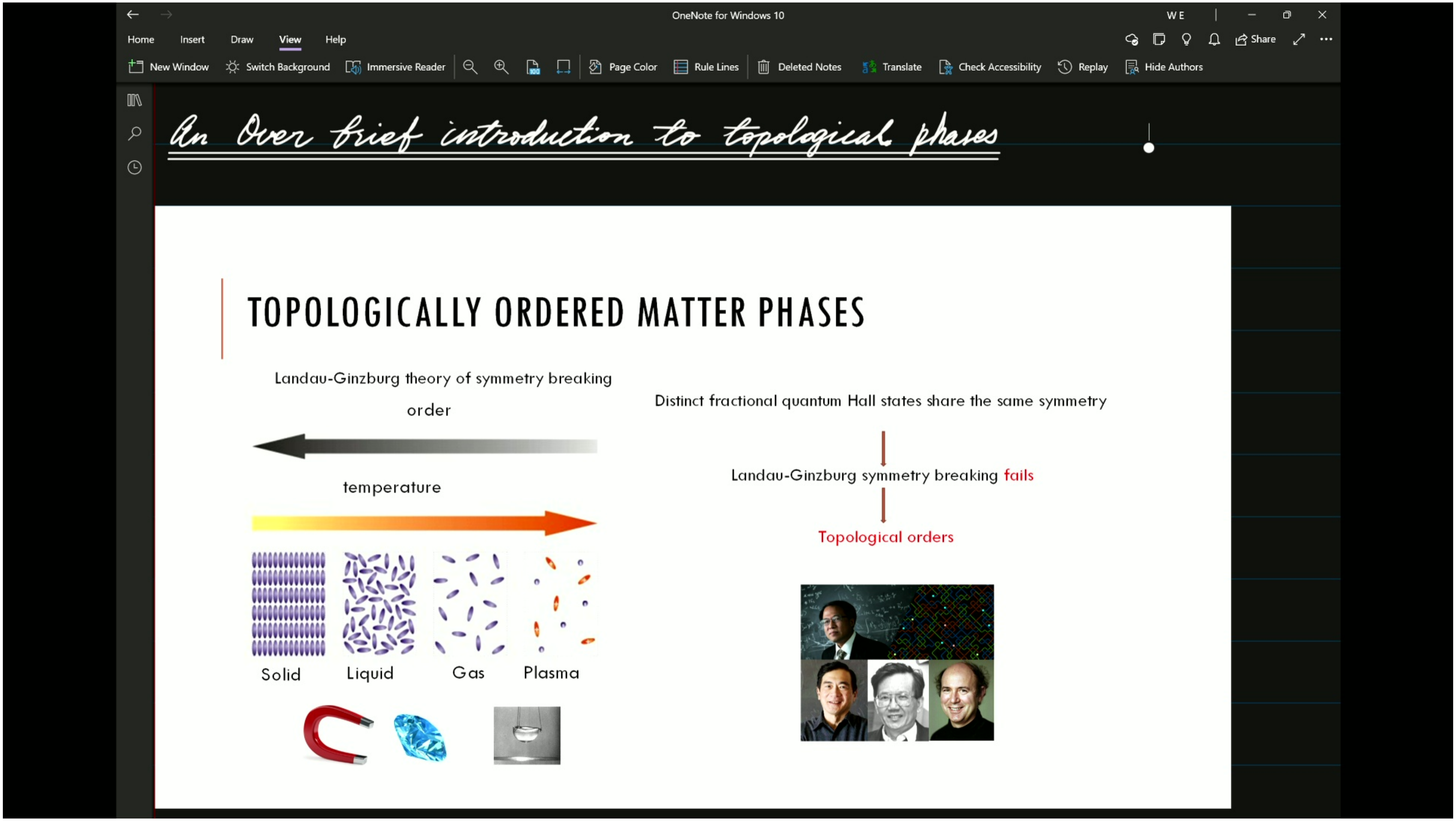
Abstract: In this mini course, I shall introduce the basic concepts in 2D topological orders by studying simple models of topological orders and then introduce topological quantum computing based on Fibonacci anyons. Here is the (not perfectly ordered) syllabus.

Overview of topological phases of matter
Z2 toric code model: the simplest model of 2D topological orders
Quick generalization to the quantum double model
Anyons, topological entanglement entropy, S and T matrices
Fusion and braiding of anyons: quantum dimensions, pentagon and hexagon identities
Fibonacci anyons
Topological quantum computing.



An Over brief introduction to topological phases

TOPOLOGICALLY ORDERED MATTER PHASES

Landau-Ginzburg theory of symmetry breaking order

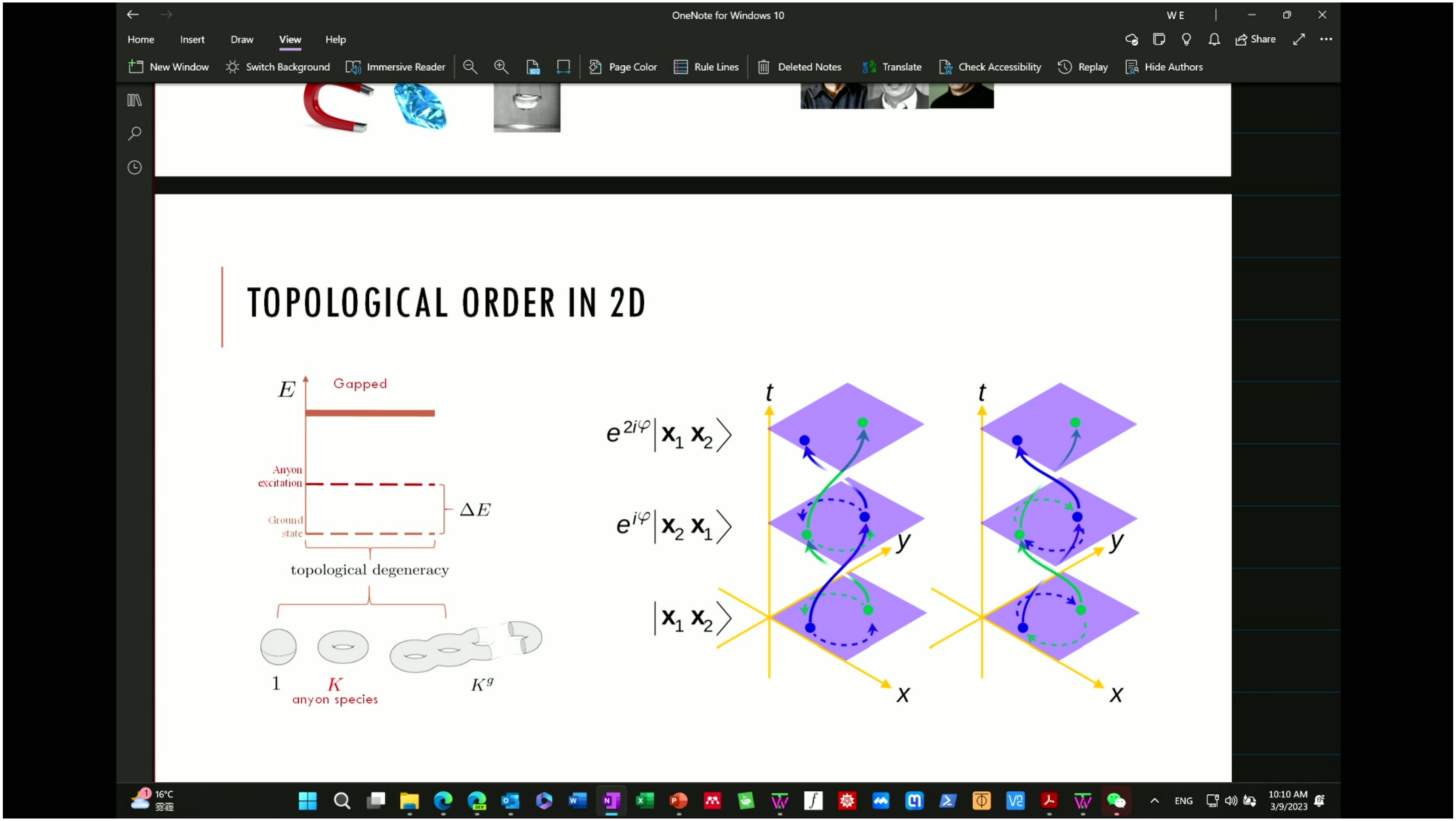
temperature

Solid Liquid Gas Plasma

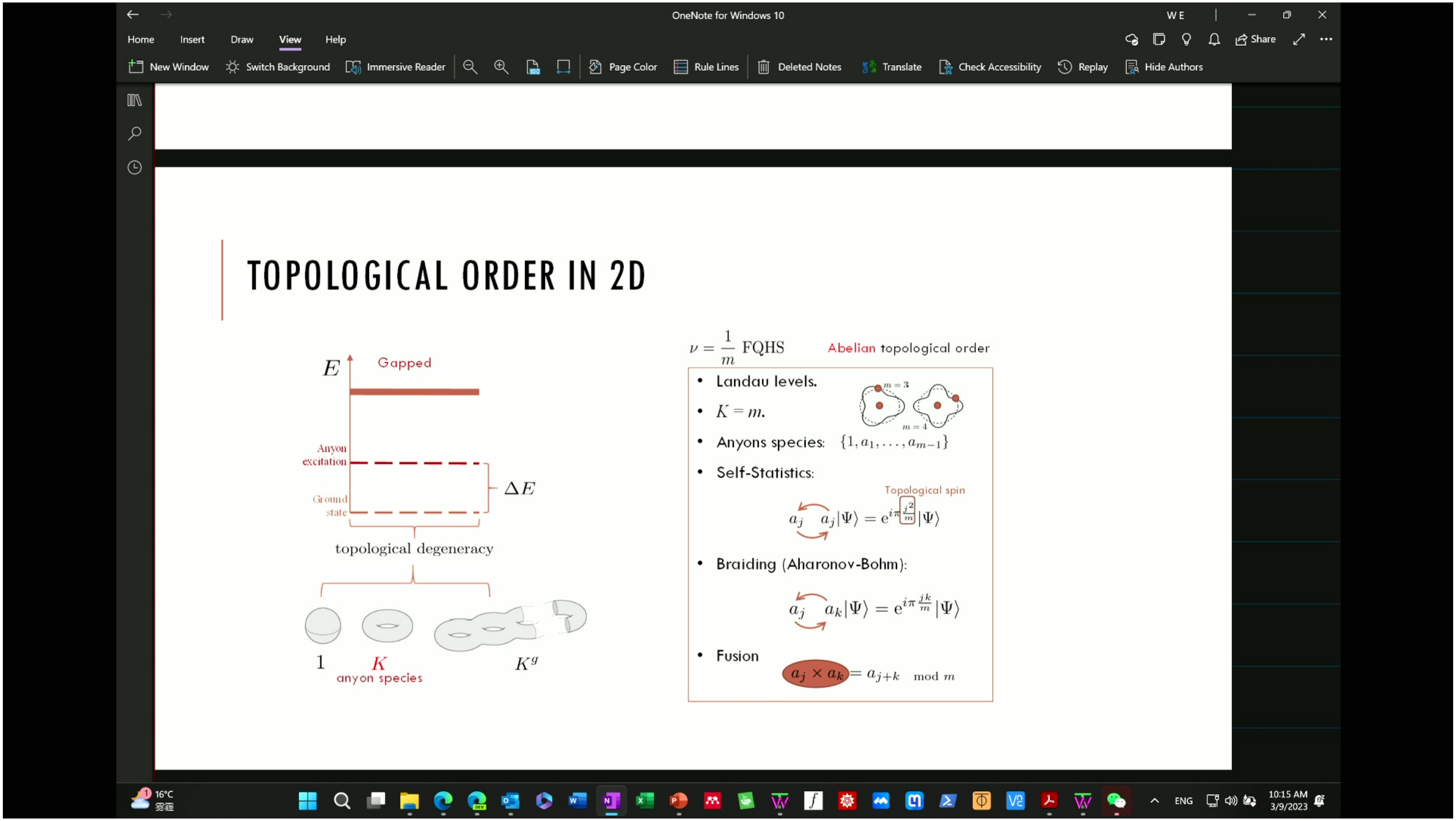
Distinct fractional quantum Hall states share the same symmetry

Landau-Ginzburg symmetry breaking fails

Topological orders



TOPOLOGICAL ORDER IN 2D



TOPOLOGICAL ORDER IN 2D

$\nu = \frac{1}{m}$ FQHS   Abelian topological order

- Landau levels.
- $K = m.$
- Anyons species: $\{1, a_1, \dots, a_{m-1}\}$
- Self-Statistics:

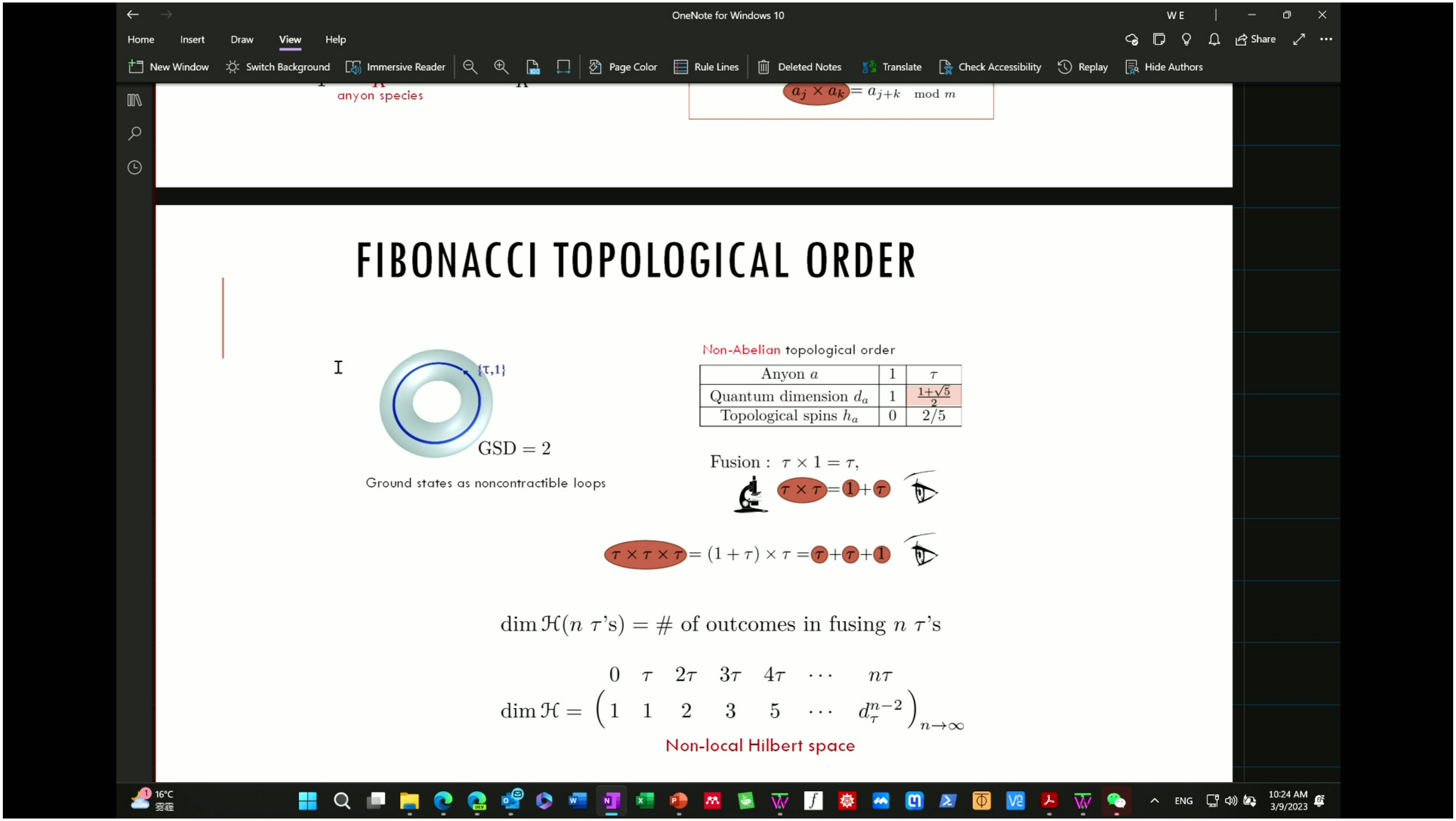
$$a_j \quad a_j|\Psi\rangle = \mathrm{e}^{i\pi \boxed{\frac{j^2}{m}}}|\Psi\rangle$$

- Braiding (Aharonov-Bohm):

$$a_j \quad a_k|\Psi\rangle = \mathrm{e}^{i\pi \frac{jk}{m}}|\Psi\rangle$$

- Fusion

$$a_j \times a_k = a_{j+k} \mod m$$



anyon species

$a_j \times a_k = a_{j+k} \mod m$

FIBONACCI TOPOLOGICAL ORDER

$\mathrm{GSD} = 2$

Ground states as noncontractible loops

Non-Abelian topological order

Anyon a	1	τ
Quantum dimension d_a	1	$\frac{1+\sqrt{5}}{2}$
Topological spins h_a	0	$2/5$

$$\text{Fusion}: \ \tau \times 1 = \tau,$$

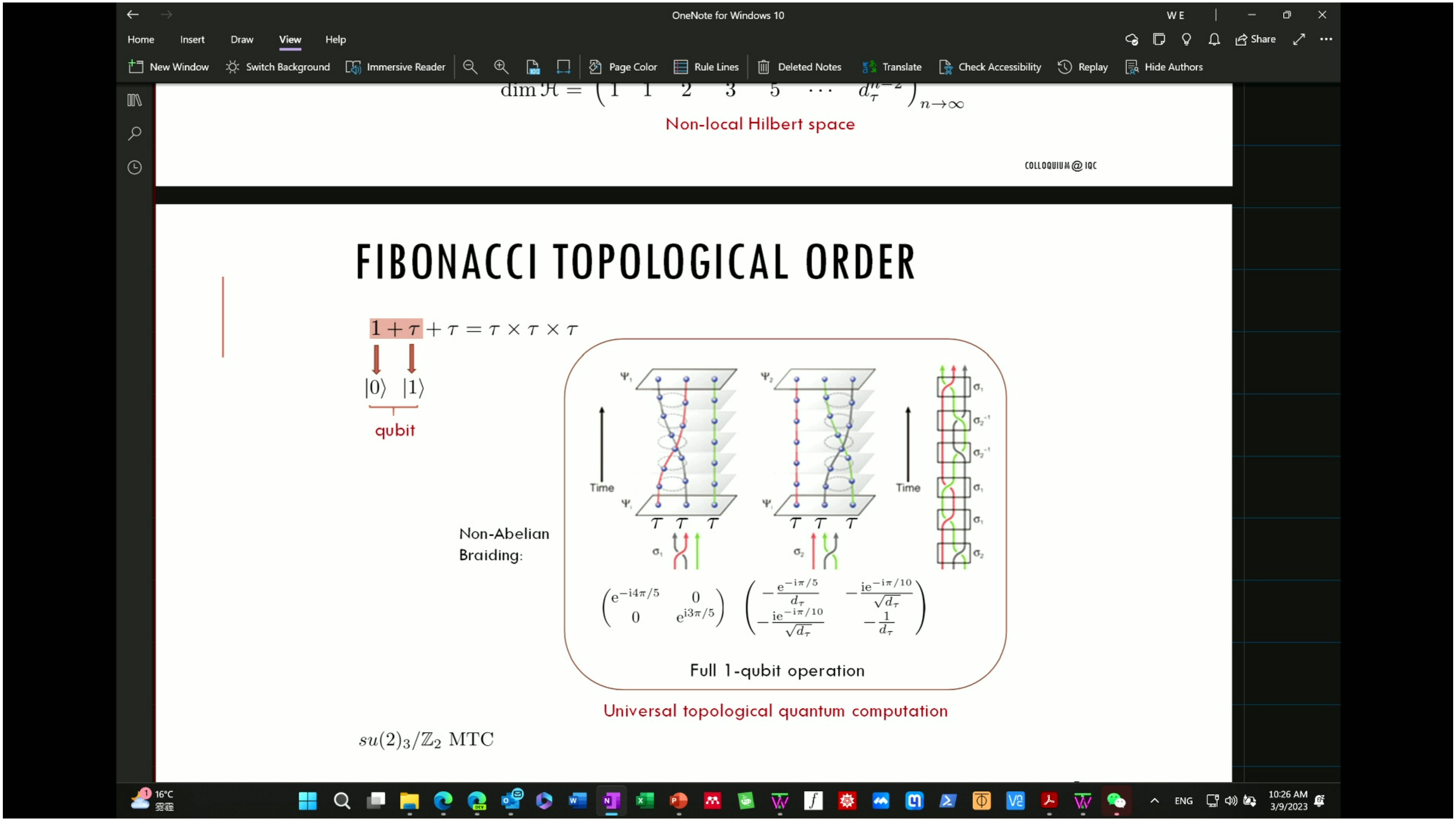
$$\tau \times \tau = 1 + \tau$$

$$\tau \times \tau \times \tau = (1+\tau) \times \tau = \tau + \tau + 1$$

$$\dim \mathcal{H}(n \ \tau\text{'s}) = \# \text{ of outcomes in fusing } n \ \tau\text{'s}$$

$$\dim \mathcal{H} = \begin{array}{c} \begin{array}{ccccccc} 0 & \tau & 2\tau & 3\tau & 4\tau & \cdots & n\tau \end{array} \\ \begin{pmatrix} 1 & 1 & 2 & 3 & 5 & \cdots & d_\tau^{n-2} \end{pmatrix}_{n \to \infty} \end{array}$$

Non-local Hilbert space



$$\dim \mathcal{H} = \begin{pmatrix} 1 & 1 & 2 & 3 & 5 & \cdots & d_\tau^{n-2} \end{pmatrix}_{n\to\infty}$$

Non-local Hilbert space

COLLOQUIUM @ IQC

FIBONACCI TOPOLOGICAL ORDER

$$\underbrace{1+\tau}+\tau=\tau\times\tau\times\tau$$

$|0\rangle$ $|1\rangle$

qubit

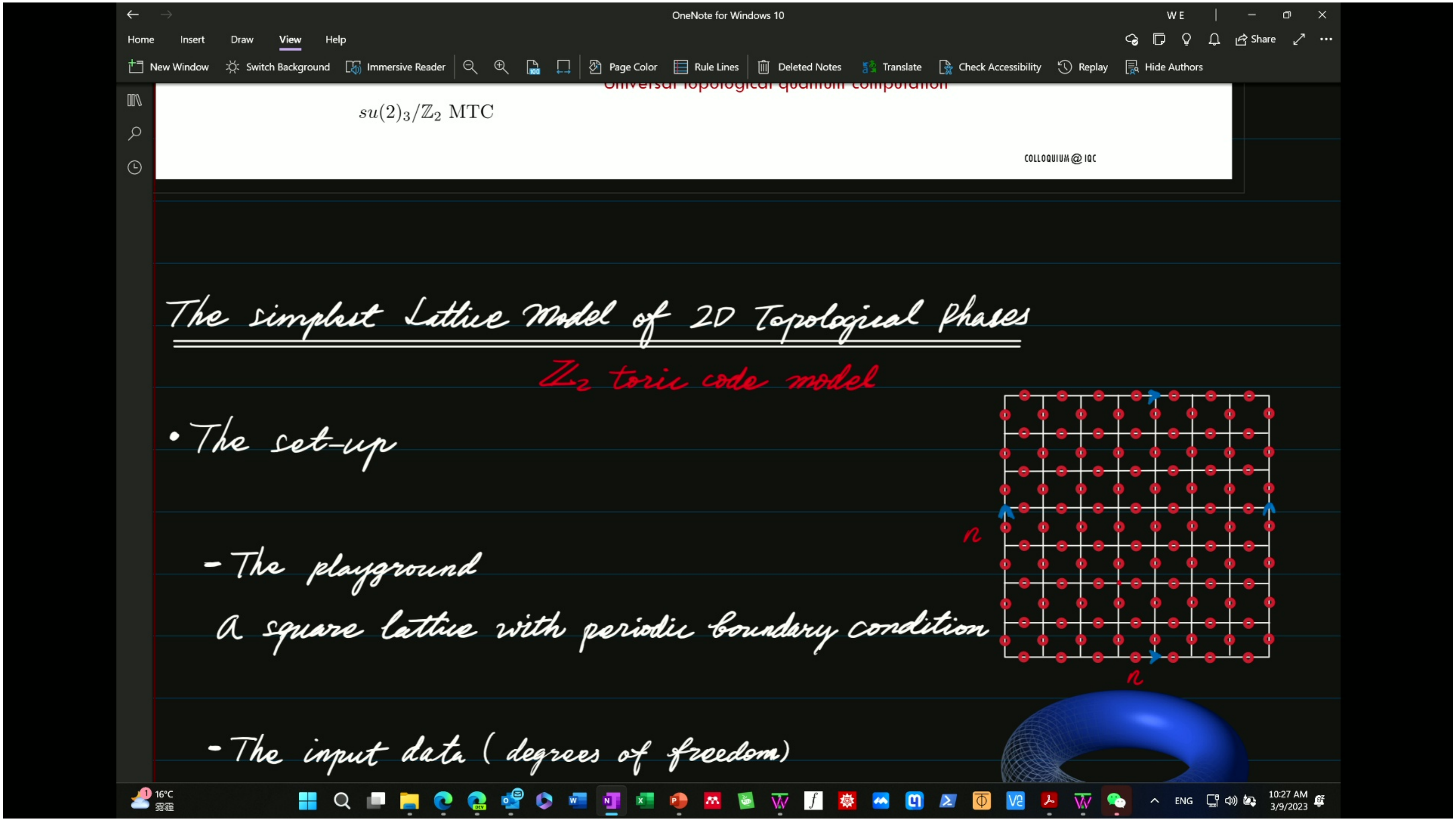
Non-Abelian Braiding:

$$\begin{pmatrix} e^{-i4\pi/5} & 0 \\ 0 & e^{i3\pi/5} \end{pmatrix} \quad \begin{pmatrix} -\frac{e^{-i\pi/5}}{d_\tau} & -\frac{ie^{-i\pi/10}}{\sqrt{d_\tau}} \\ -\frac{ie^{-i\pi/10}}{\sqrt{d_\tau}} & -\frac{1}{d_\tau} \end{pmatrix}$$

Full 1-qubit operation

Universal topological quantum computation

$su(2)_3/\mathbb{Z}_2$ MTC



Universal topological quantum computation

$su(2)_3/\mathbb{Z}_2$ MTC

COLLOQUIUM @ IQC

The simplest Lattice Model of 2D Topological Phases

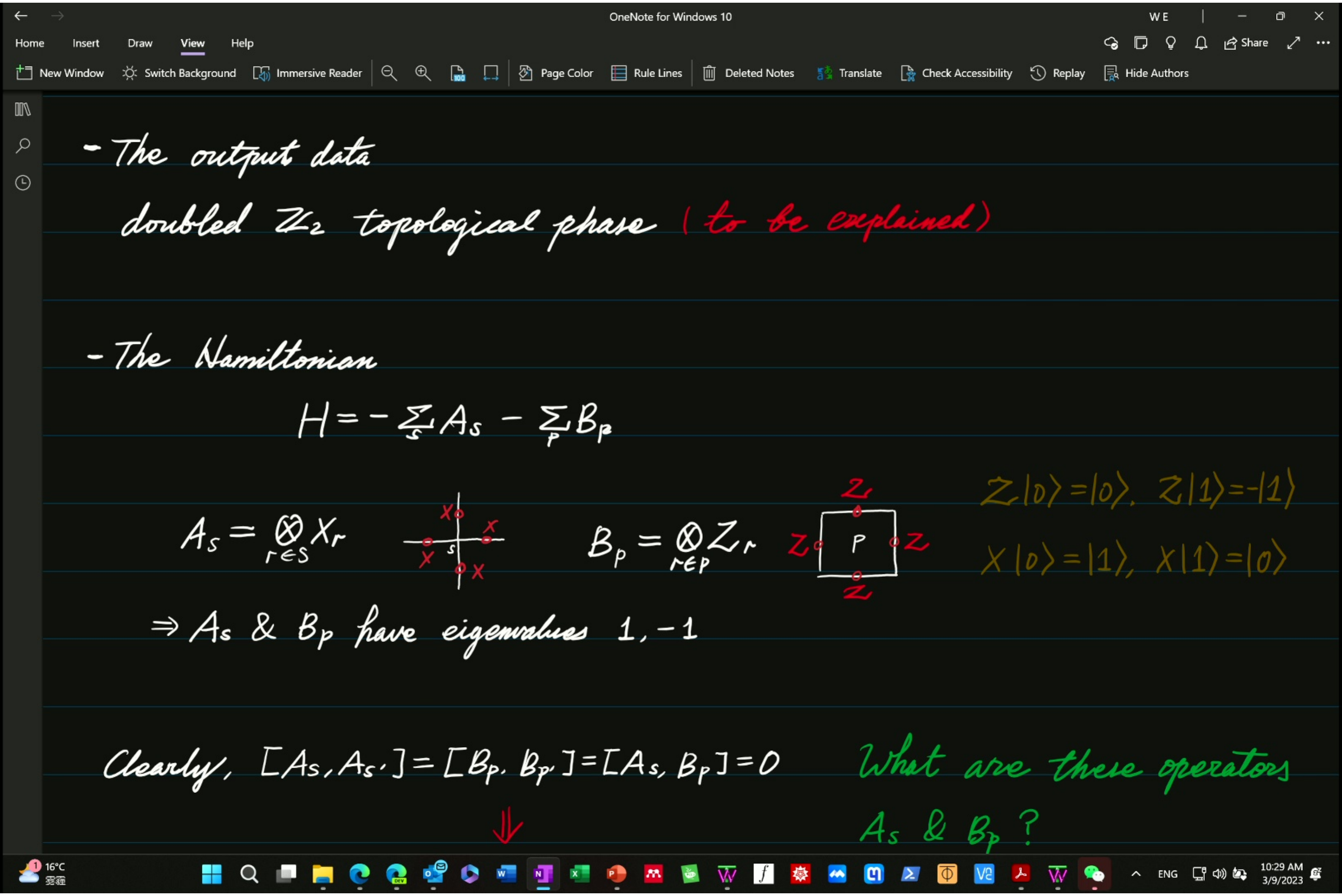
$\mathbb{Z}_2$ toric code model

- The set-up

  - The playground

    A square lattice with periodic boundary condition

  - The input data (degrees of freedom)



- The output data

doubled $\mathbb{Z}_2$ topological phase (to be explained)

- The Hamiltonian

$$H = -\sum_s A_s - \sum_p B_p$$

$$A_s = \bigotimes_{r \in s} X_r \qquad B_p = \bigotimes_{r \in p} Z_r$$

$Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$

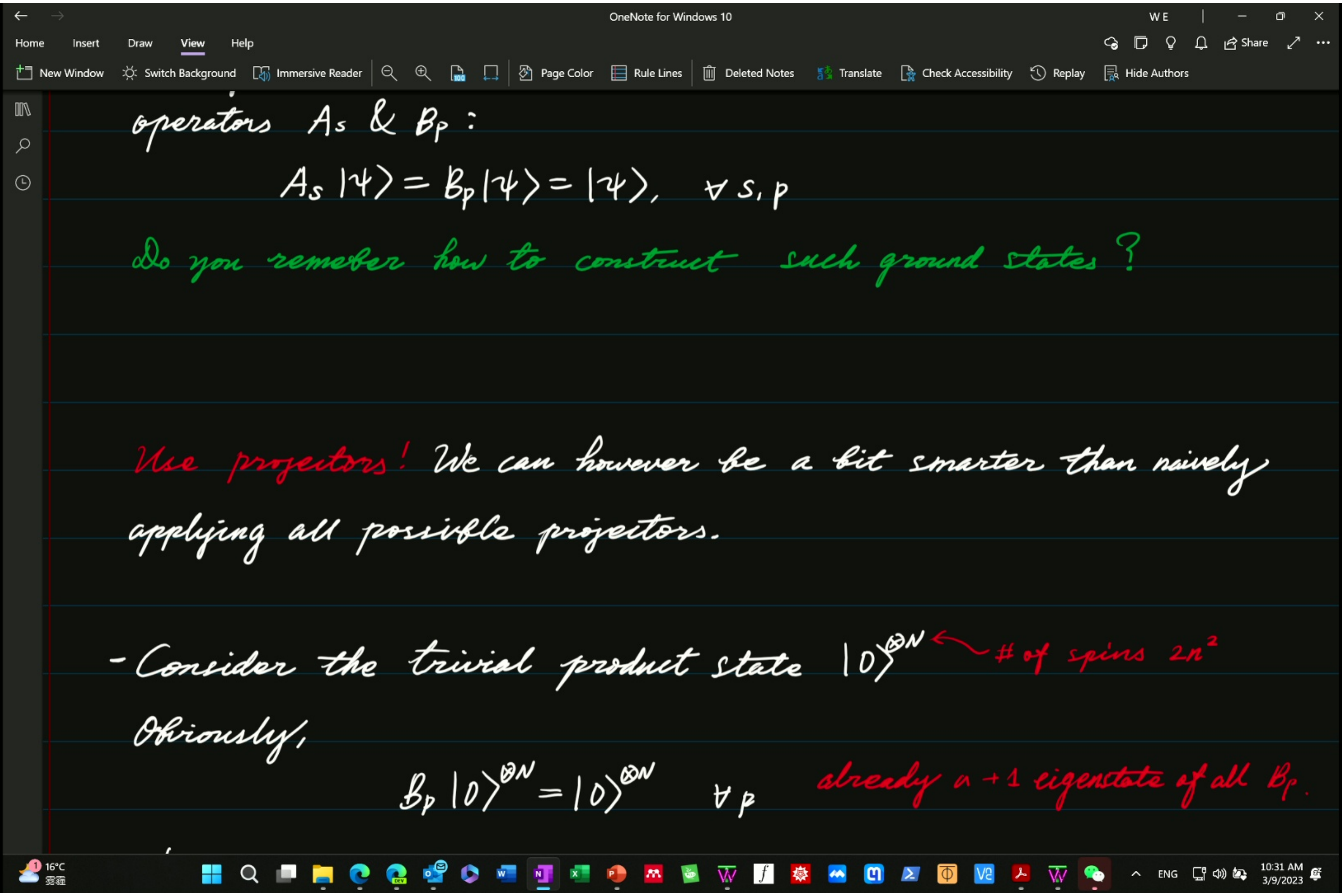
$X|0\rangle = |1\rangle$, $X|1\rangle = |0\rangle$

$\Rightarrow A_s$ & B_p have eigenvalues $1, -1$

Clearly, $[A_s, A_{s'}] = [B_p, B_{p'}] = [A_s, B_p] = 0$

$\Downarrow$

What are these operators A_s & B_p?



operators A_s & B_p :

$$A_s|\psi\rangle = B_p|\psi\rangle = |\psi\rangle, \quad \forall s, p$$

Do you remember how to construct such ground states?

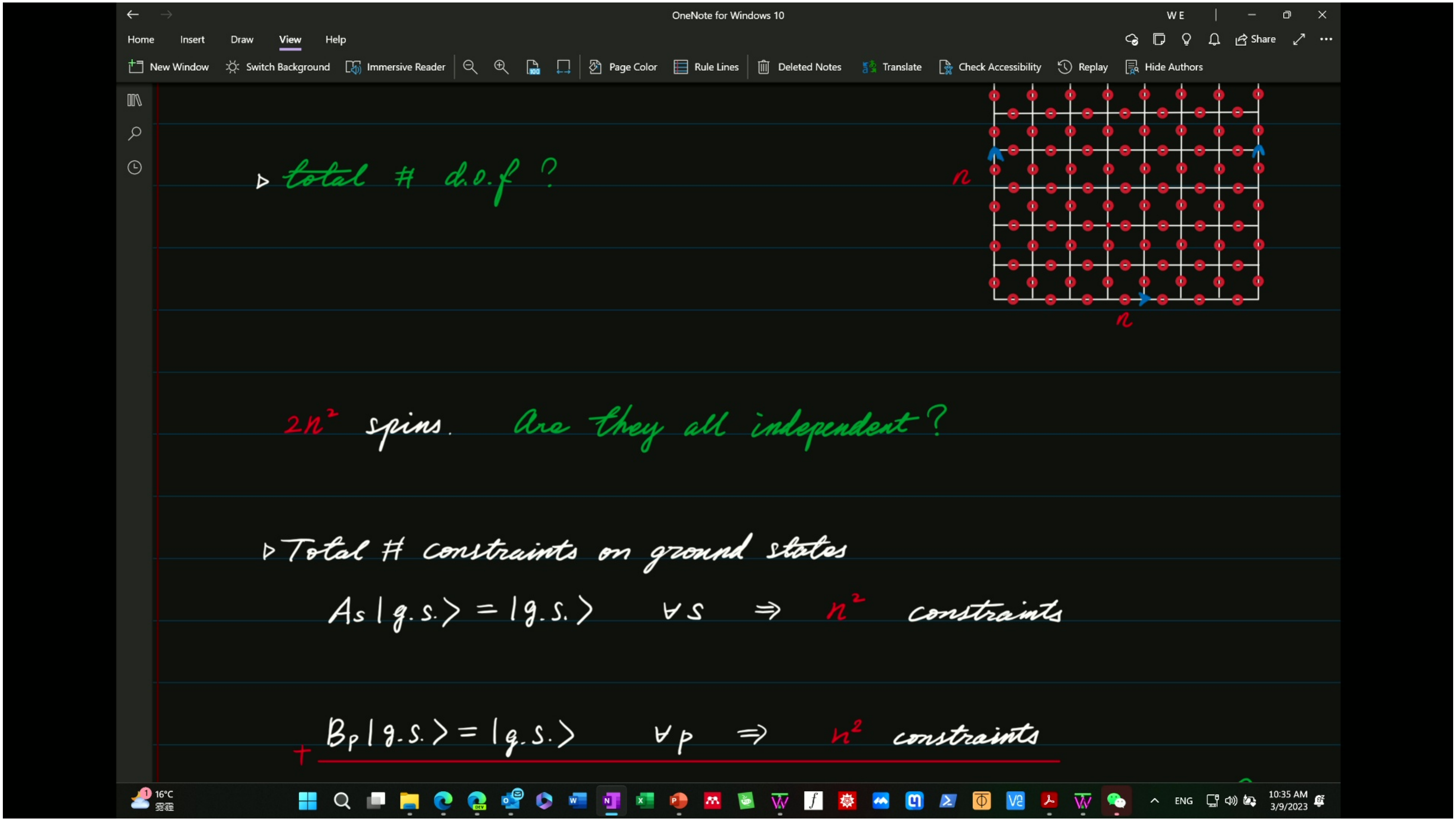
Use projectors! We can however be a bit smarter than naively applying all possible projectors.

- Consider the trivial product state $|0\rangle^{\otimes N}$ ← # of spins $2n^2$

Obviously,

$$B_p|0\rangle^{\otimes N} = |0\rangle^{\otimes N} \quad \forall p$$

already a +1 eigenstate of all B_p.



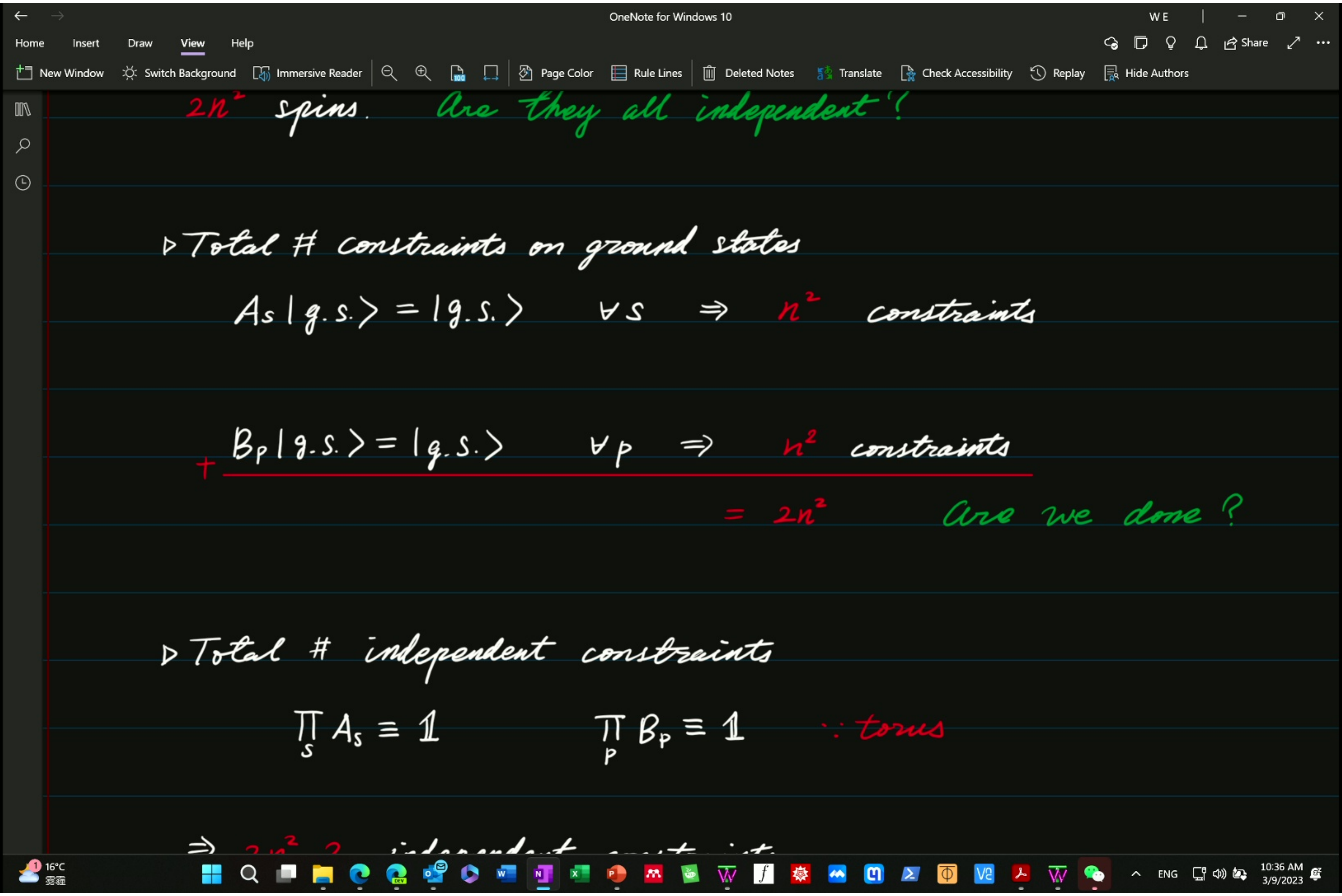
▷ total # d.o.f ?

$2n^2$ spins. Are they all independent?

▷ Total # constraints on ground states

$$A_s|g.s.\rangle = |g.s.\rangle \quad \forall s \quad \Rightarrow \quad n^2 \text{ constraints}$$

$$+ \quad \underline{B_p|g.s.\rangle = |g.s.\rangle \quad \forall p \quad \Rightarrow \quad n^2 \text{ constraints}}$$



$2n^2$ spins. Are they all independent?

▷ Total # constraints on ground states

$A_s |g.s.\rangle = |g.s.\rangle \quad \forall s \quad \Rightarrow \quad n^2$ constraints

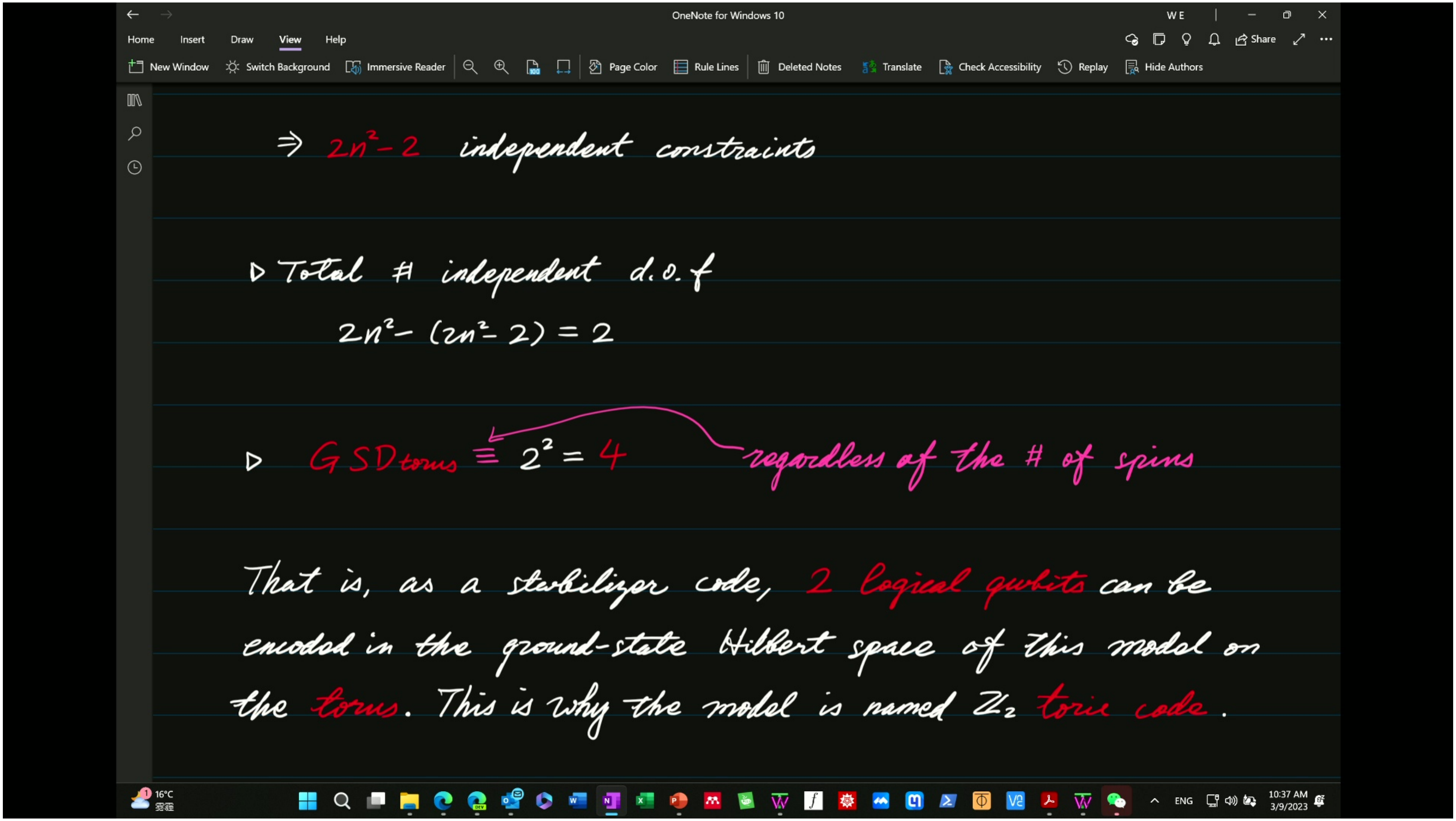
$$\underline{+ \quad B_p |g.s.\rangle = |g.s.\rangle \quad \forall p \quad \Rightarrow \quad n^2 \text{ constraints}}$$

$= 2n^2$ Are we done?

▷ Total # independent constraints

$\prod_s A_s \equiv \mathbb{1} \qquad \prod_p B_p \equiv \mathbb{1}$ $\because$ torus

$\Rightarrow$ $2n^2$ 2 independent constraints



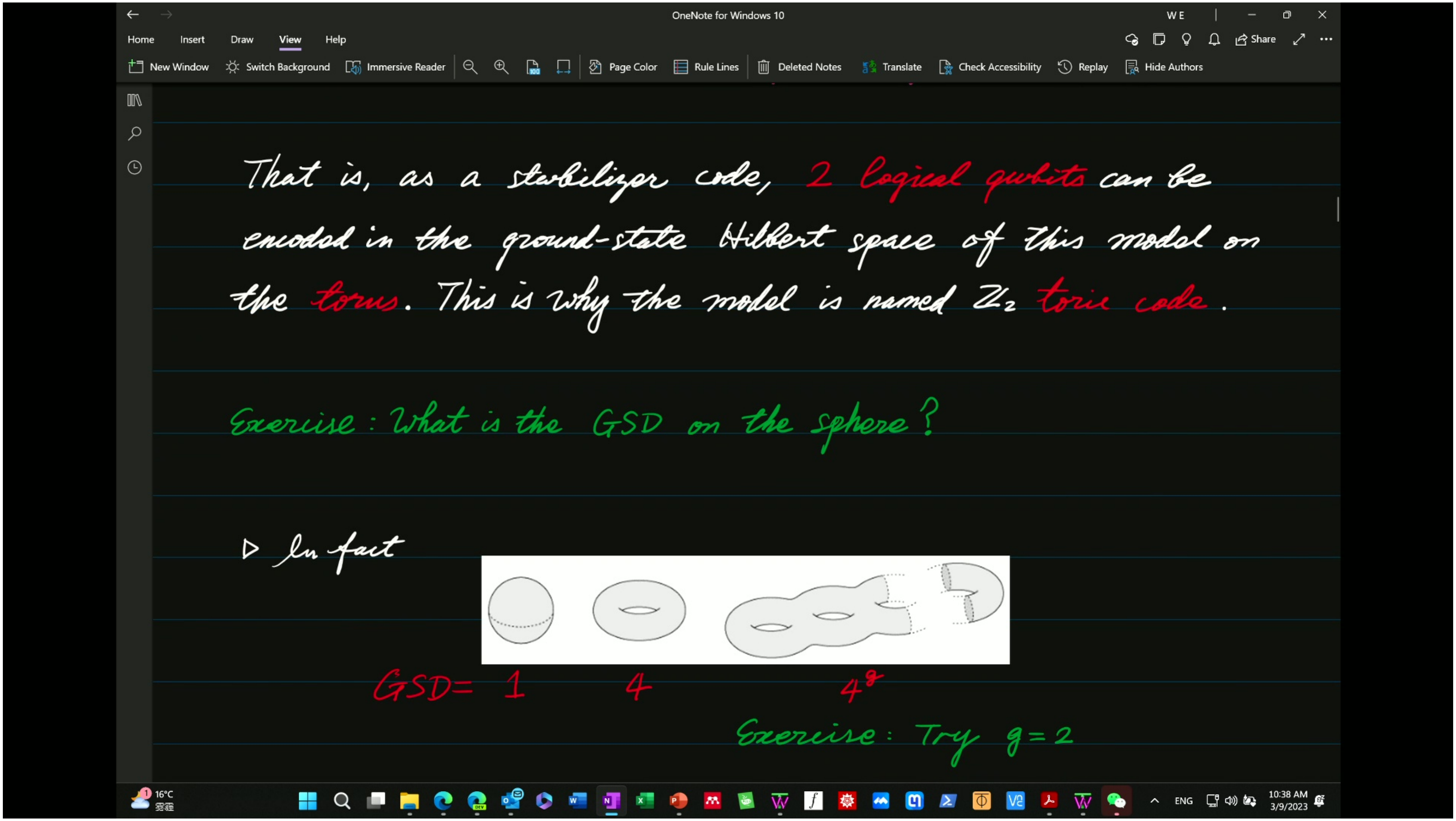
$\Rightarrow$ $2n^2-2$ independent constraints

▷ Total # independent d.o.f

$$2n^2-(2n^2-2)=2$$

▷ $GSD_{torus} \equiv 2^2 = 4$ regardless of the # of spins

That is, as a stabilizer code, 2 logical qubits can be encoded in the ground-state Hilbert space of this model on the torus. This is why the model is named $\mathbb{Z}_2$ toric code.



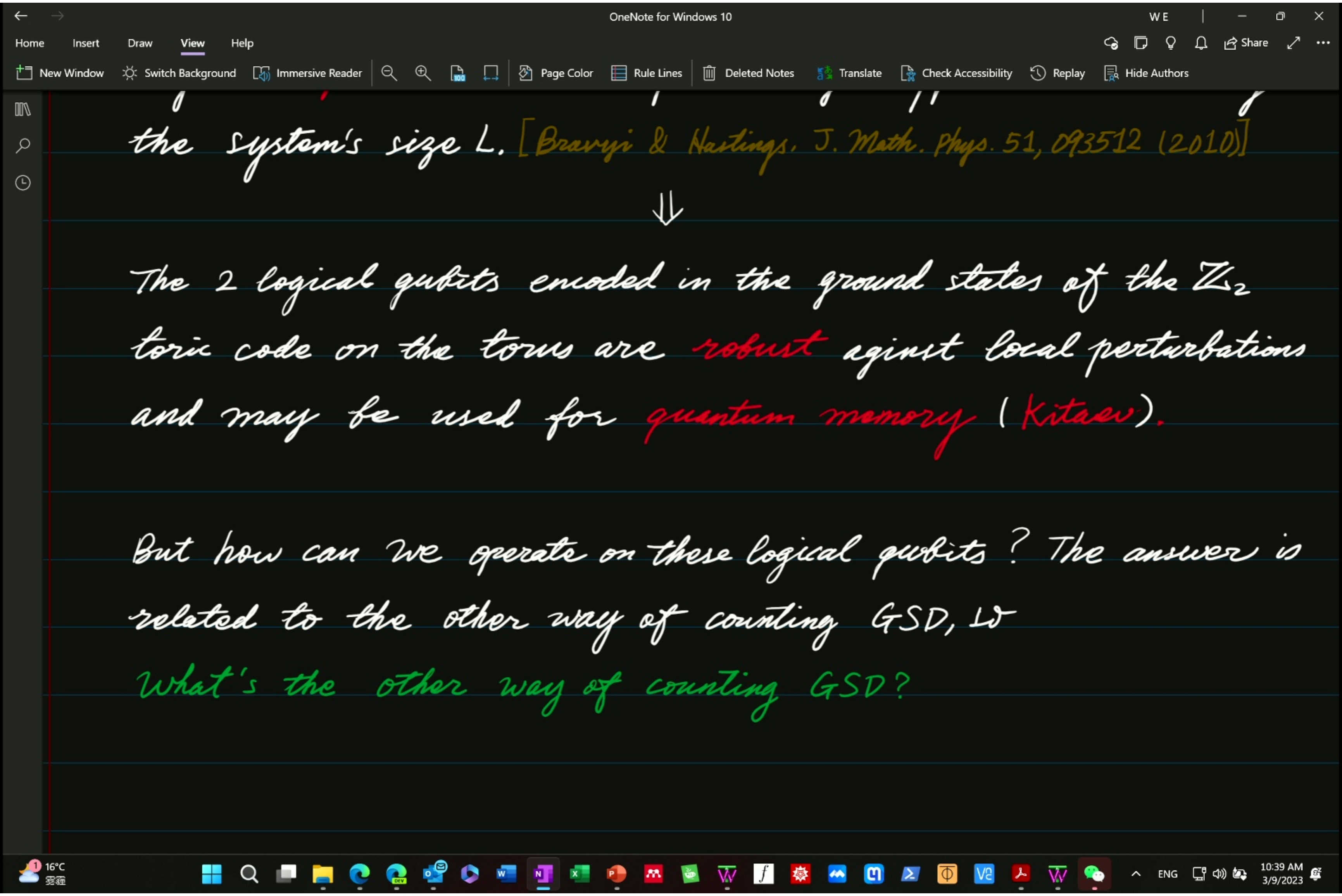
That is, as a stabilizer code, 2 logical qubits can be encoded in the ground-state Hilbert space of this model on the torus. This is why the model is named $\mathbb{Z}_2$ toric code.

Exercise: What is the GSD on the sphere?

▷ In fact

GSD= 1 4 4^g

Exercise: Try $g=2$



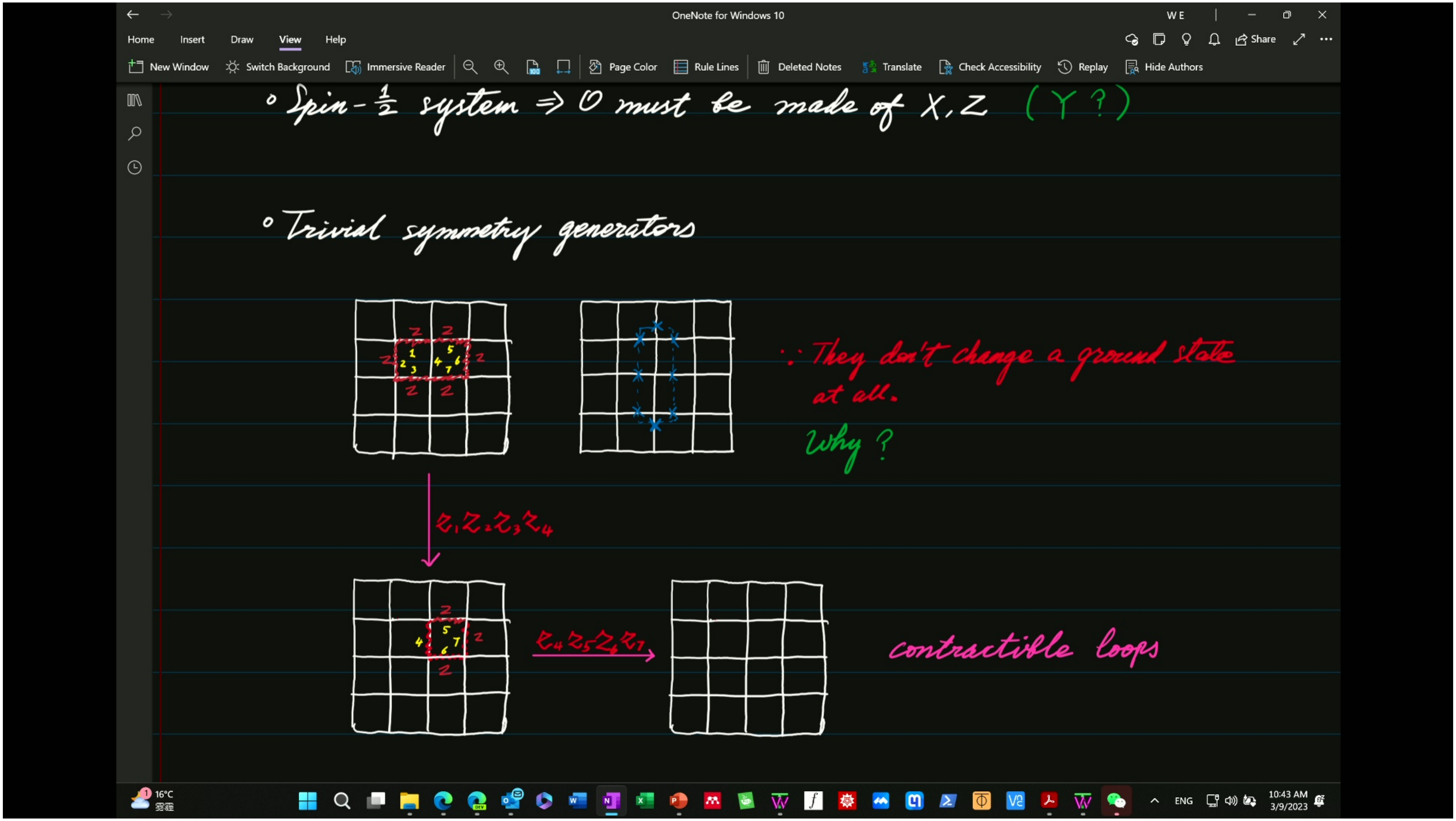
the system's size L. [Bravyi & Hastings, J. Math. Phys. 51, 093512 (2010)]

⇓

The 2 logical qubits encoded in the ground states of the $\mathbb{Z}_2$ toric code on the torus are robust aginst local perturbations and may be used for quantum memory (Kitaev).

But how can we operate on these logical qubits? The answer is related to the other way of counting GSD, Lv

What's the other way of counting GSD?



∘ Spin-$\frac{1}{2}$ system ⇒ $\mathcal{O}$ must be made of X, Z (Y ?)

∘ Trivial symmetry generators

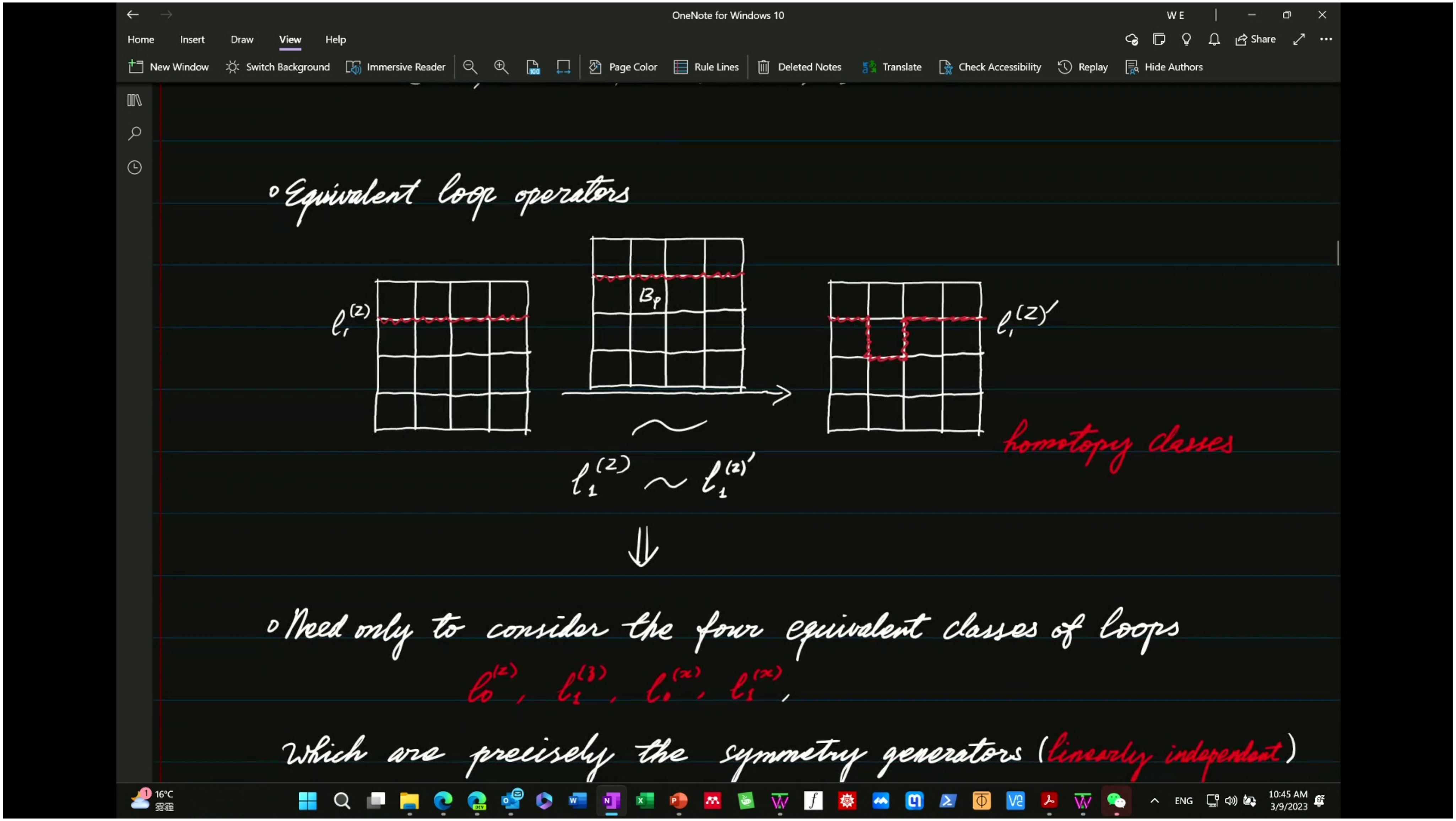
∵ They don't change a ground state at all.

Why ?

$Z_1Z_2Z_3Z_4$

$Z_4Z_5Z_6Z_7$

contractible loops



∘ Equivalent loop operators

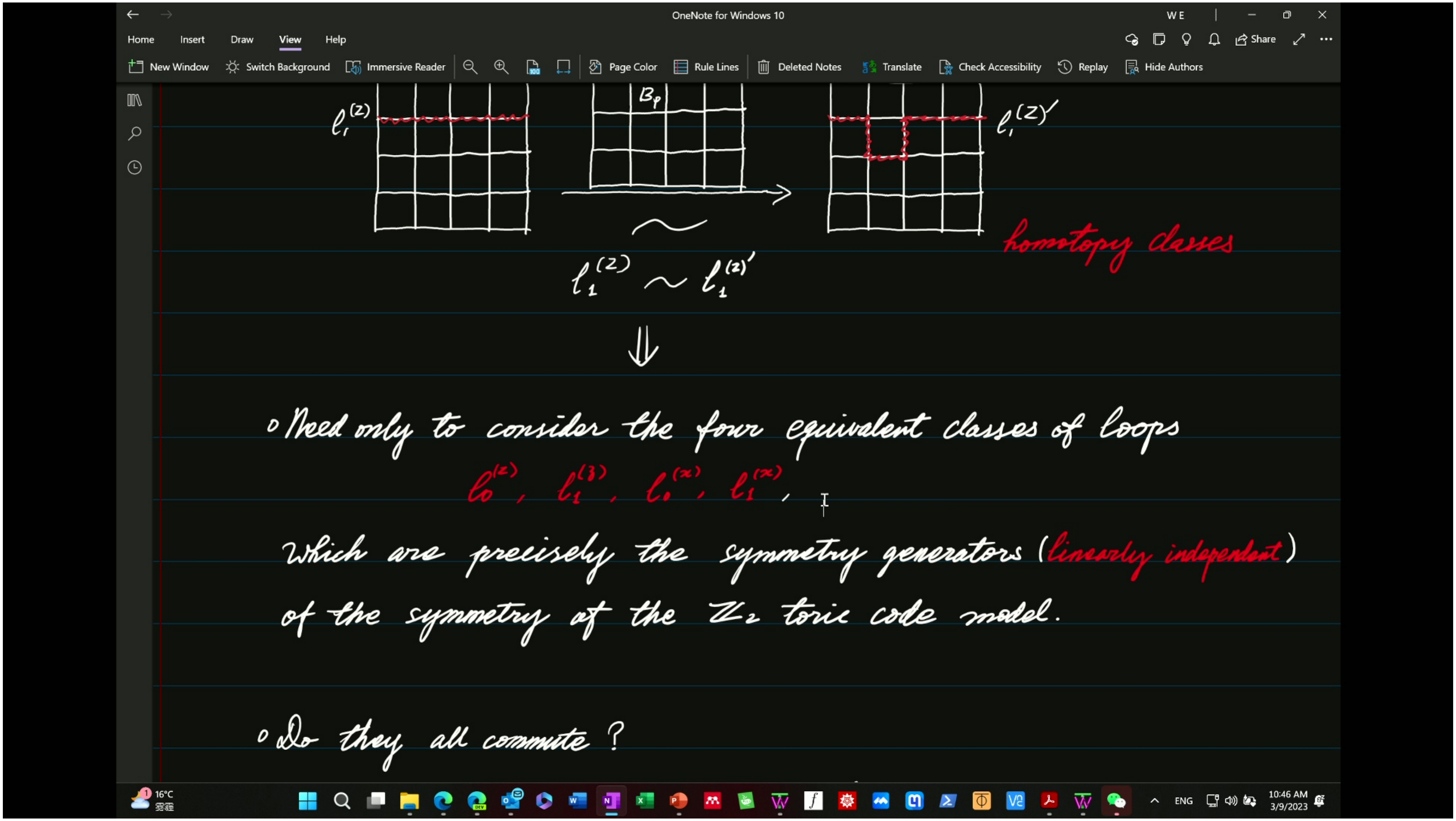
$$\ell_1^{(z)} \sim \ell_1^{(z)'}$$

$$\Downarrow$$

∘ Need only to consider the four equivalent classes of loops

$\ell_0^{(z)}, \ \ell_1^{(z)}, \ \ell_0^{(x)}, \ \ell_1^{(x)},$

Which are precisely the symmetry generators (linearly independent)



$$\ell_1^{(z)} \sim \ell_1^{(z)'}$$

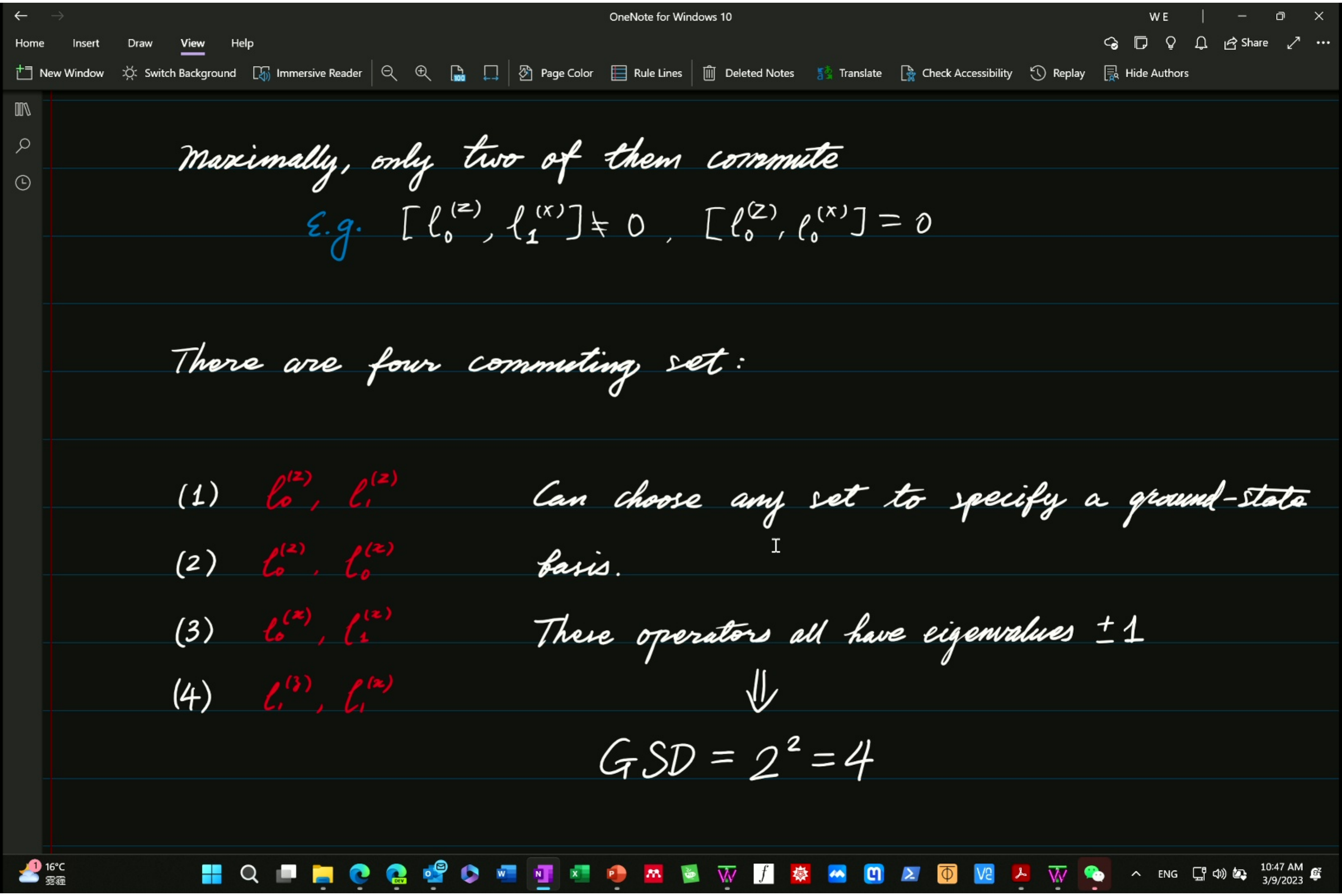
$$\Downarrow$$

- Need only to consider the four equivalent classes of loops

$\ell_0^{(z)}, \ell_1^{(z)}, \ell_0^{(x)}, \ell_1^{(x)},$

Which are precisely the symmetry generators (linearly independent) of the symmetry of the $\mathbb{Z}_2$ toric code model.

- Do they all commute ?



Maximally, only two of them commute

E.g. $[\ell_0^{(z)}, \ell_1^{(x)}] \neq 0$, $[\ell_0^{(z)}, \ell_0^{(x)}] = 0$

There are four commuting set:

(1) $\ell_0^{(z)}, \ell_1^{(z)}$

(2) $\ell_0^{(z)}, \ell_0^{(z)}$

(3) $\ell_0^{(x)}, \ell_1^{(z)}$

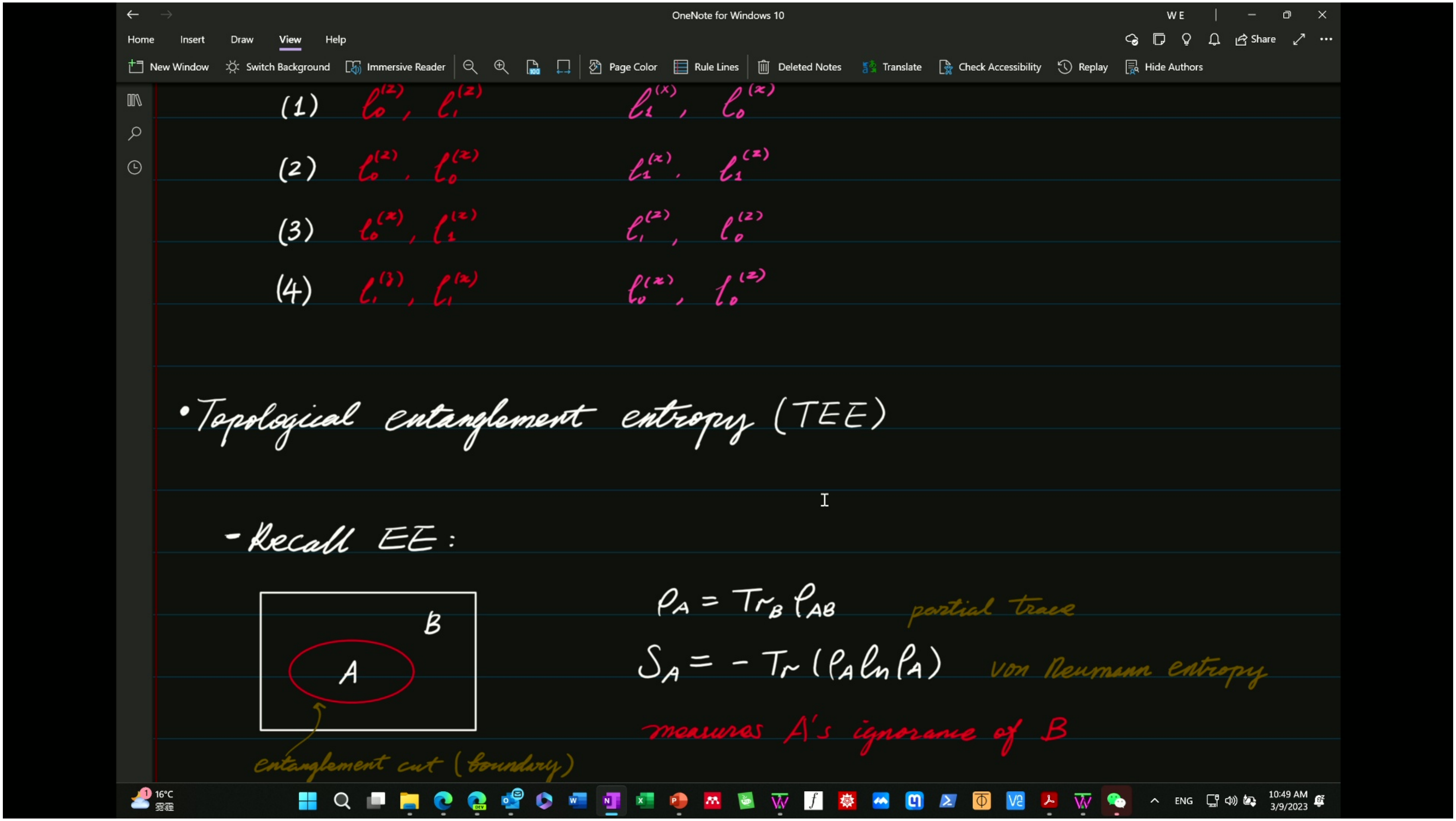
(4) $\ell_1^{(3)}, \ell_1^{(x)}$

Can choose any set to specify a ground-state basis.

There operators all have eigenvalues ±1

$$\Downarrow$$

$$GSD = 2^2 = 4$$



(1) $\ell_0^{(z)}, \ell_1^{(z)}$ $\ell_1^{(x)}, \ell_0^{(x)}$

(2) $\ell_0^{(z)}, \ell_0^{(x)}$ $\ell_1^{(x)}, \ell_1^{(z)}$

(3) $\ell_0^{(x)}, \ell_1^{(z)}$ $\ell_1^{(z)}, \ell_0^{(z)}$

(4) $\ell_1^{(3)}, \ell_1^{(x)}$ $\ell_0^{(x)}, \ell_0^{(z)}$

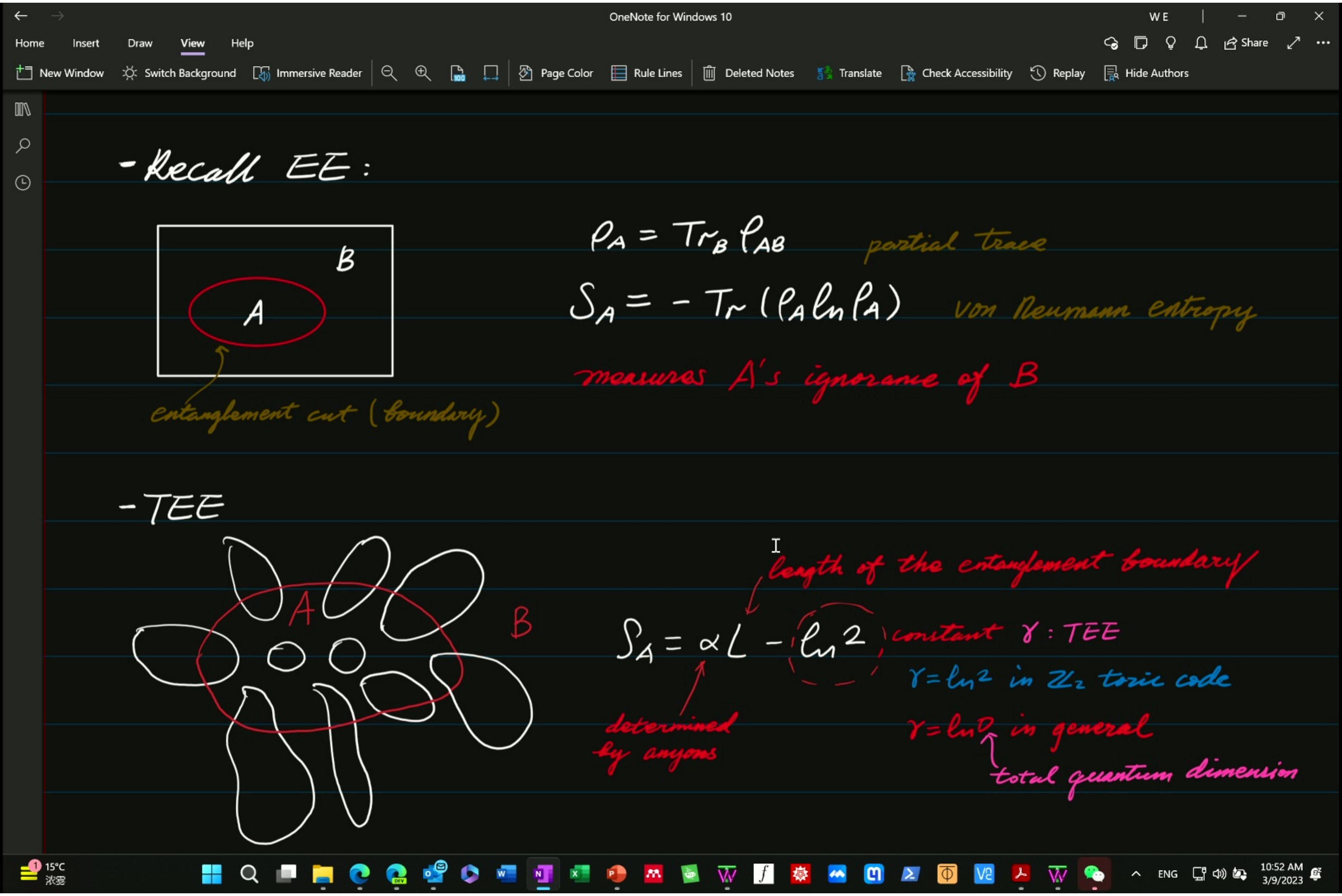
- Topological entanglement entropy (TEE)

- Recall EE:

$\rho_A = Tr_B \, \rho_{AB}$ partial trace

$S_A = -Tr(\rho_A \ln \rho_A)$ von Neumann entropy

measures A's ignorance of B



- Recall EE:

$\rho_A = \mathrm{Tr}_B\, \rho_{AB}$ partial trace

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measures A's ignorance of B

entanglement cut (boundary)

- TEE

length of the entanglement boundary

$$S_A = \alpha L - \ln 2$$

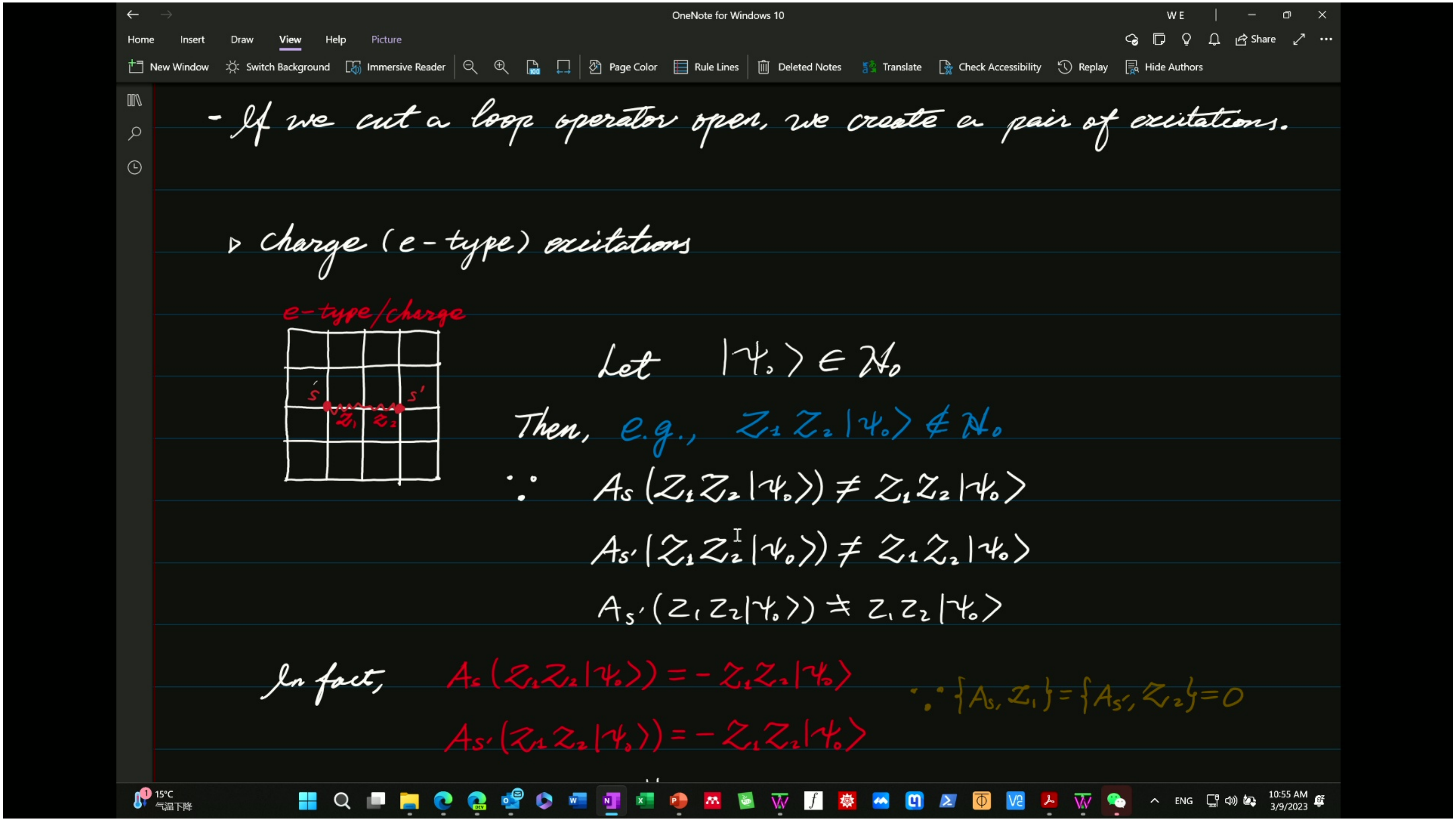
constant γ : TEE

$\gamma = \ln 2$ in $\mathbb{Z}_2$ toric code

$\gamma = \ln \mathcal{D}$ in general

total quantum dimension

determined by anyons



- If we cut a loop operator open, we create a pair of excitations.

▷ Charge (e-type) excitations

Let $|\psi_0\rangle \in \mathcal{H}_0$

Then, e.g., $Z_1 Z_2 |\psi_0\rangle \notin \mathcal{H}_0$

$$\because \quad A_s (Z_1 Z_2 |\psi_0\rangle) \neq Z_1 Z_2 |\psi_0\rangle$$

$$A_{s'} (Z_1 Z_2 |\psi_0\rangle) \neq Z_1 Z_2 |\psi_0\rangle$$

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In fact, $A_s (Z_1 Z_2 |\psi_0\rangle) = - Z_1 Z_2 |\psi_0\rangle$ $\quad \because \{A_s, Z_1\} = \{A_{s'}, Z_2\} = 0$

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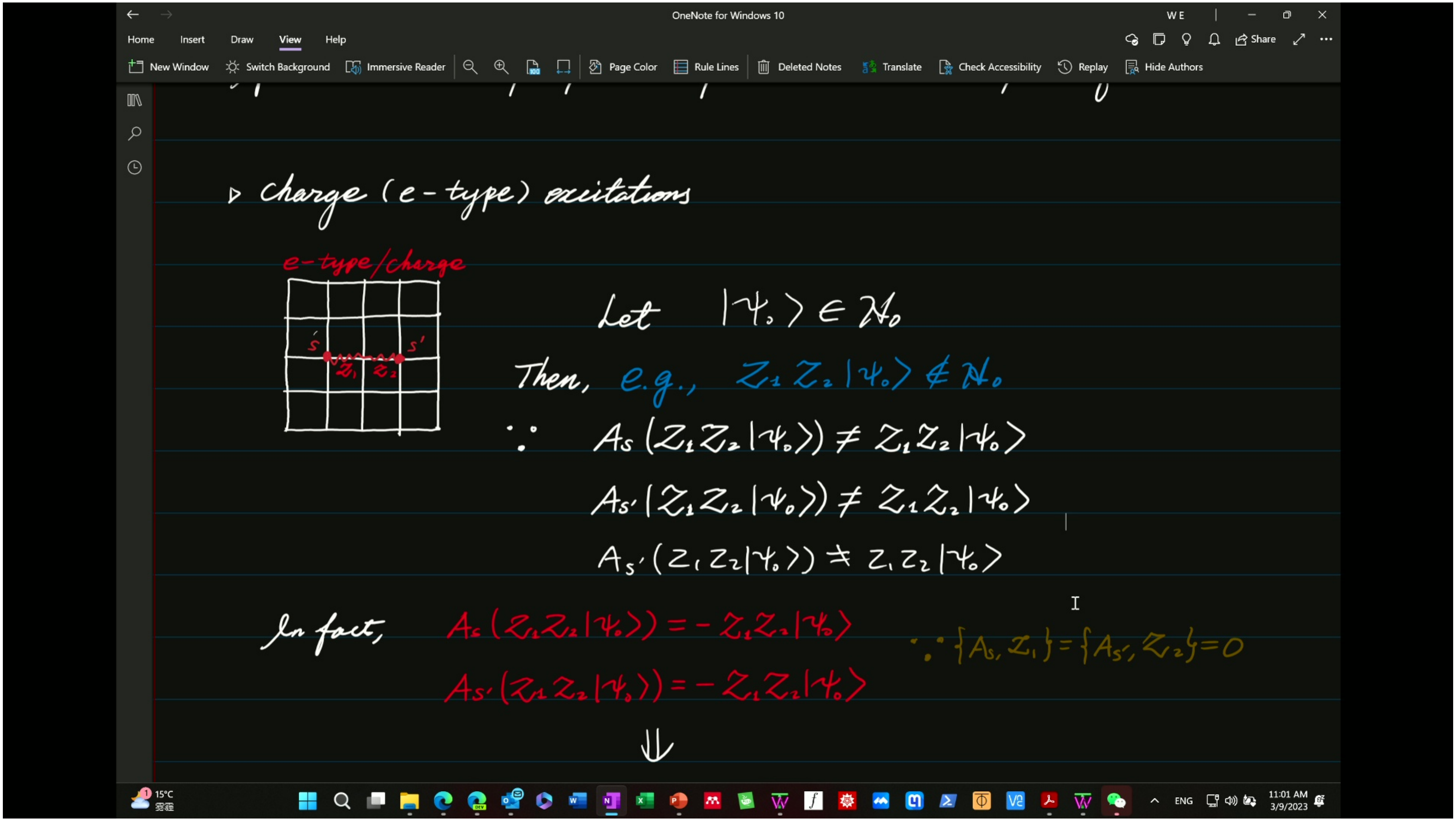

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$A_{s'}(Z_1 Z_2|\psi_0\rangle) \neq Z_1 Z_2 |\psi_0\rangle$

$A_{s'}(Z_1 Z_2|\psi_0\rangle) \not\asymp Z_1 Z_2 |\psi_0\rangle$

In fact, $A_s(Z_1 Z_2|\psi_0\rangle) = -Z_1 Z_2 |\psi_0\rangle$

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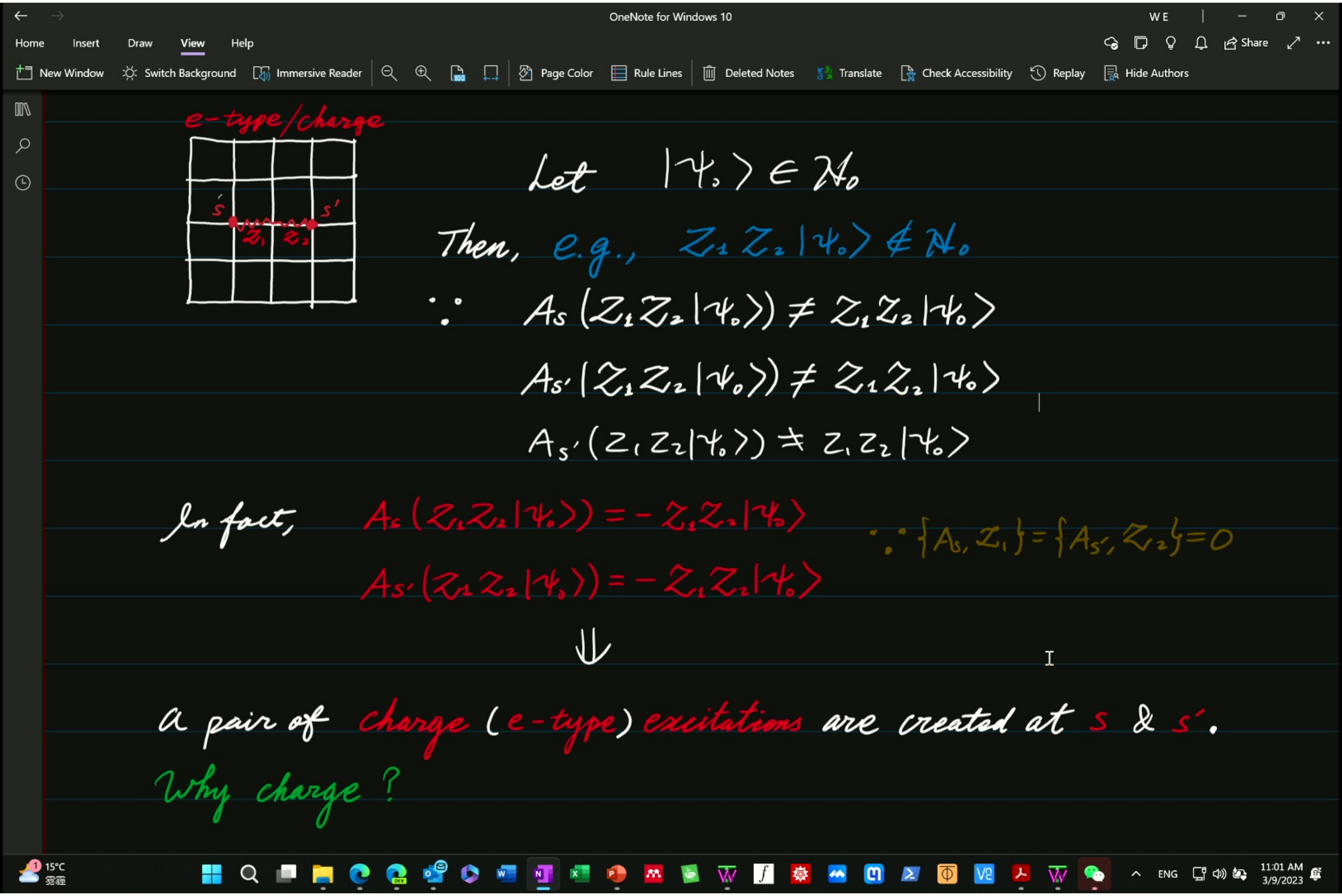
$\because \{A_s, Z_1\} = \{A_{s'}, Z_2\} = 0$

$\Downarrow$



e-type/charge

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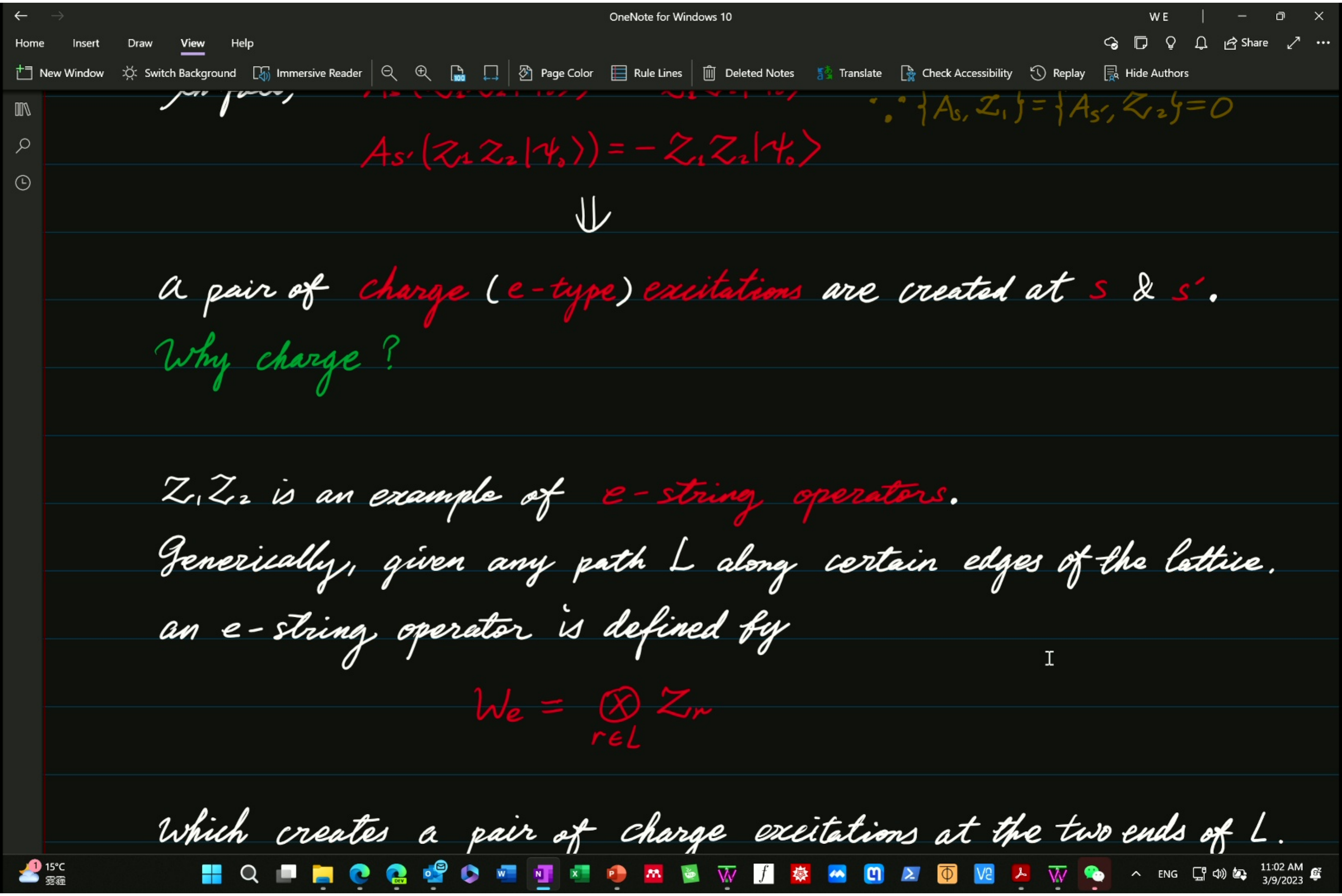
$A_{s'}(Z_1 Z_2|\psi_0\rangle) = -Z_1 Z_2|\psi_0\rangle$

$\Downarrow$

a pair of charge (e-type) excitations are created at s & s'.

Why charge ?



$$\because \{A_s, Z_1\} = \{A_{s'}, Z_2\} = 0$$

$$A_{s'}(Z_1 Z_2|\psi_0\rangle) = -Z_1 Z_2|\psi_0\rangle$$

$$\Downarrow$$

a pair of charge (e-type) excitations are created at s & s'.

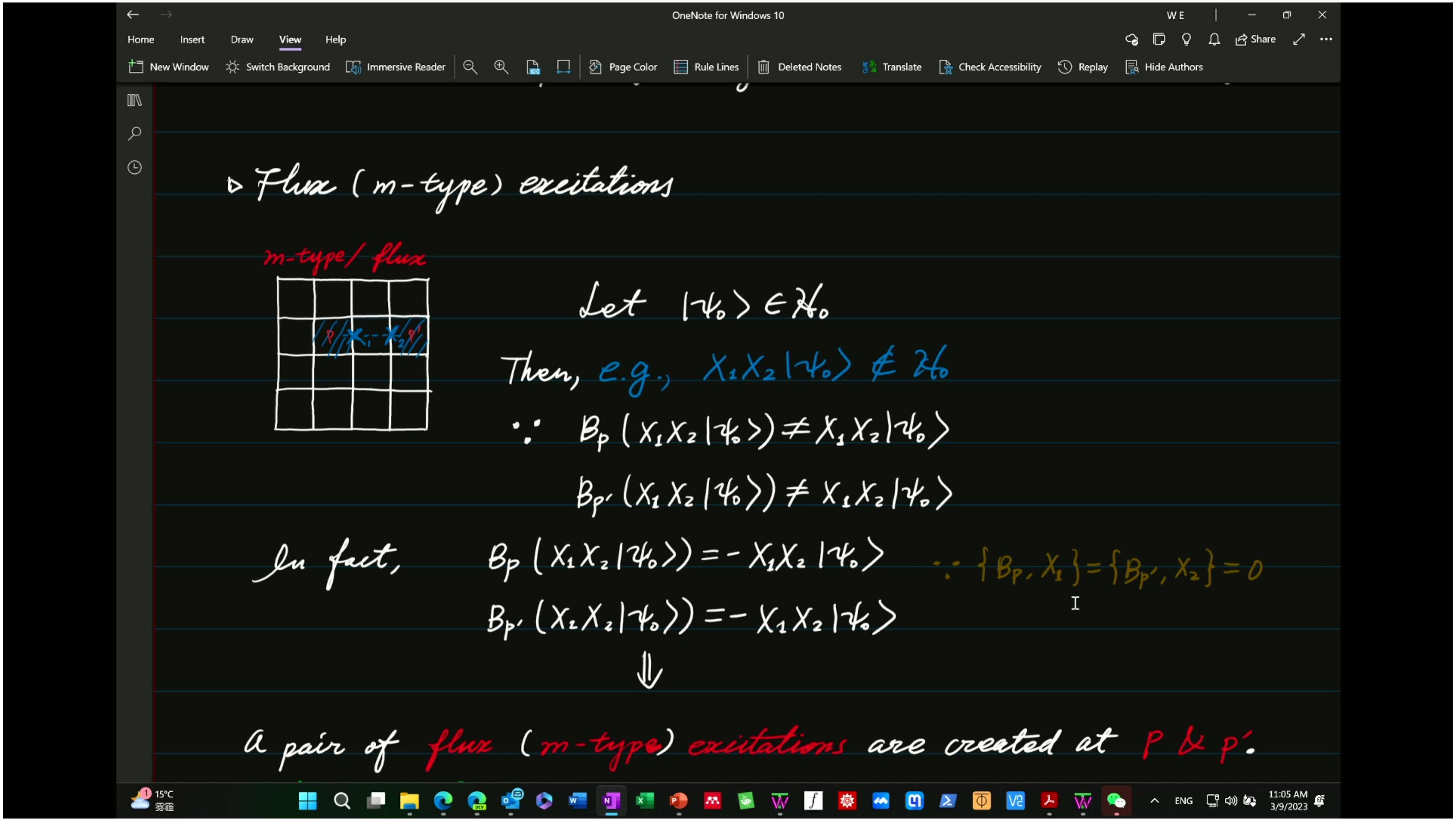
Why charge ?

$Z_1 Z_2$ is an example of e-string operators.

Generically, given any path L along certain edges of the lattice, an e-string operator is defined by

$$W_e = \bigotimes_{r \in L} Z_r$$

which creates a pair of charge excitations at the two ends of L.



▷ Flux (m-type) excitations

Let $|\psi_0\rangle \in \mathcal{H}_0$

Then, e.g., $X_1 X_2 |\psi_0\rangle \notin \mathcal{H}_0$

$$\because \quad B_p (X_1 X_2 |\psi_0\rangle) \neq X_1 X_2 |\psi_0\rangle$$

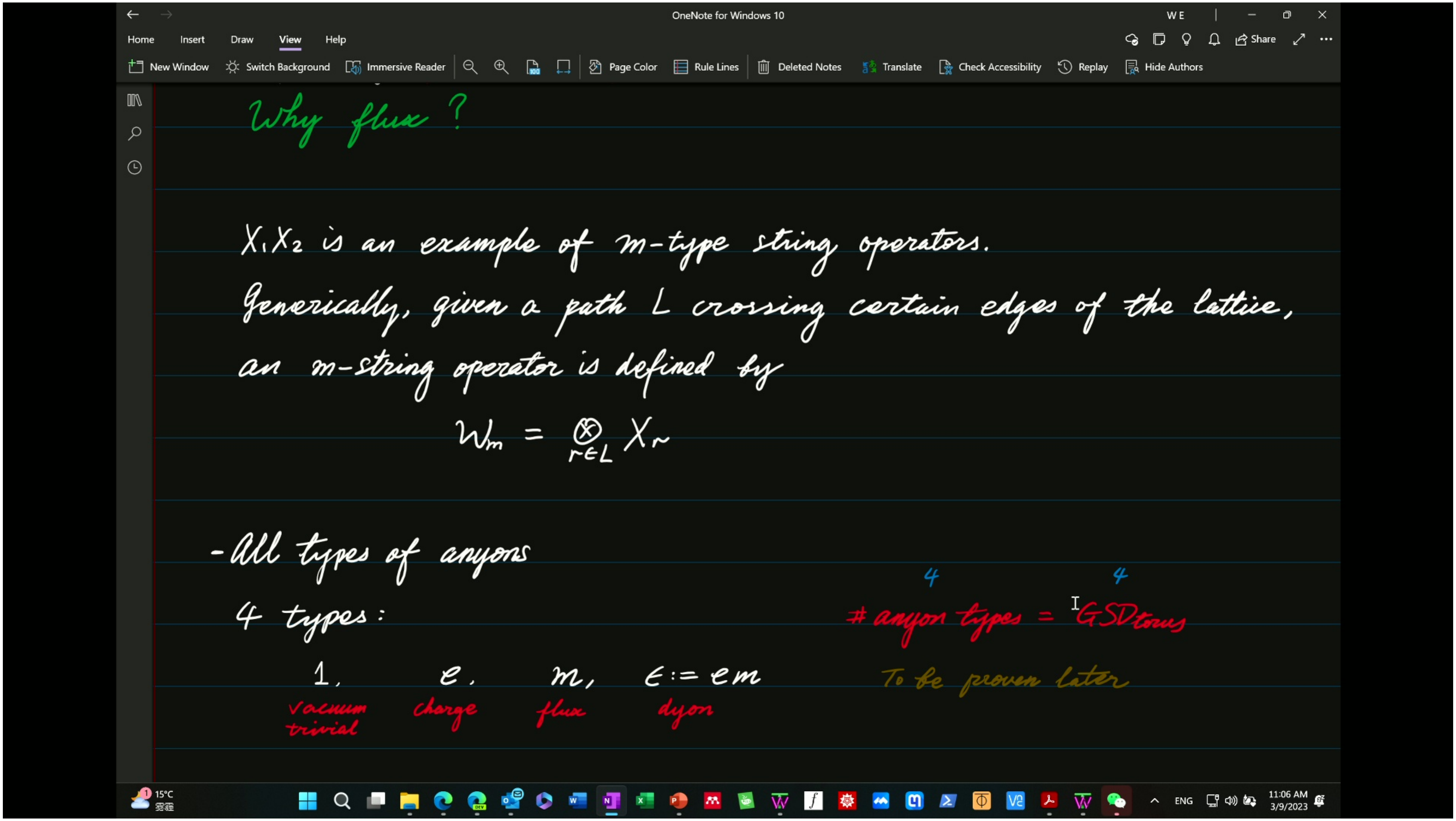
$$B_{p'} (X_1 X_2 |\psi_0\rangle) \neq X_1 X_2 |\psi_0\rangle$$

In fact, $B_p (X_1 X_2 |\psi_0\rangle) = - X_1 X_2 |\psi_0\rangle$ $\quad \because \{B_p, X_1\} = \{B_{p'}, X_2\} = 0$

$$B_{p'} (X_1 X_2 |\psi_0\rangle) = - X_1 X_2 |\psi_0\rangle$$

$$\Downarrow$$

A pair of flux (m-type) excitations are created at p & p'.



Why flux ?

$X_1 X_2$ is an example of m-type string operators.

Generically, given a path L crossing certain edges of the lattice, an m-string operator is defined by

$$W_m = \bigotimes_{r \in L} X_r$$

- All types of anyons

4 types:

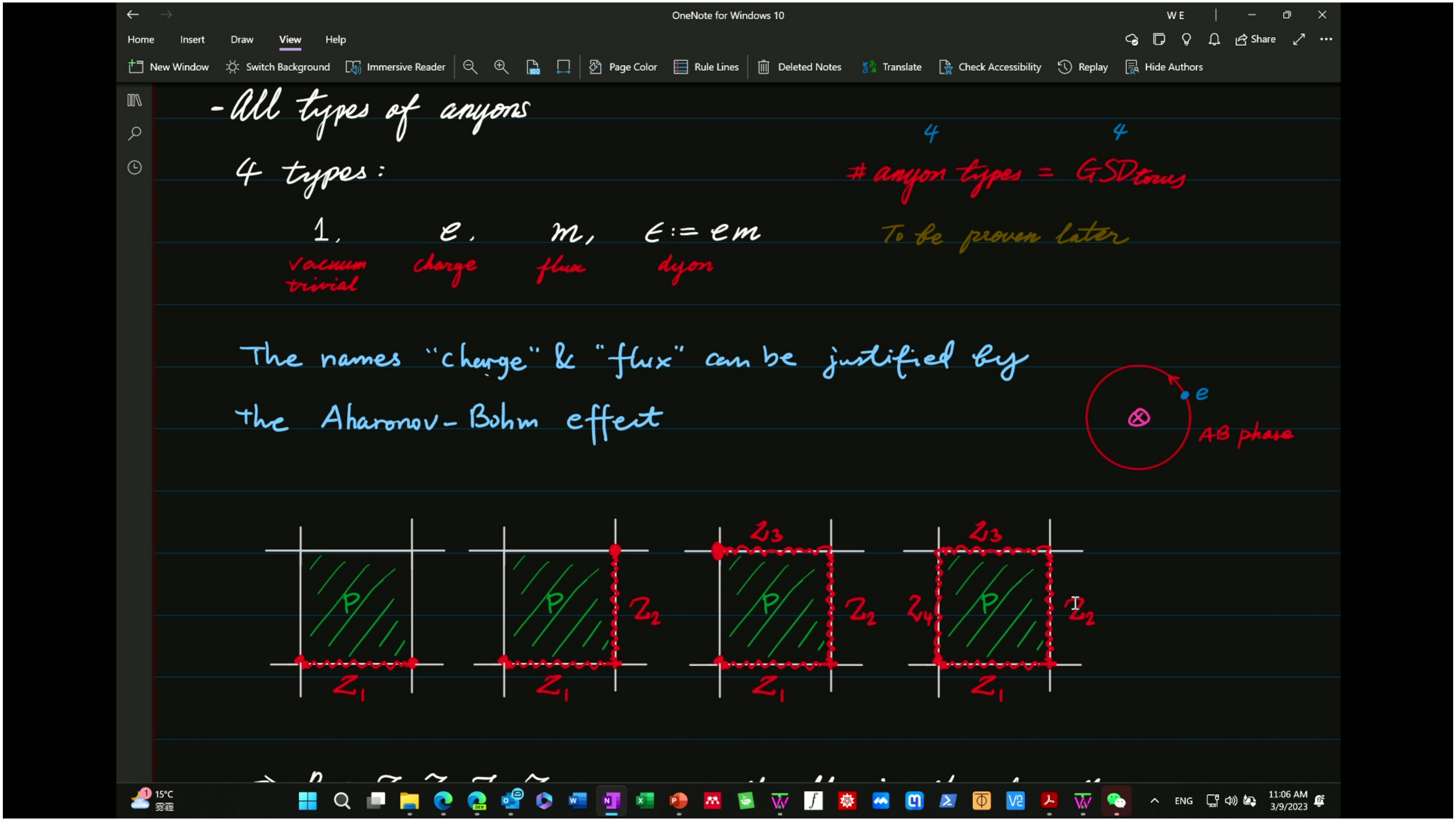
1, e, m, $\epsilon := em$

vacuum trivial, charge, flux, dyon

$$\# \text{ anyon types} = \text{GSD}_{\text{torus}}$$

$$\overset{4}{} \qquad \overset{4}{}$$

To be proven later



- All types of anyons

4 types:

1, e, m, $\epsilon := em$

vacuum trivial, charge, flux, dyon

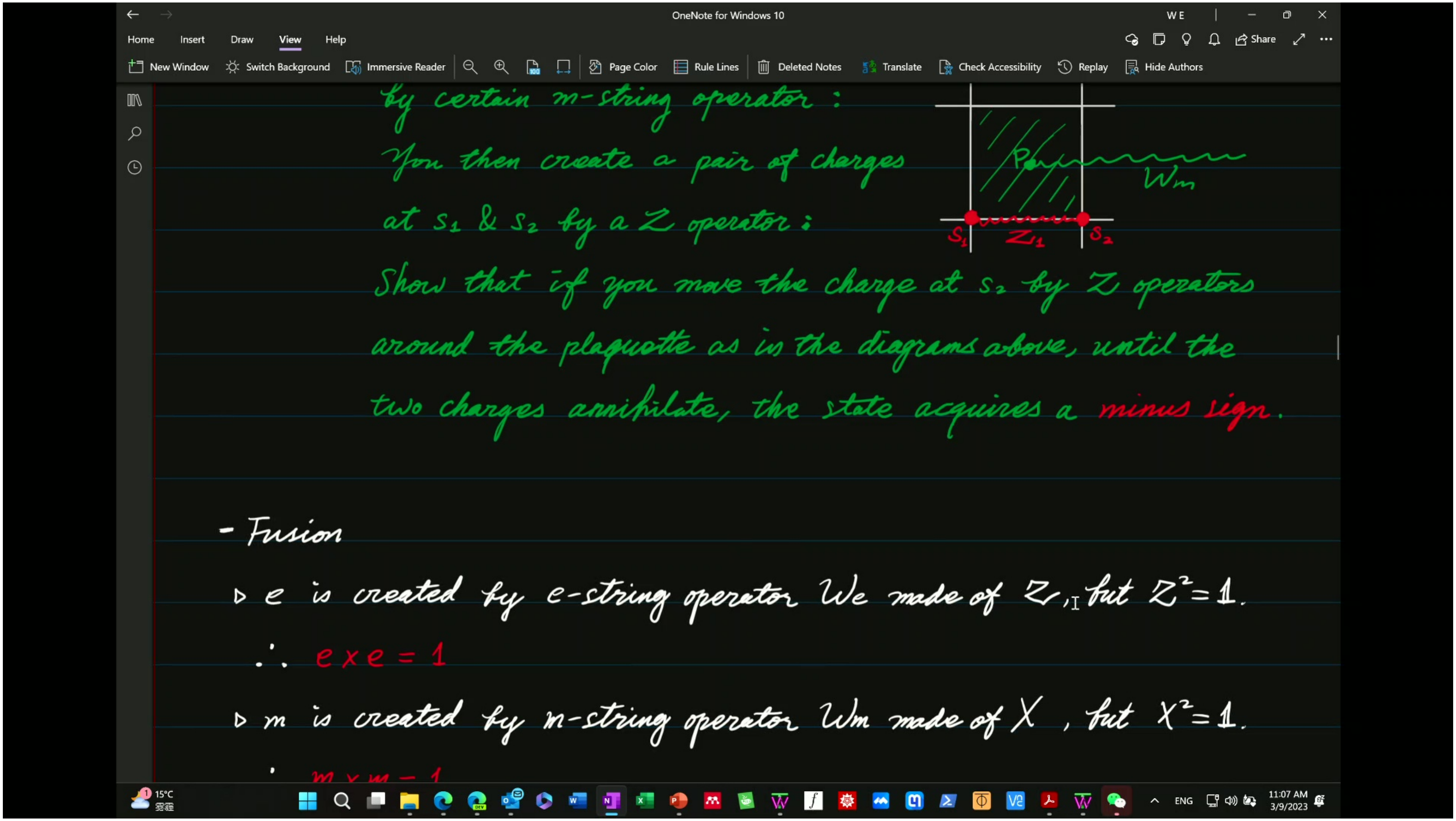
$\# \text{anyon types} = GSD_{torus}$ ($4 = 4$)

To be proven later

The names "charge" & "flux" can be justified by the Aharonov-Bohm effect

e

AB phase



by certain m-string operator :

You then create a pair of charges

at s_1 & s_2 by a Z operator :

Show that if you move the charge at s_2 by Z operators around the plaquette as in the diagrams above, until the two charges annihilate, the state acquires a minus sign.

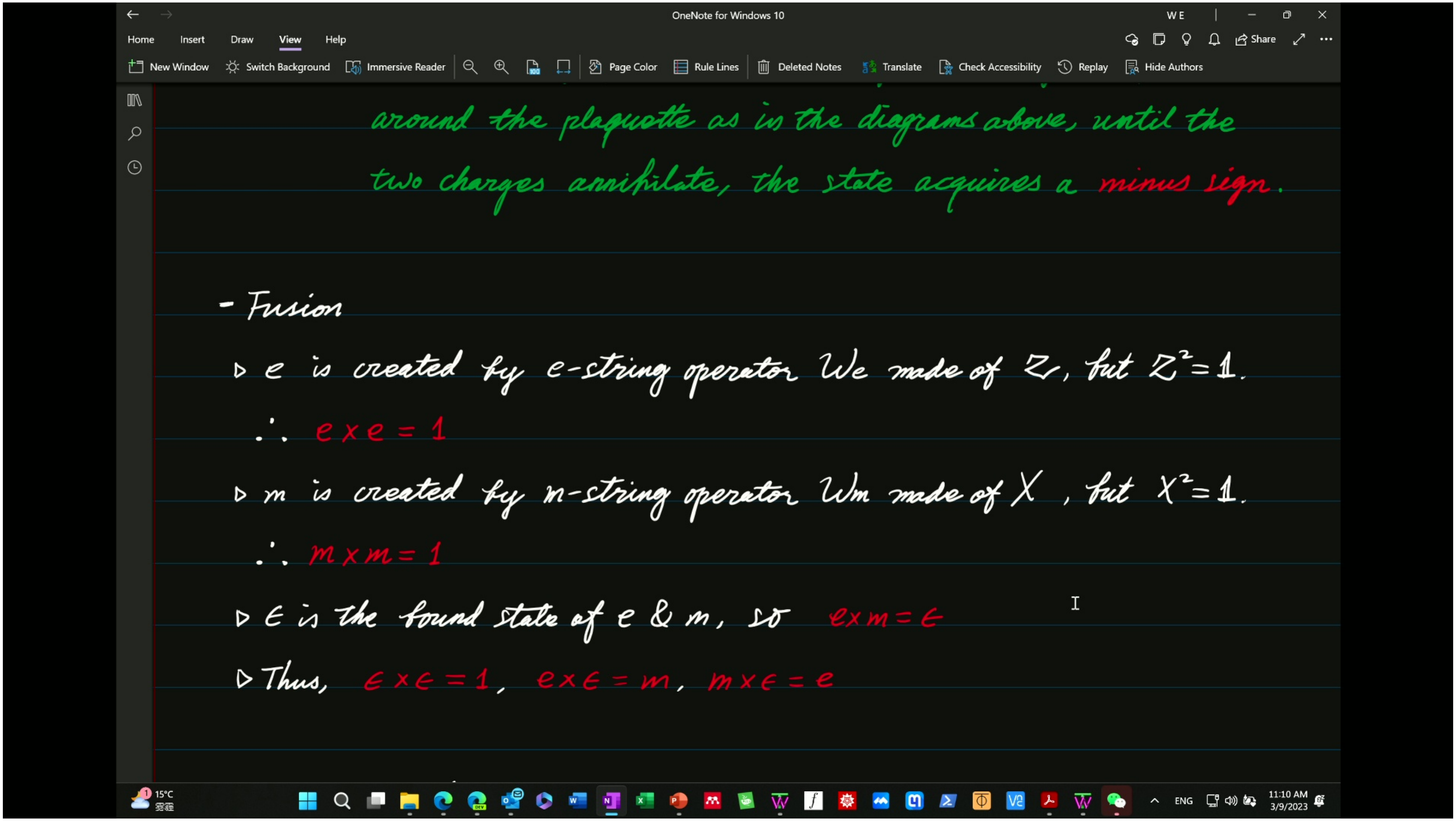
- Fusion

▷ e is created by e-string operator We made of Z, but $Z^2 = 1$.

$\therefore$ $e \times e = 1$

▷ m is created by m-string operator Wm made of X, but $X^2 = 1$.

$\therefore$ $m \times m = 1$



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- Fusion

▷ e is created by e-string operator W_e made of Z, but $Z^2=1$.

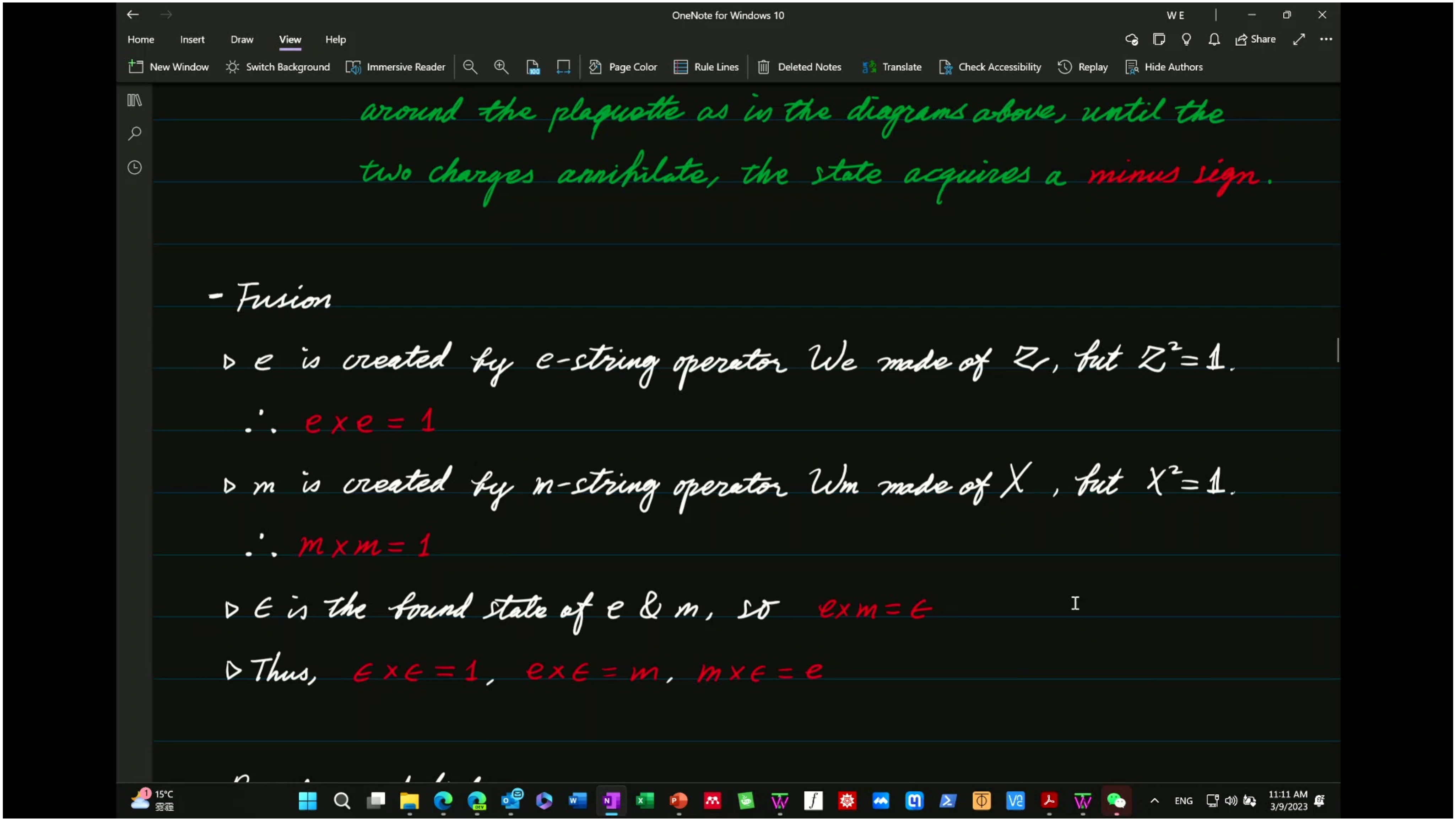
∴ $e \times e = 1$

▷ m is created by m-string operator W_m made of X, but $X^2=1$.

∴ $m \times m = 1$

▷ ϵ is the bound state of e & m, so $e \times m = \epsilon$

▷ Thus, $\epsilon \times \epsilon = 1$, $e \times \epsilon = m$, $m \times \epsilon = e$



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- Fusion

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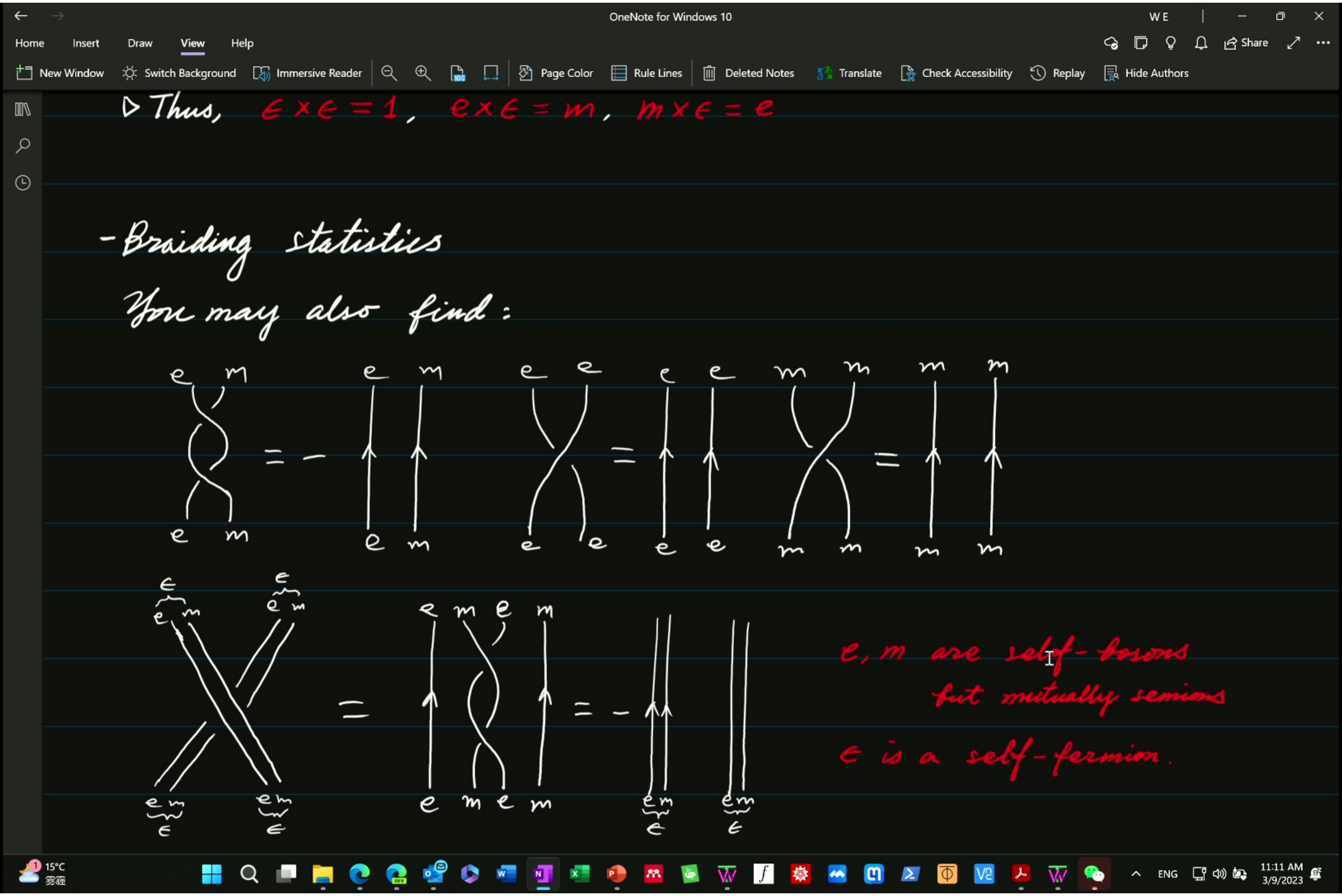


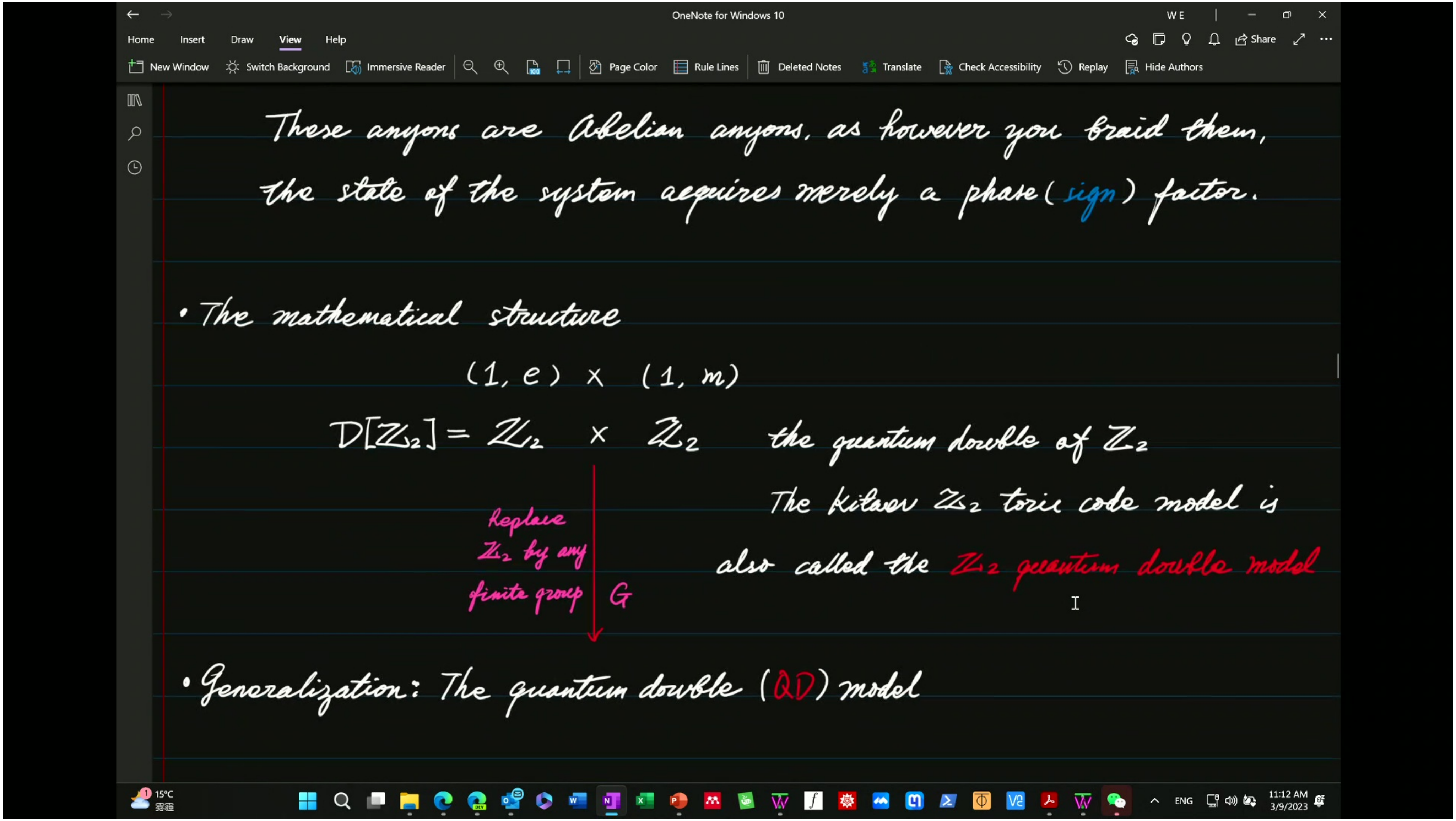
▷ Thus, $\epsilon \times \epsilon = 1$, $e \times \epsilon = m$, $m \times \epsilon = e$

- Braiding statistics

You may also find:

e, m are self-bosons but mutually semions

ϵ is a self-fermion.



These anyons are Abelian anyons, as however you braid them, the state of the system acquires merely a phase (sign) factor.

- The mathematical structure

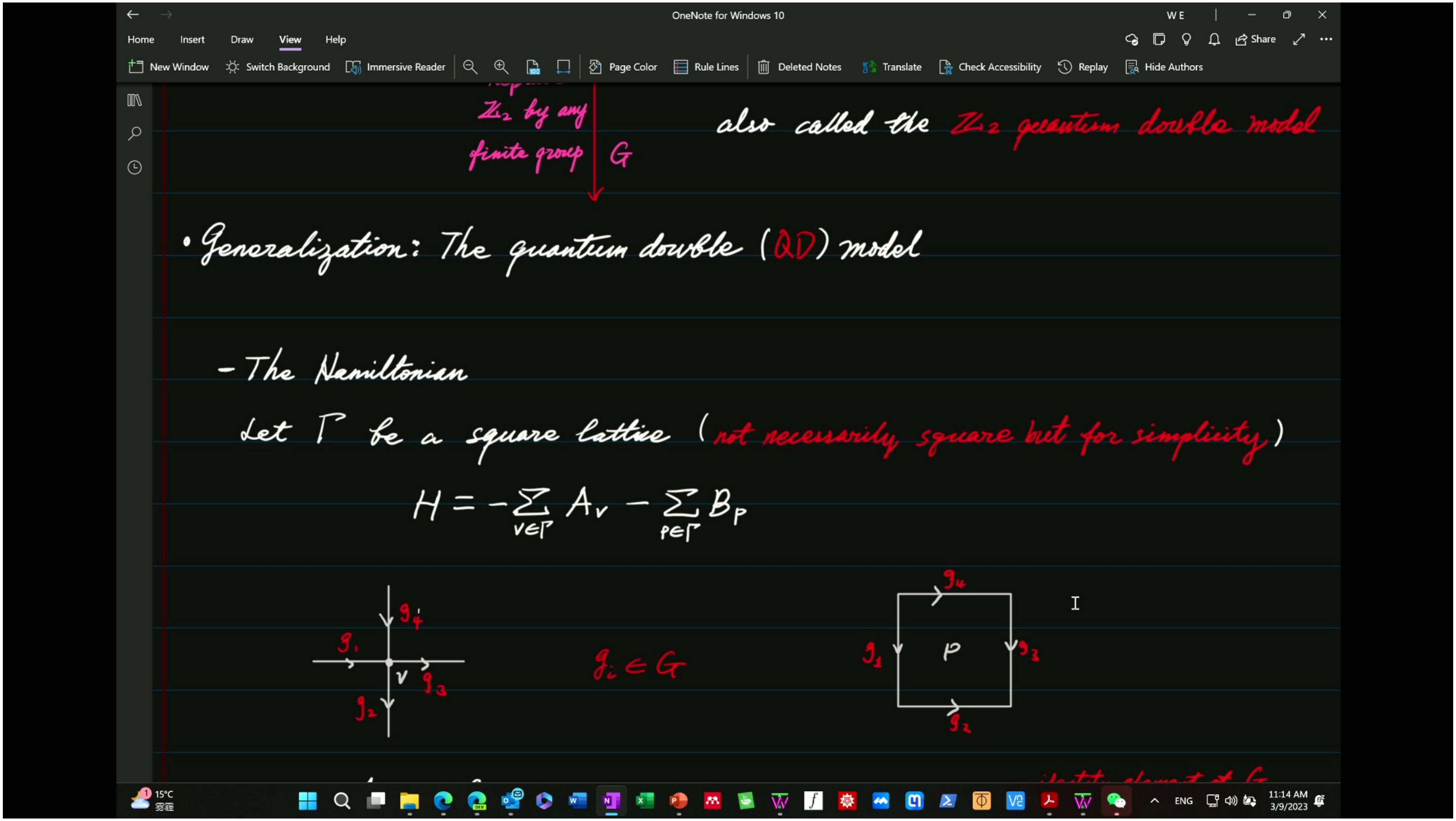
$$(1, e) \times (1, m)$$

$D[\mathbb{Z}_2] = \mathbb{Z}_2 \times \mathbb{Z}_2$ the quantum double of $\mathbb{Z}_2$

The Kitaev $\mathbb{Z}_2$ toric code model is also called the $\mathbb{Z}_2$ quantum double model

Replace $\mathbb{Z}_2$ by any finite group G $\big\downarrow$

- Generalization: The quantum double (QD) model



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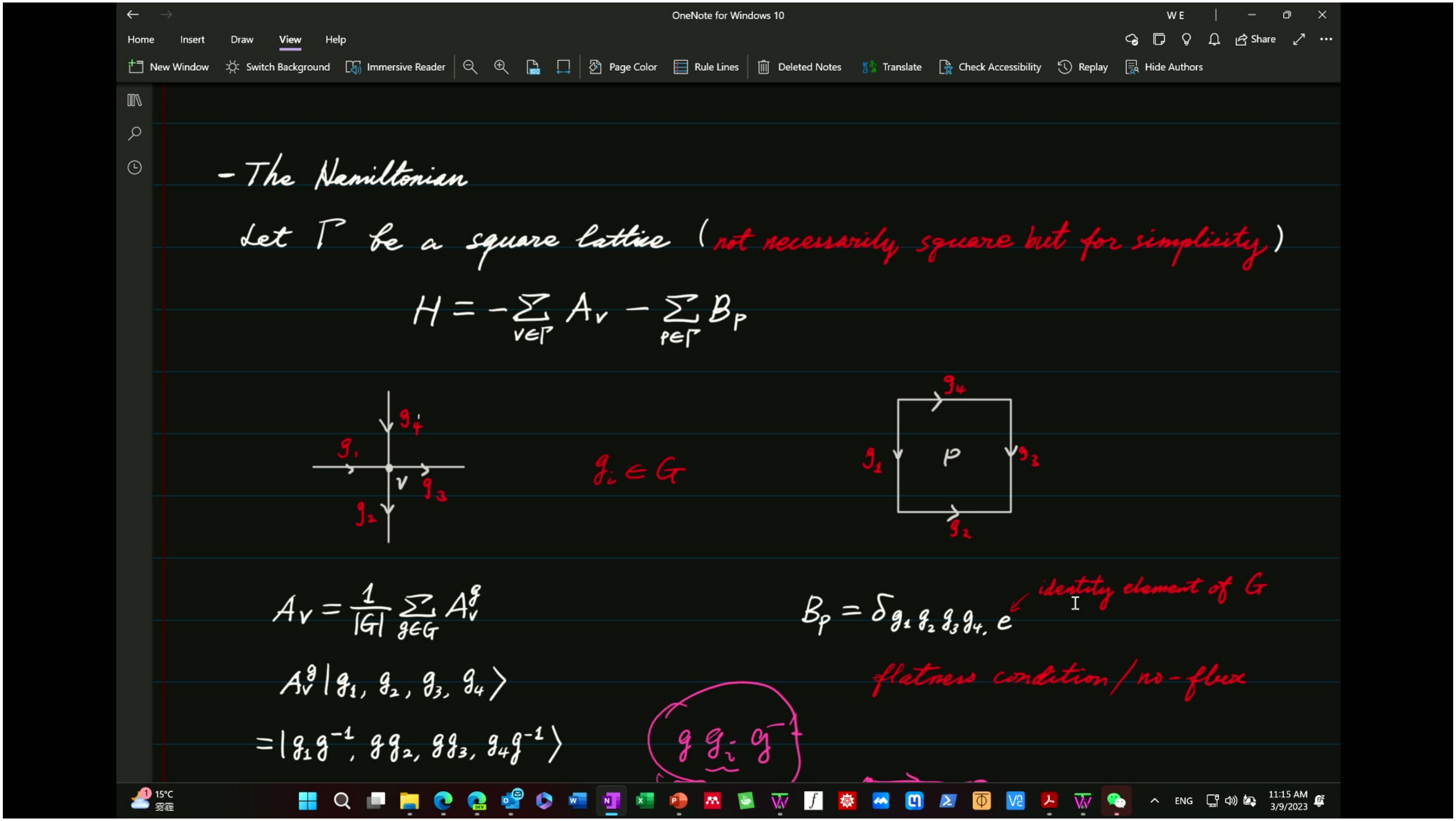
- Generalization: The quantum double (QD) model

– The Hamiltonian

Let Γ be a square lattice (not necessarily square but for simplicity)

$$H = -\sum_{v\in\Gamma} A_v - \sum_{p\in\Gamma} B_p$$

$g_i \in G$



- The Hamiltonian

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$$A_v = \frac{1}{|G|}\sum_{g\in G} A_v^g$$

$$B_p = \delta_{g_1 g_2 g_3 g_4,\, e}$$

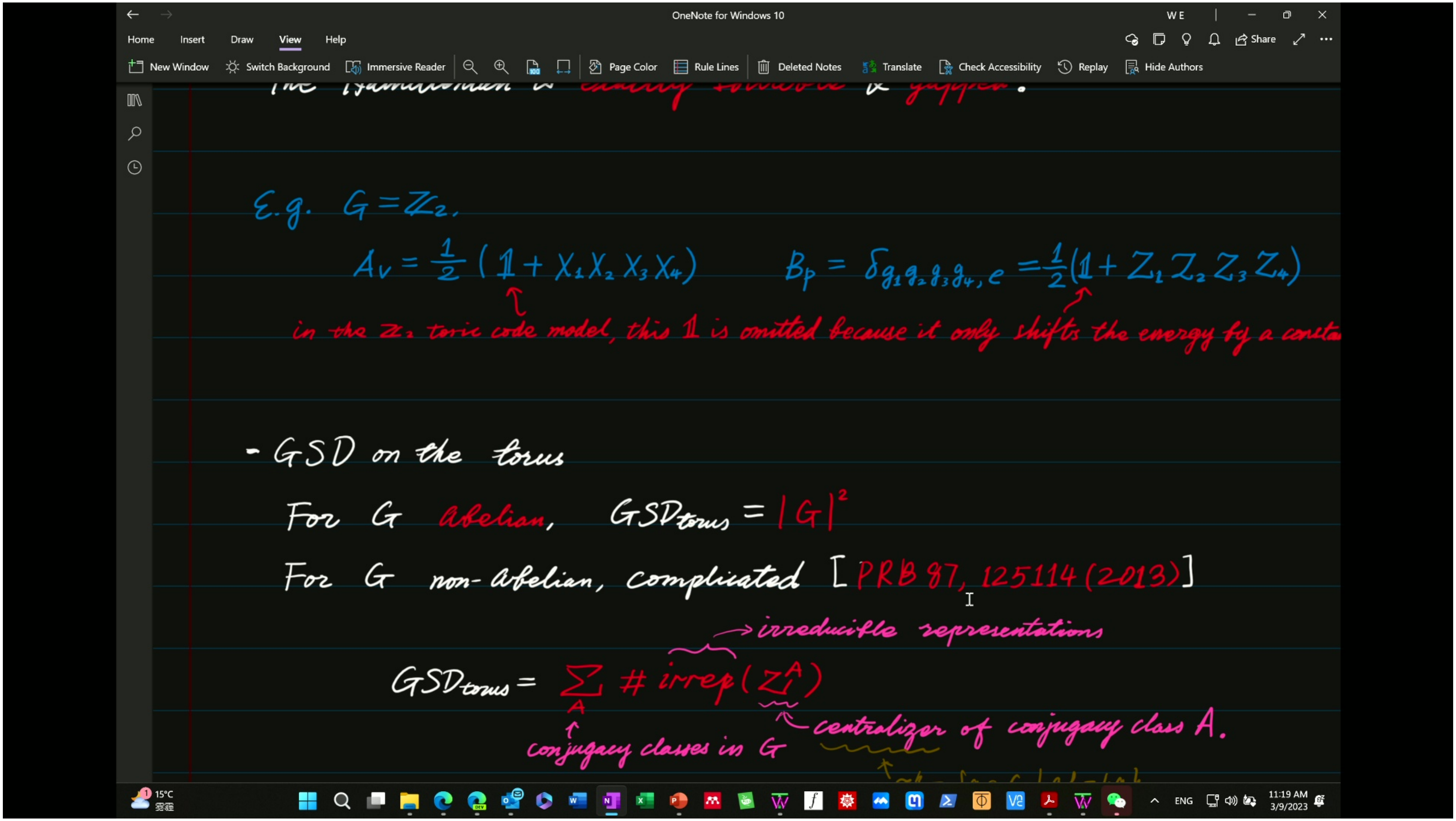
identity element of G

flatness condition / no-flux

$$A_v^g |g_1, g_2, g_3, g_4\rangle$$

$$= |g_1 g^{-1}, g g_2, g g_3, g_4 g^{-1}\rangle$$

$$\boxed{g g_i g^{-1}}$$



The Hamiltonian is exactly solvable & gapped.

E.g. $G = \mathbb{Z}_2$.

$$A_v = \frac{1}{2}\left(\mathbb{1} + X_1 X_2 X_3 X_4\right) \qquad B_p = \delta_{g_1 g_2 g_3 g_4, e} = \frac{1}{2}\left(\mathbb{1} + Z_1 Z_2 Z_3 Z_4\right)$$

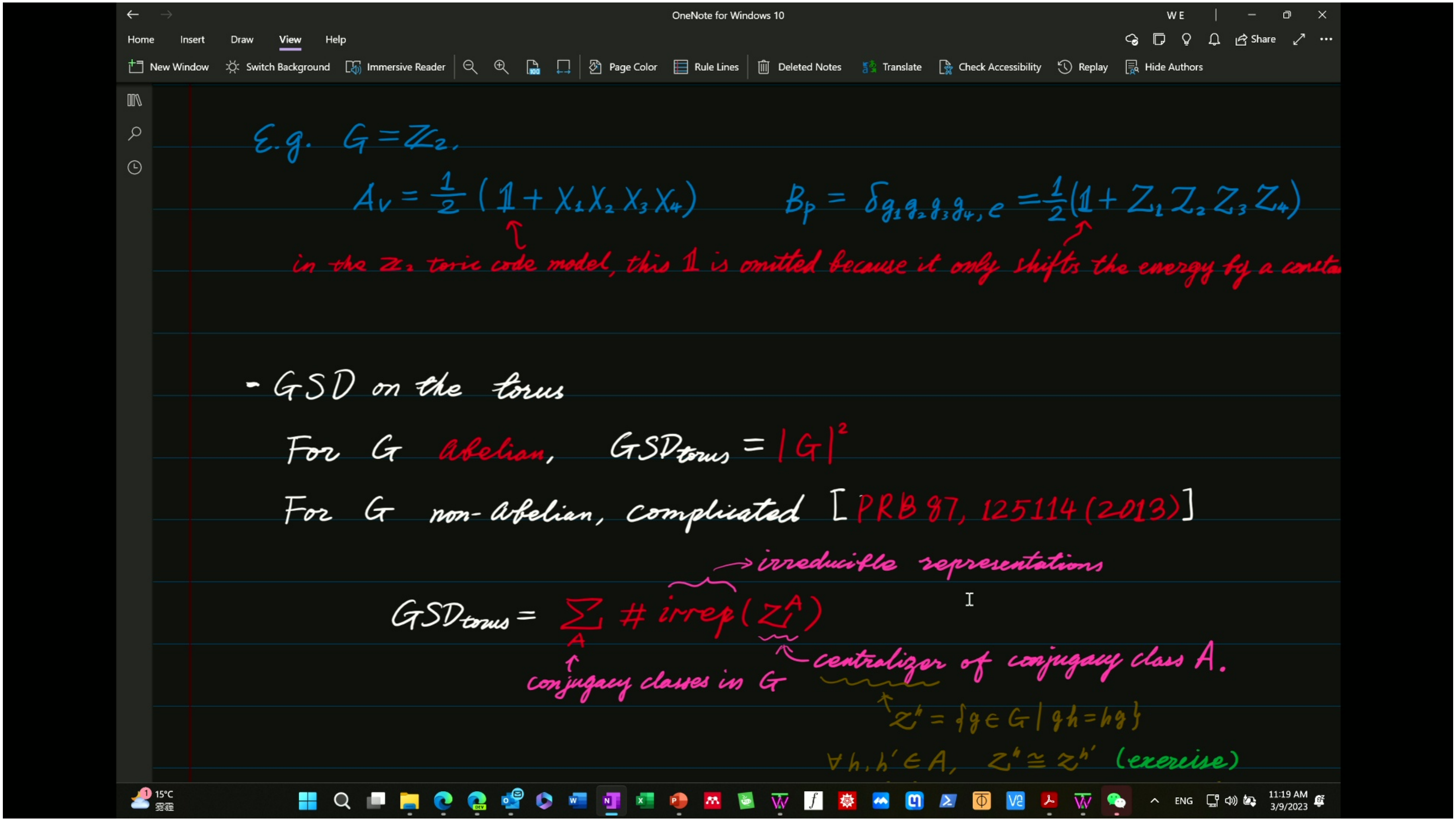
in the $\mathbb{Z}_2$ toric code model, this $\mathbb{1}$ is omitted because it only shifts the energy by a consta

- GSD on the torus

For G abelian, $GSD_{torus} = |G|^2$

For G non-abelian, complicated [PRB 87, 125114 (2013)]

$$GSD_{torus} = \underset{\substack{\uparrow \\ \text{conjugacy classes in } G}}{\sum_{A}} \# \overbrace{\text{irrep}}^{\to \text{irreducible representations}}\big(\underbrace{Z^A}_{\leftarrow \text{centralizer of conjugacy class } A.}\big)$$



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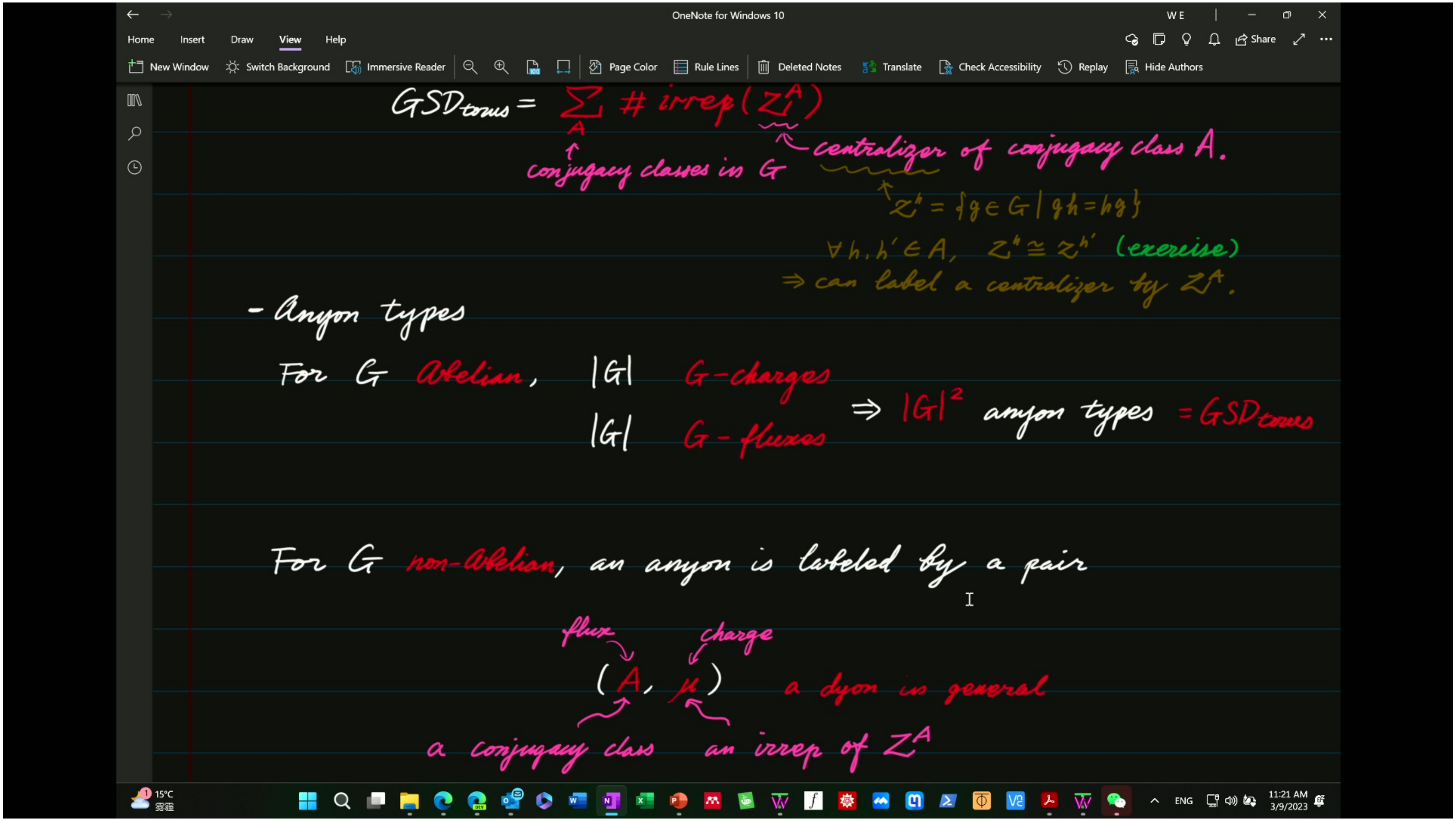
$Z^h = \{g \in G \mid gh = hg\}$

$\forall h, h' \in A$, $Z^h \cong Z^{h'}$ (exercise)



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↑ conjugacy classes in G

↖ centralizer of conjugacy class A.

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⇒ can label a centralizer by Z^A.

- Anyon types

For G Abelian, $|G|$ G-charges

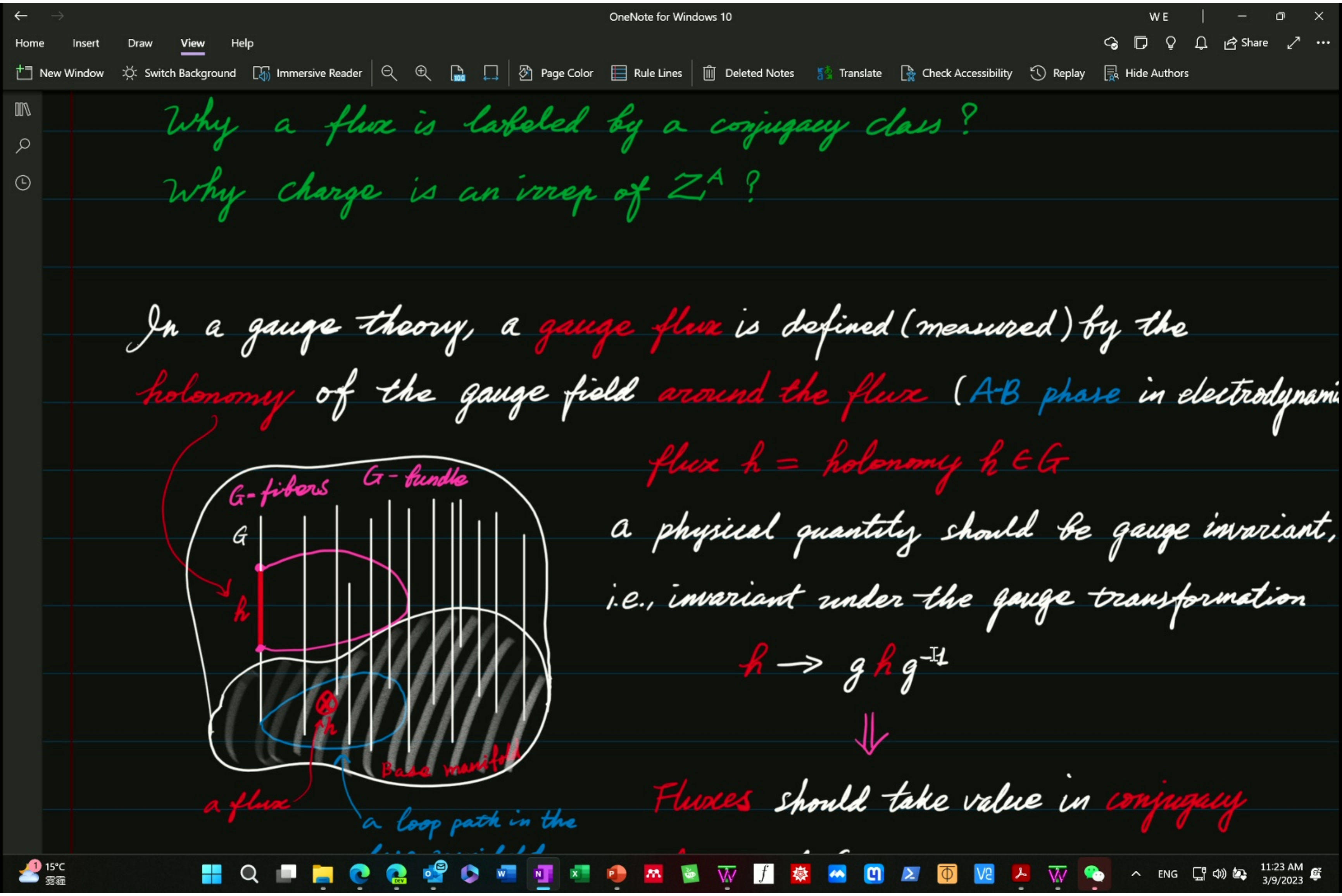
$|G|$ G-fluxes ⇒ $|G|^2$ anyon types $= GSD_{torus}$

For G non-Abelian, an anyon is labeled by a pair

flux, charge

(A, μ) a dyon in general

a conjugacy class, an irrep of Z^A



Why a flux is labeled by a conjugacy class?

Why charge is an irrep of Z^A?

In a gauge theory, a gauge flux is defined (measured) by the holonomy of the gauge field around the flux (A-B phase in electrodynami

flux h = holonomy $h \in G$

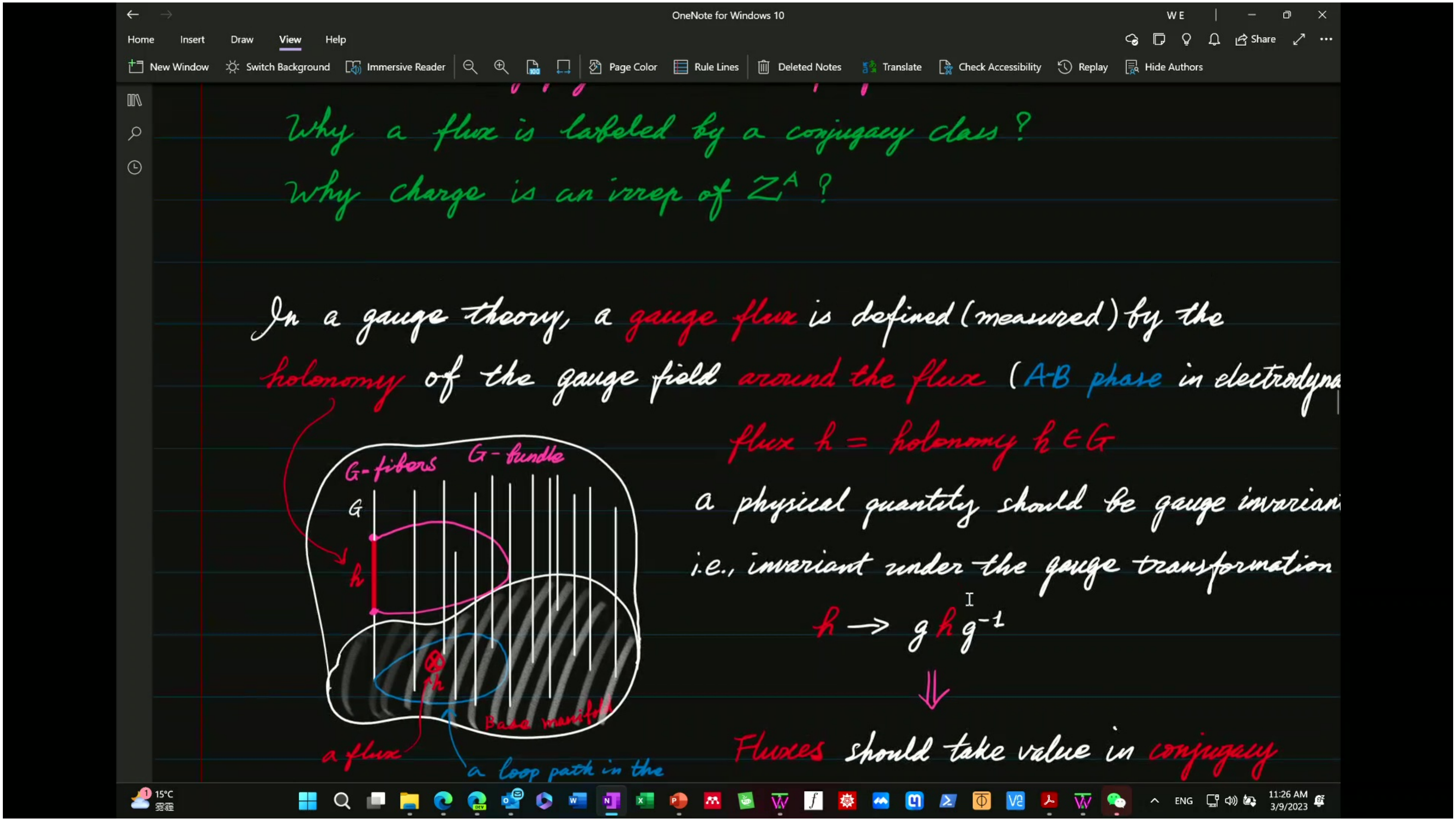
a physical quantity should be gauge invariant, i.e., invariant under the gauge transformation

$$h \to g h g^{-1}$$

$$\Downarrow$$

Fluxes should take value in conjugacy

a loop path in the



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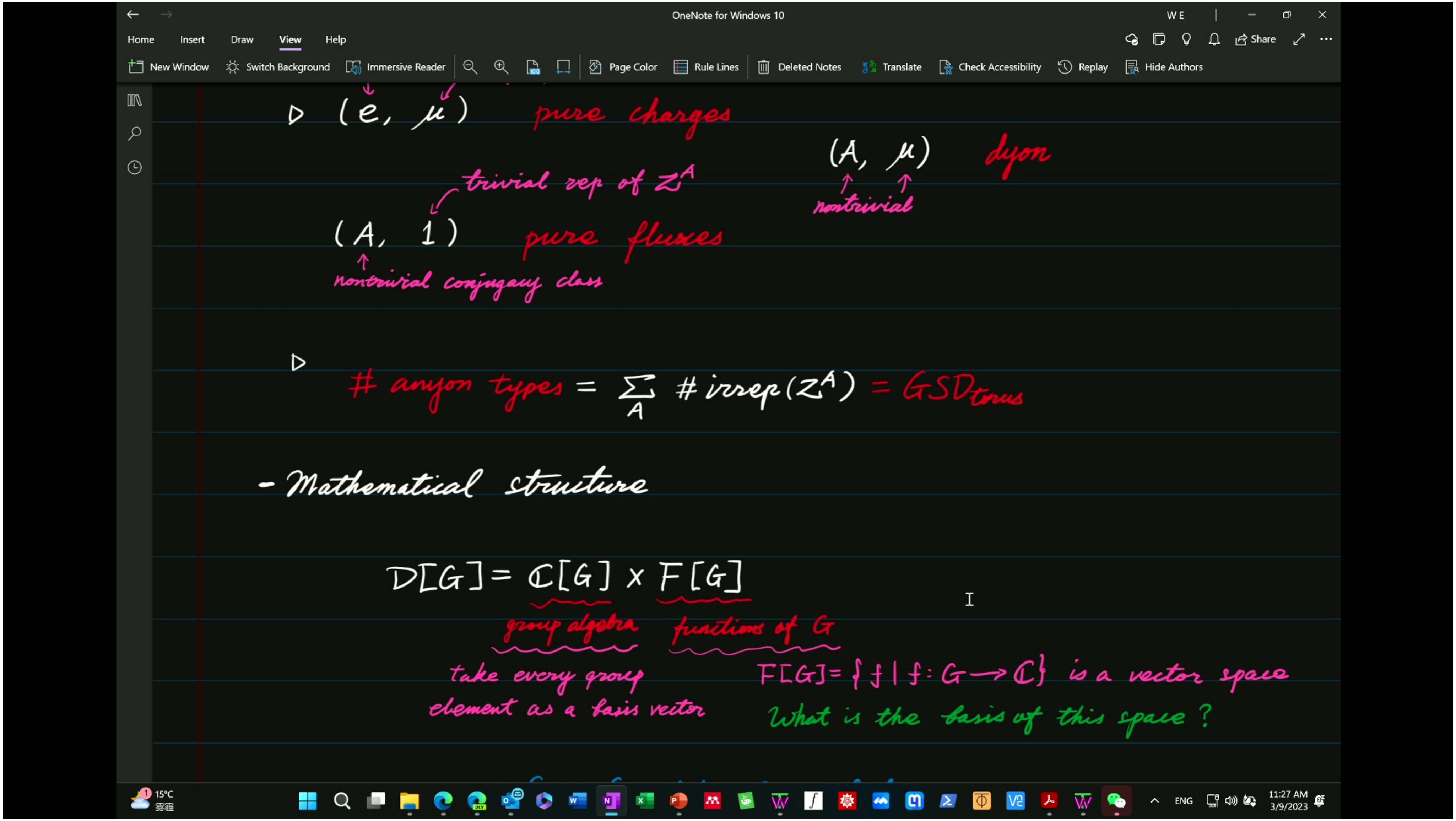


▷ (e, μ) pure charges

$(\underset{\text{nontrivial}}{A}, \underset{\text{nontrivial}}{\mu})$ dyon

$(\underset{\text{nontrivial conjugacy class}}{A}, \overset{\text{trivial rep of } Z^A}{1})$ pure fluxes

▷ $\#$ anyon types $= \sum_A \# \text{irrep}(Z^A) = GSD_{\text{torus}}$

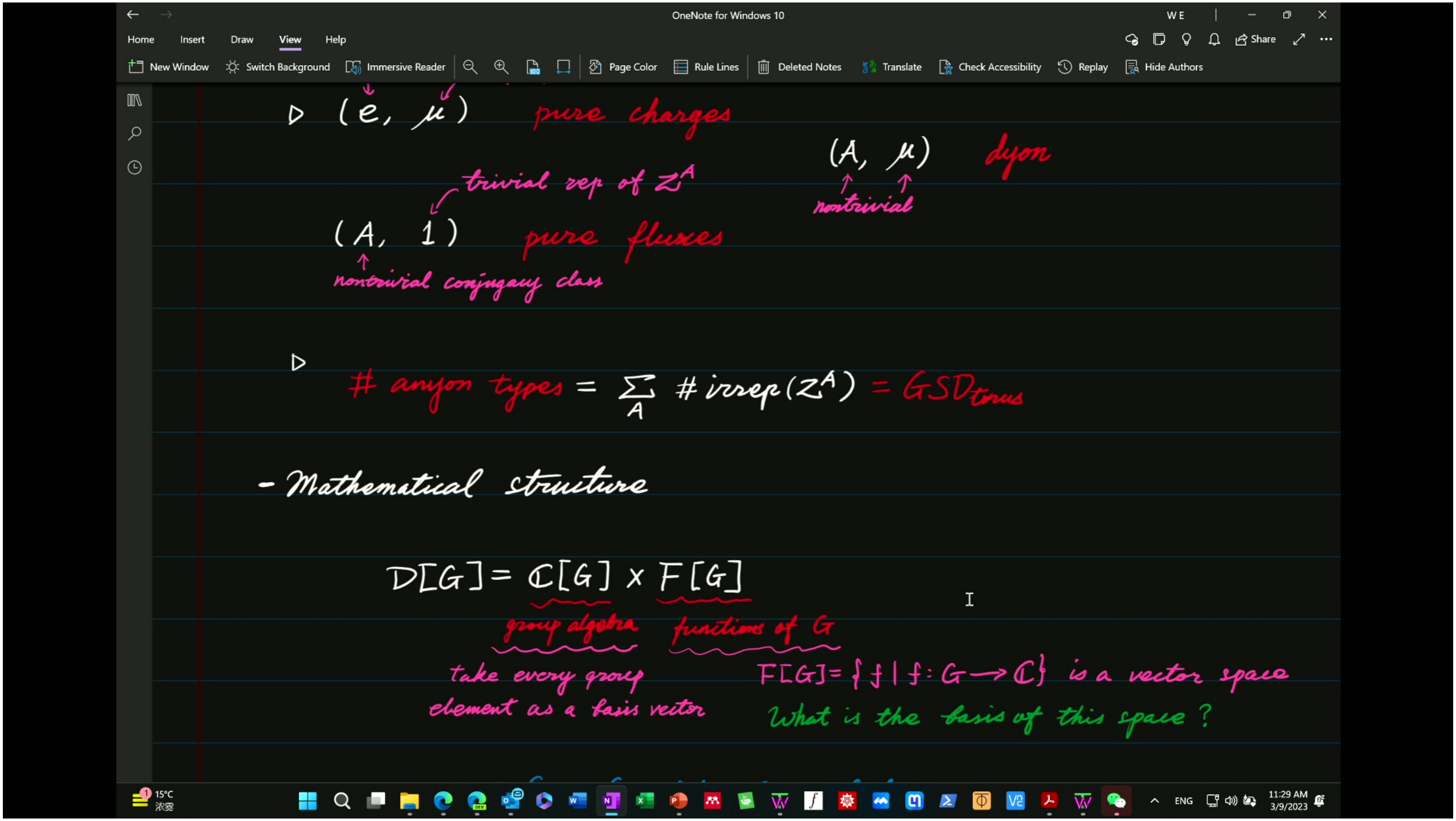
- Mathematical structure

$$D[G] = \underbrace{\mathbb{C}[G]}_{\text{group algebra}} \times \underbrace{F[G]}_{\text{functions of } G}$$

take every group element as a basis vector

$F[G] = \{ f \mid f : G \to \mathbb{C} \}$ is a vector space

What is the basis of this space ?



▷ (e, μ) pure charges

(A, μ) dyon

$\underset{\text{nontrivial}}{(A,\ \mu)}$

trivial rep of Z^A

$(A, 1)$ pure fluxes

nontrivial conjugacy class

▷ $\# \text{ anyon types} = \sum_A \# \text{irrep}(Z^A) = GSD_{\text{torus}}$

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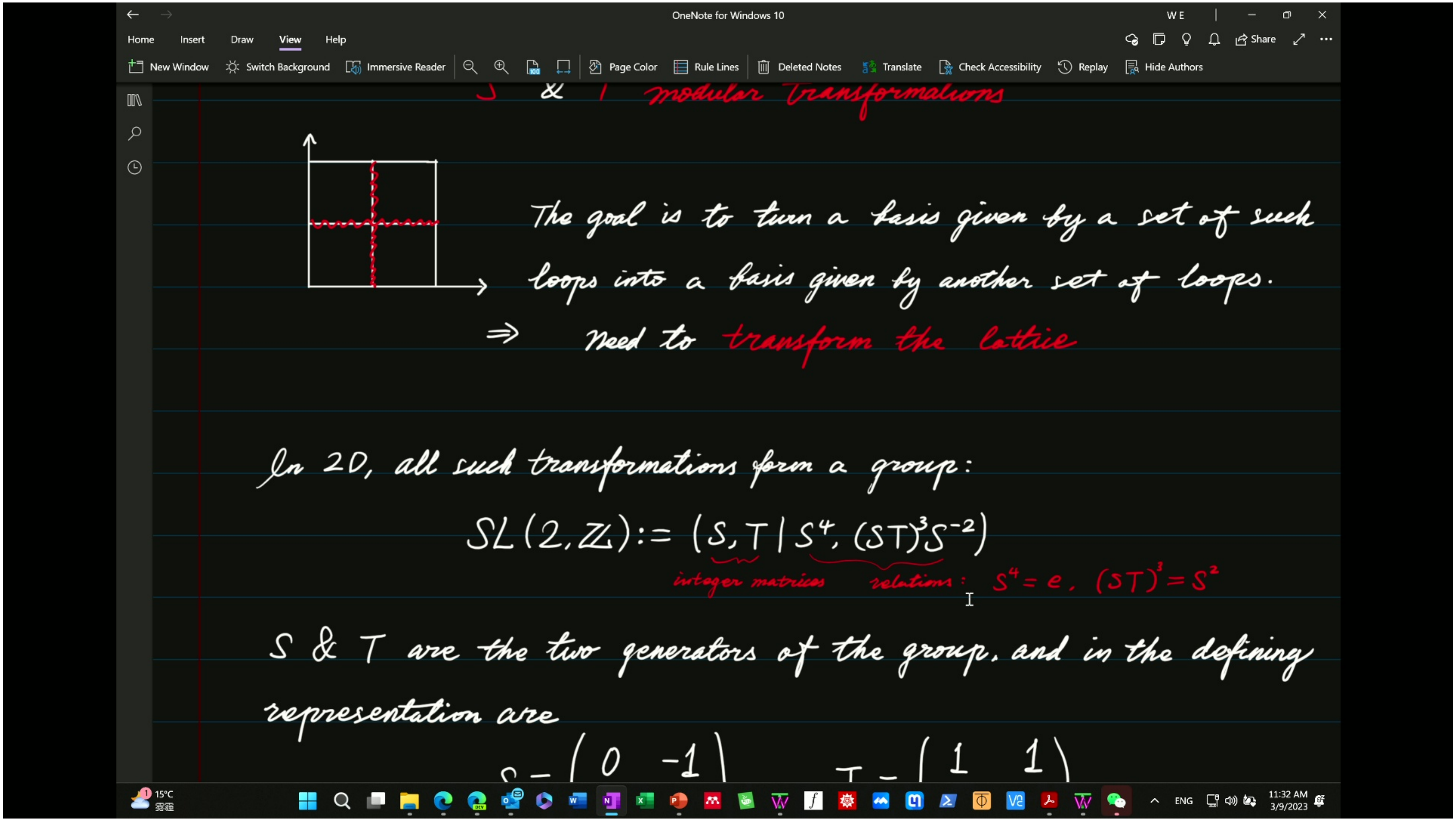


S & T modular transformations

The goal is to turn a basis given by a set of such loops into a basis given by another set of loops.

⇒ need to transform the lattice

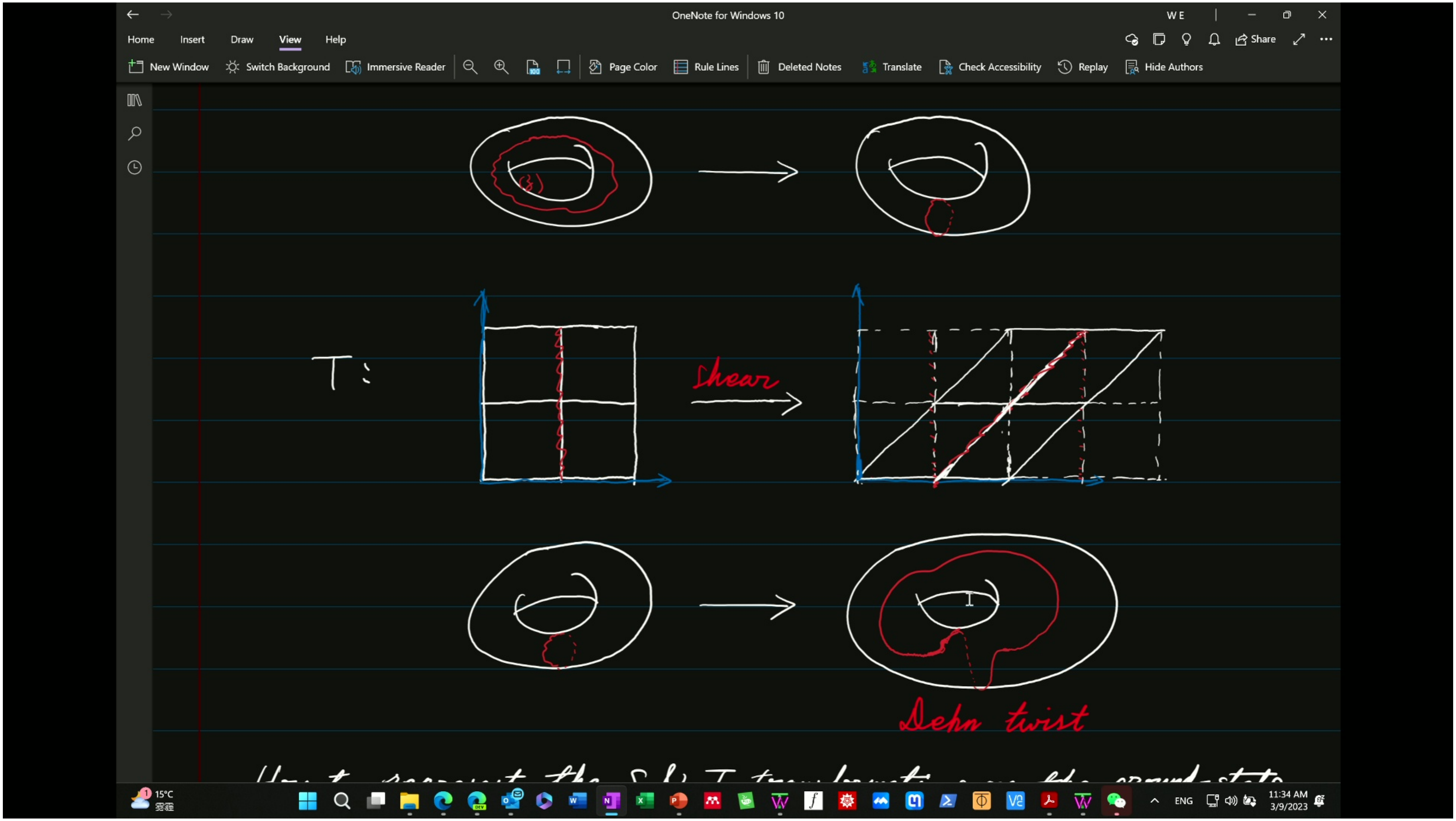
In 2D, all such transformations form a group:

$$SL(2,\mathbb{Z}) := (\underbrace{S,T}_{\text{integer matrices}} \mid \underbrace{S^4, (ST)^3 S^{-2}}_{\text{relations:}})$$

$S^4 = e,\ (ST)^3 = S^2$

S & T are the two generators of the group, and in the defining representation are

$$S = \begin{pmatrix} 0 & -1 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \end{pmatrix}$$


OneNote for Windows 10
Home
Insert
Draw
View
Help
Share
New Window
Switch Background
Immersive Reader
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Rule Lines
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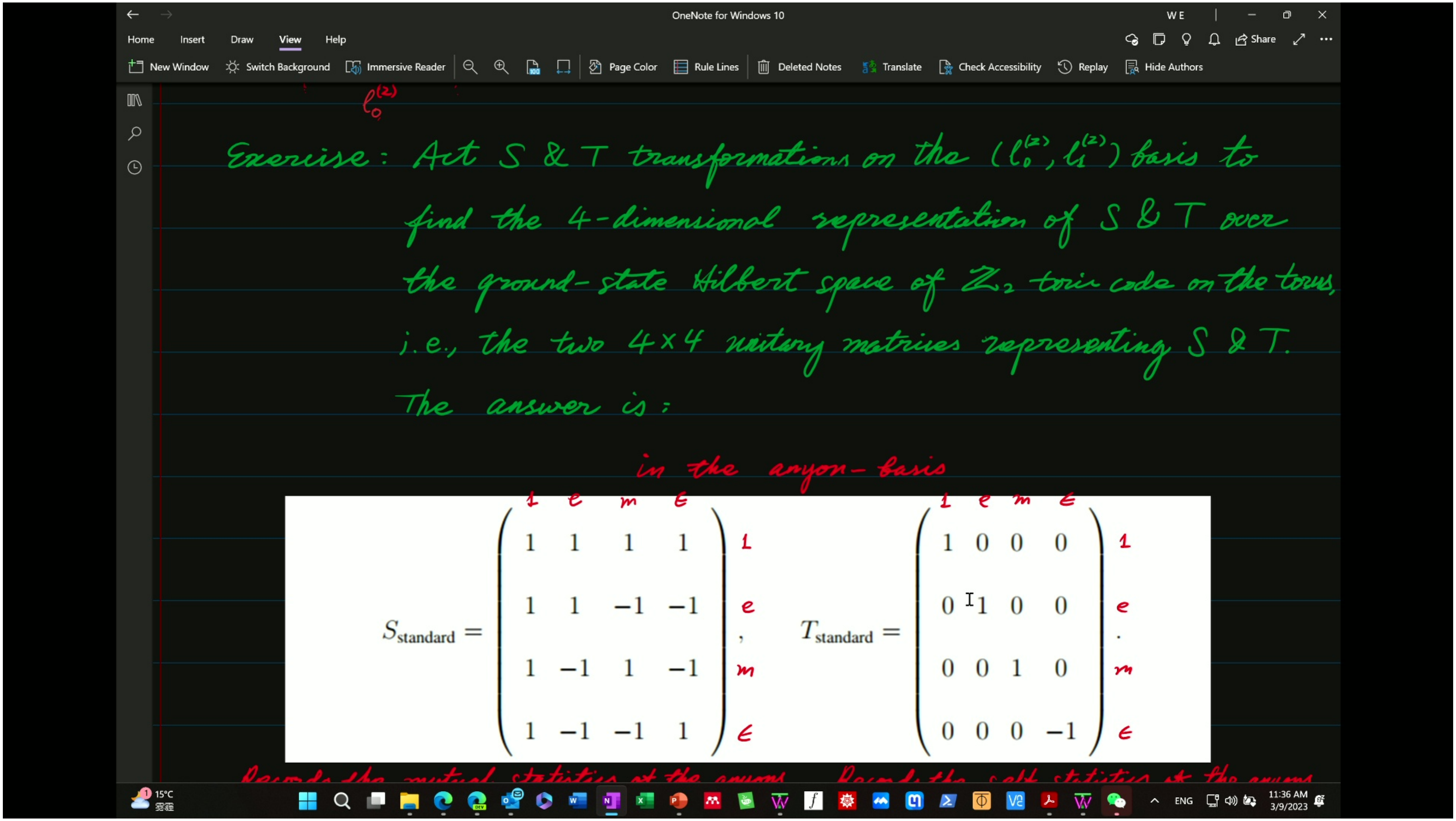


$\ell_0^{(2)}$

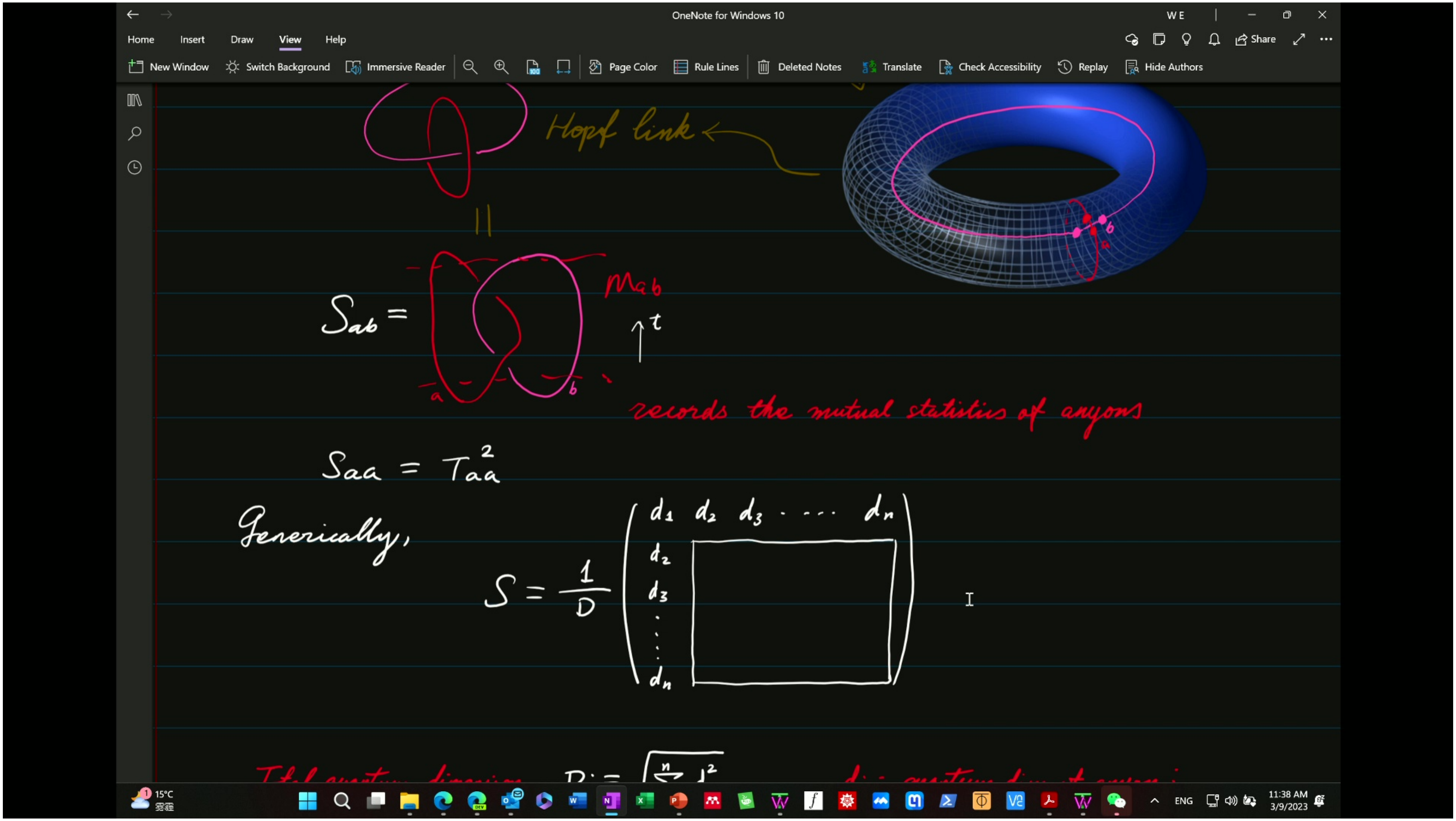
Exercise: Act S & T transformations on the $(\ell_0^{(2)}, \ell_1^{(2)})$ basis to find the 4-dimensional representation of S & T over the ground-state Hilbert space of $\mathbb{Z}_2$ toric code on the torus, i.e., the two 4×4 unitary matrices representing S & T.

The answer is:

in the anyon-basis

$$S_{\text{standard}} = \begin{array}{c} \begin{array}{cccc} 1 & e & m & \epsilon \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ e \\ m \\ \epsilon \end{array} , \quad T_{\text{standard}} = \begin{array}{c} \begin{array}{cccc} 1 & e & m & \epsilon \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ e \\ m \\ \epsilon \end{array} .$$

Records the mutual statistics of the anyons Records the self statistics of the anyons



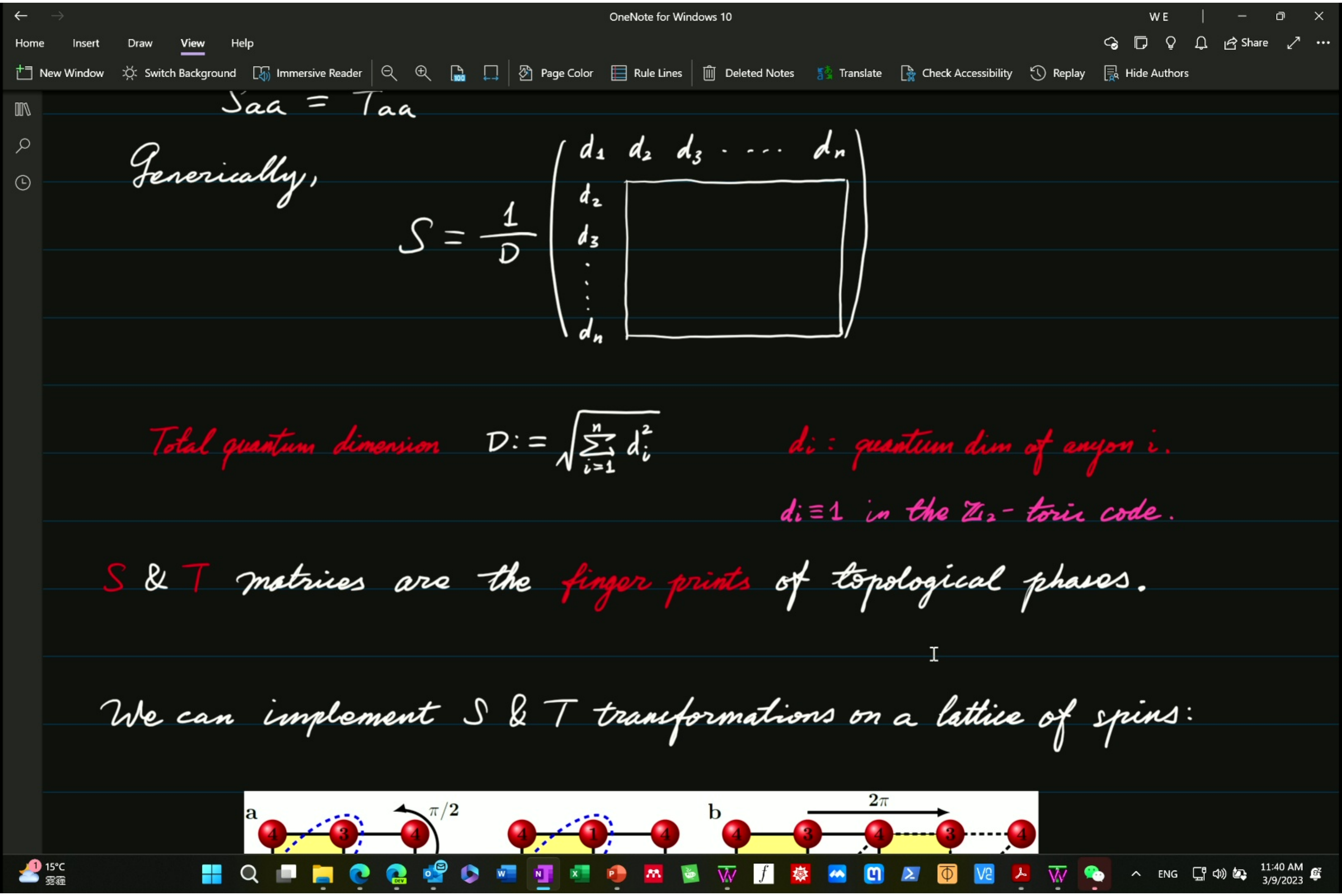
$$S_{ab} = M_{ab}$$

records the mutual statistics of anyons

$$S_{aa} = T_{aa}^2$$

Generically,

$$S = \frac{1}{D}\begin{pmatrix} d_1 & d_2 & d_3 & \cdots & d_n \\ d_2 & & & & \\ d_3 & & & & \\ \vdots & & & & \\ d_n & & & & \end{pmatrix}$$



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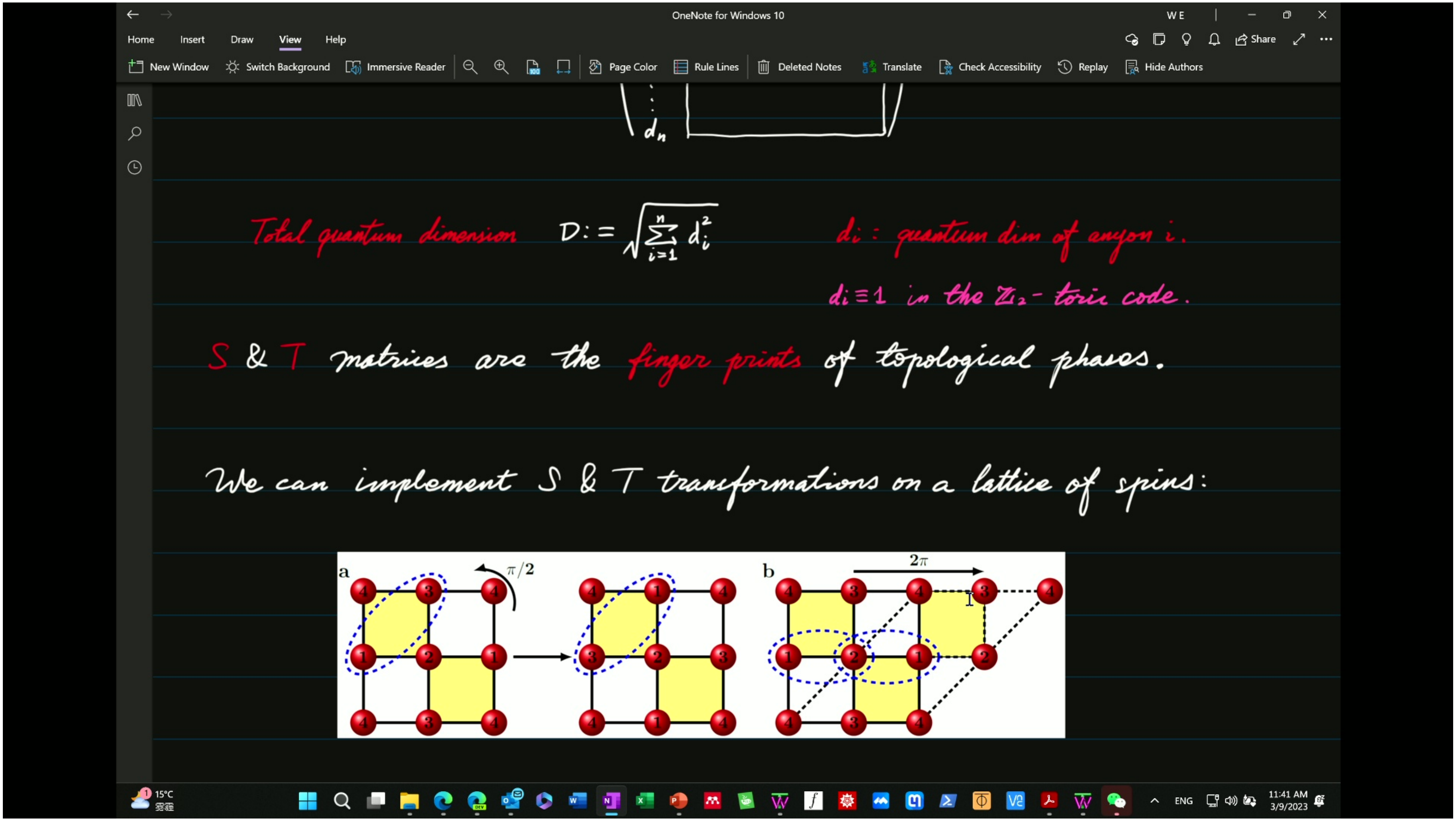
Total quantum dimension $D := \sqrt{\sum_{i=1}^{n} d_i^2}$

d_i : quantum dim of anyon i.

$d_i \equiv 1$ in the $\mathbb{Z}_2$- toric code.

S & T matrices are the finger prints of topological phases.

We can implement S & T transformations on a lattice of spins:



$$\begin{pmatrix} \vdots \\ d_n \end{pmatrix} \quad \boxed{}$$

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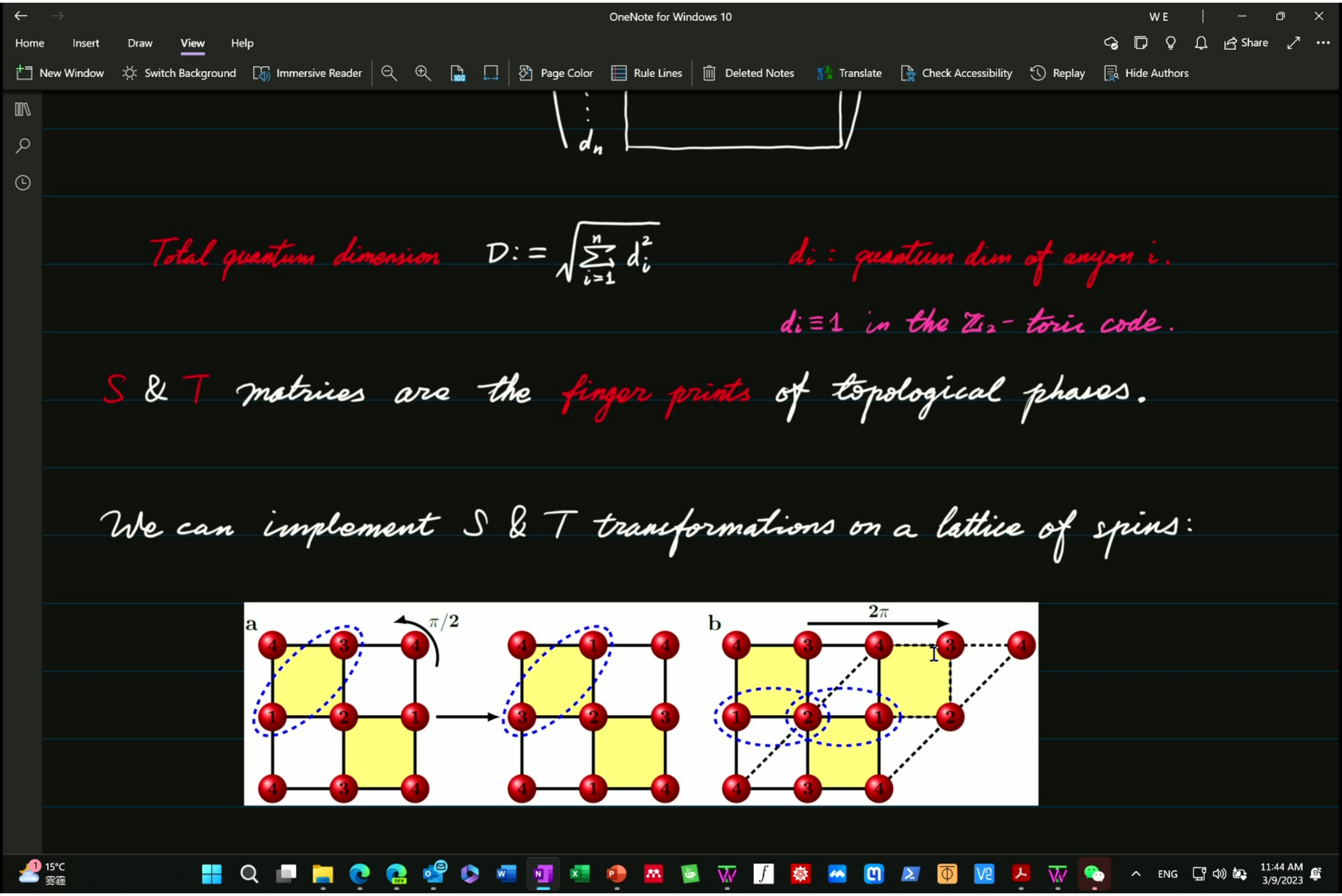

d_n

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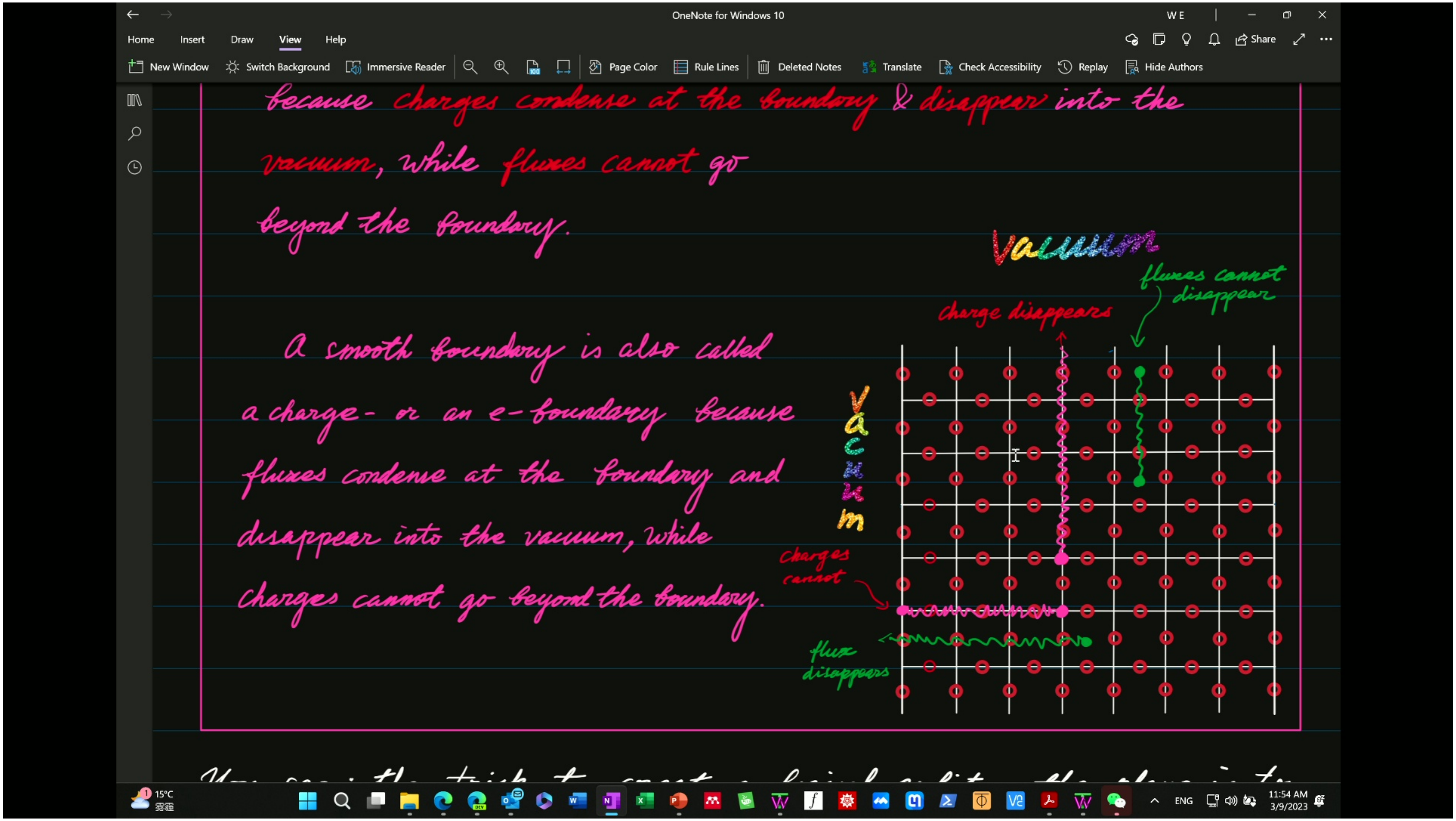
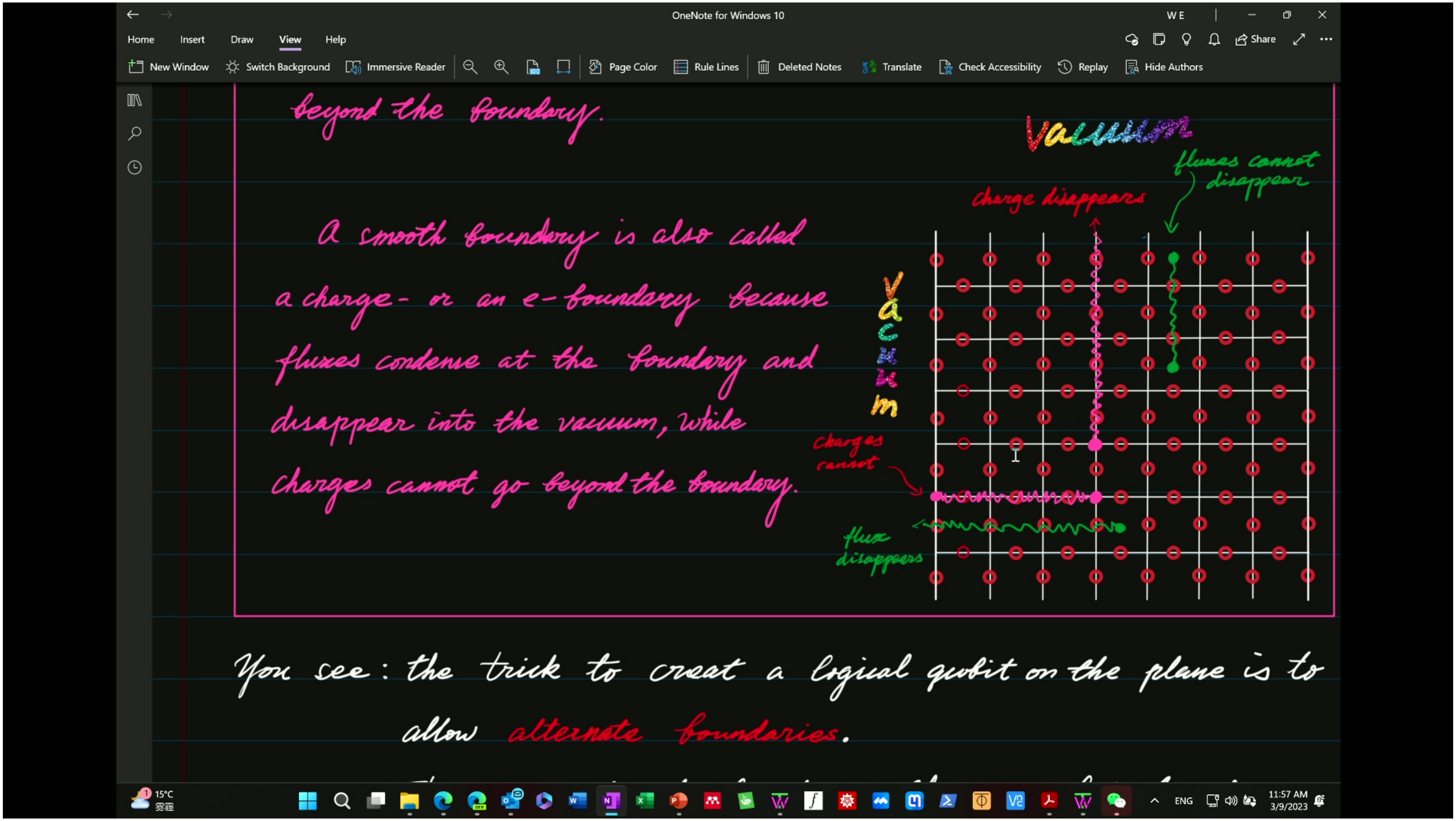


because charges condense at the boundary & disappear into the vacuum, while fluxes cannot go beyond the boundary.

A smooth boundary is also called a charge- or an e-boundary because fluxes condense at the boundary and disappear into the vacuum, while charges cannot go beyond the boundary.



beyond the boundary.

A smooth boundary is also called a charge- or an e-boundary because fluxes condense at the boundary and disappear into the vacuum, while charges cannot go beyond the boundary.

You see: the trick to creat a logical qubit on the plane is to allow alternate boundaries.