Title: Symmetry Lost and Found

Speakers: Shu-Heng Shao

Series: Quantum Fields and Strings

Date: March 28, 2023 - 2:00 PM

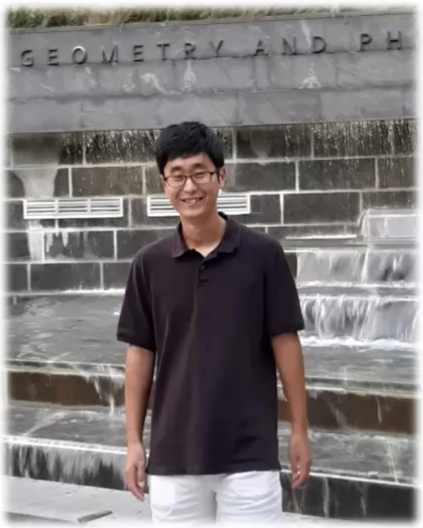
URL: https://pirsa.org/23030078

Abstract: In massless QED, we find that the classical U(1) axial symmetry is not completely broken by the Adler-Bell-Jackiw anomaly. Rather, it is resurrected as a generalized global symmetry labeled by the rational numbers. Intuitively, this new global symmetry in QED is a composition of the naive axial rotation and a fractional quantum Hall state. The conserved symmetry operators do not obey a group multiplication law, but a non-invertible fusion algebra. We further generalize our construction to QCD, and show that the neutral pion decay can be derived from a matching condition of the non-invertible global symmetry. Finally, we find a non-invertible Gauss law in axion-Maxwell theory.

Zoom link: https://pitp.zoom.us/j/93832561140?pwd=czFFSkVvYS9RbXRjOTJPQVFhL2hGZz09

Symmetry Lost and Found

Shu-Heng Shao
YITP, Stony Brook University



Yichul Choi
Stony Brook

Ho Tat Lam
MIT

Mainly based on
[Choi-Lam-SHS 2205.05086 (PRL)]
[Choi-Lam-SHS 2208.04331 (PRL)]
[Choi-Lam-SHS, 2212.04499]

See also
[Cordova-Ohmori 2205.06243]

And
[Choi-Lam-SHS 2208.04331]
[Roumpedakis-Seifnashri-SHS 2204.02407]
[Choi-Cordova-Hsin-Lam-SHS 2204.09025]

Chiral symmetry in QED

- Consider QED with a massless, unit charge Dirac fermion and $U(1)$ gauge group.

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\Psi}(\partial_\mu - iA_\mu)\gamma^\mu \Psi$$

- The classical $U(1)_A$ chiral symmetry acts as

$$\Psi \to \exp\left(\frac{i\alpha}{2}\gamma_5\right)\Psi \quad , \qquad \alpha \sim \alpha + 2\pi$$

- Note that $\alpha = 2\pi$ corresponds to the fermion parity, which is part of the gauge symmetry.
- The Adler-Bell-Jackiw anomaly implies that the classical $U(1)_A$ chiral symmetry **fails** to be a global symmetry quantum mechanically.

ABJ anomaly

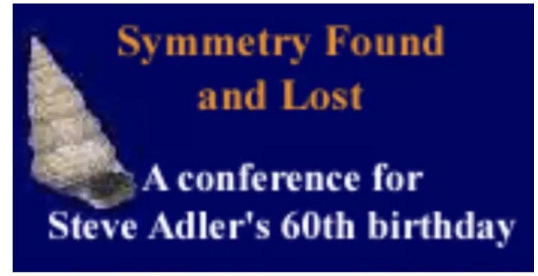
- The ABJ anomaly was discovered in the late 60s to explain the neutral pion decay, $\pi^0 \to \gamma\gamma$.

- It successfully determined the coupling

$$\frac{i}{8\pi^2 f_\pi}\pi^0 F \wedge F$$

  in the pion Lagrangian.

ABJ anomaly?



- Conceptually, there is something *slightly* counterintuitive though.
- Usually, we celebrate when we discover the **existence** of a global symmetry.
- ABJ anomaly states that there is **not** a global symmetry that one would have naively expected.
- So how come we can derive all these quantitative results from the **absence** of a global symmetry?
- Of course, since the fine structure constant is small, we can treat the electromagnetic gauge field as background. The $\pi^0 F \wedge F$ term follows from the Wess-Zumino term, which captures all the ‘t Hooft anomalies, in the chiral Lagrangian [Witten 1983].
- But wouldn’t it be nice if we can reinterpret these classic results from the existence of a generalized global symmetry (rather than the absence thereof)?

Is $U(1)_A$ a symmetry in massless QED?

- Different responses:

1. "No. Period."
2. "Yes, it is a symmetry in flat spacetime."
3. "Yes, but it is not gauge-invariant."
4. "Yes, it is an anomalous symmetry."
5. "Yes, it is a background symmetry."
6. …

- Something is conserved (e.g., helicity), but there isn't an ordinary symmetry.
- Is there a straight answer to this question?

Non-invertible global symmetries

- We will show that the continuous, invertible $U(1)_A$ chiral symmetry is broken by the ABJ anomaly to a discrete, non-invertible global symmetry labeled by the rational numbers.

- In the pion Lagrangian, the coupling $\pi^0 F \wedge F$ can be derived by matching the non-invertible global symmetry in the UV QCD.

- Therefore, the neutral pion decay $\pi^0 \to \gamma\gamma$ can be understood in terms of the non-invertible global symmetry.

Noether current

- Consider a conserved Noether current

$$\partial^\mu j_\mu = -\partial_t j_t + \partial_i j_i = 0$$

$$\mu = t, x, y, z$$
$$i = x, y, z$$

- The charge is defined as

$$Q = \int d^3x \, j_t$$

- Thanks to the conservation equation, it is conserved

$$\partial_t Q = \int d^3x \, \partial_t j_t = \int d^3x \, \partial_i j_i = 0$$

- The $U(1)$ unitary symmetry operator (the exponentiated charge) is

$$U_\vartheta = \exp(i\vartheta Q) = \exp(i\vartheta \int d^3x \, j_t) \quad , \qquad \partial_t U_\vartheta = 0$$

Symmetry and topology

- For relativistic systems in Euclidean signature, the time direction is on the same footing as any other spatial direction.
- We can therefore integrate the current on a general closed 3-manifold $M^{(3)}$ in 4-dimensional Euclidean spacetime:

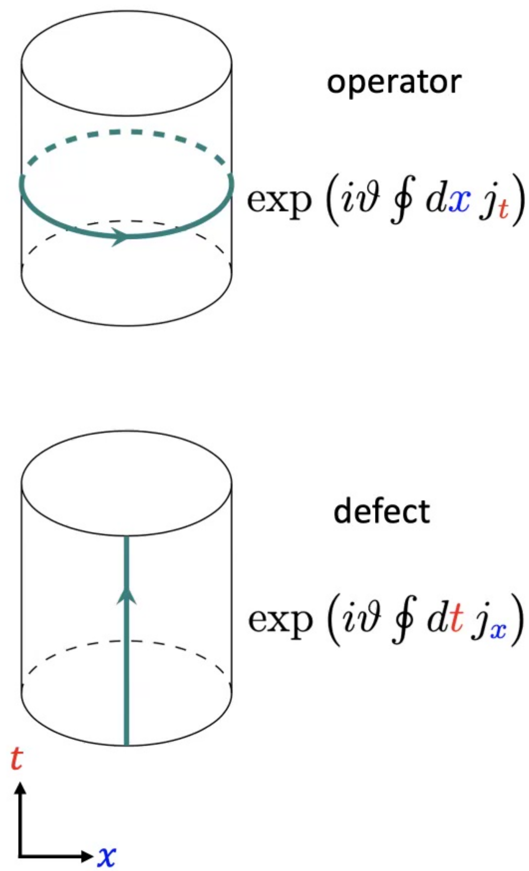
$$U_\vartheta = \exp(i\vartheta \int d^3x \, j_t)$$

$$\downarrow$$

$$U_\vartheta\big(M^{(3)}\big) = \exp(i\vartheta \oint_{M^{(3)}} j_\mu dn^\mu)$$

- The conservation equation $\partial_t U_\vartheta = 0$ is now **upgraded** to the fact that $U_\vartheta(M^{(3)})$ depends on $M^{(3)}$ only topologically because $\partial_\mu j^\mu = 0$ (divergence theorem).

Conserved → *Topological*



QED

- The axial current $j_\mu^A = \frac{1}{2}\overline{\Psi}\gamma_5\gamma_\mu\Psi$ obeys the anomalous conservation equation

$$d \star j^A = \frac{1}{8\pi^2} F \wedge F$$

Here the field strength is normalized such that $\oint F \in 2\pi\mathbb{Z}$.

- Naively, we can define the symmetry operator

$$U_\alpha(M) = \exp(i\alpha \oint_M \star j^A)$$

- However, it is **not** conserved.
- Adler defined a symmetry operator that is formally conserved, but is **not** gauge invariant:

$$\text{“ } \widehat{U}_\alpha(M) = \exp[i\alpha \oint_M (\star j^A - \frac{1}{8\pi^2} AdA)] \text{ ”}$$

Fact: The Chern-Simons action $\exp[i \oint_M (\frac{N}{4\pi} AdA)]$ is gauge invariant iff N is an integer.

Rational angles

- Let us be less ambitious, and assume the chiral rotation angle is a fraction:

$$\alpha = \frac{2\pi}{N}$$

$$\text{“ } \hat{U}_{\frac{2\pi}{N}}(M) = \exp[\oint_M (\frac{2\pi i}{N} \star j^A - \frac{i}{4\pi N} AdA)] \text{ ”}$$

- The operator $\hat{U}_{\frac{2\pi}{N}}(M)$ is still not gauge invariant because of the fractional Chern-Simons term.

Fractional quantum Hall state

$$\text{“} -\frac{i}{4\pi N}\oint_M AdA \text{”}$$

- In condensed matter physics, this action is commonly used to describe the $\nu = 1/N$ fractional quantum Hall effect (FQHE) in 2+1d.
- It is however not gauge invariant. Fortunately, there is a well-known fix.
- The more precise, gauge invariant Lagrangian for the FQHE is

$$\oint_M \left(\frac{iN}{4\pi} ada + \frac{i}{2\pi} adA\right)$$

  where a is a dynamical $U(1)$ gauge field living on the 2+1d manifold M.
- The two actions are related by illegally integrating out a to obtain

$$\text{“} a = -\frac{A}{N} \text{”}$$

Rational angles

- Let us be less ambitious, and assume the chiral rotation angle is a fraction:

$$\alpha = \frac{2\pi}{N}$$

$$\text{“ } \hat{U}_{\frac{2\pi}{N}}(M) = \exp[\oint_M (\frac{2\pi i}{N} \star j^A - \frac{i}{4\pi N} AdA)] \text{ ”}$$

- The operator $\hat{U}_{\frac{2\pi}{N}}(M)$ is still not gauge invariant because of the fractional Chern-Simons term.

Fractional quantum Hall state

$$“-\frac{i}{4\pi N}\oint_M AdA”$$

- In condensed matter physics, this action is commonly used to describe the $\nu = 1/N$ fractional quantum Hall effect (FQHE) in 2+1d.
- It is however not gauge invariant. Fortunately, there is a well-known fix.
- The more precise, gauge invariant Lagrangian for the FQHE is

$$\oint_M \left(\frac{iN}{4\pi}ada + \frac{i}{2\pi}adA\right)$$

  where a is a dynamical $U(1)$ gauge field living on the 2+1d manifold M.
- The two actions are related by illegally integrating out a to obtain

$$“a = -\frac{A}{N}”$$

QED

- The axial current $j^A_\mu = \frac{1}{2}\bar{\Psi}\gamma_5\gamma_\mu\Psi$ obeys the anomalous conservation equation

$$d \star j^A = \frac{1}{8\pi^2} F \wedge F$$

Here the field strength is normalized such that $\oint F \in 2\pi\mathbb{Z}$.

- Naively, we can define the symmetry operator

$$U_\alpha(M) = \exp(i\alpha \oint_M \star j^A)$$

- However, it is **not** conserved.
- Adler defined a symmetry operator that is formally conserved, but is **not** gauge invariant:

$$\text{“} \hat{U}_\alpha(M) = \exp[i\alpha \oint_M (\star j^A - \frac{1}{8\pi^2} AdA)] \text{”}$$

Fact: The Chern-Simons action $\exp[i \oint_M (\frac{N}{4\pi} AdA)]$ is gauge invariant iff N is an integer.

Fractional quantum Hall state

$$“-\frac{i}{4\pi N}\oint_M AdA”$$

- In condensed matter physics, this action is commonly used to describe the $\nu = 1/N$ fractional quantum Hall effect (FQHE) in 2+1d.
- It is however not gauge invariant. Fortunately, there is a well-known fix.
- The more precise, gauge invariant Lagrangian for the FQHE is

$$\oint_M \left(\frac{iN}{4\pi}ada + \frac{i}{2\pi}adA\right)$$

  where a is a dynamical $U(1)$ gauge field living on the 2+1d manifold M.
- The two actions are related by illegally integrating out a to obtain

$$“a = -\frac{A}{N}”$$

Back to QED

[Choi-Lam-SHS 2205.05086 (PRL), Cordova-Ohmori 2022]

- Motivated by the discussion of FQHE in 2+1d, we define a new operator in 3+1d QED:

$$\text{“ } \hat{U}_{\frac{2\pi}{N}}(M) = \exp[\oint_M (\frac{2\pi i}{N} \star j^A - \frac{i}{4\pi N} AdA)] \text{ ”}$$

a: auxiliary field on M
A: bulk gauge field

$$\downarrow$$

$$\mathcal{D}_{1/N}(M) \equiv \int [Da]_M \exp\left[\oint_M (\frac{2\pi i}{N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA) + \cdots\right]$$

- The new operator is gauge-invariant *and* conserved (topological).

*The FQH state **"cures"** the ABJ anomaly.*

Non-invertible chiral symmetry in QED

[Choi-Lam-SHS 2205.05086 (PRL)]

- The price we pay is that it **NOT** unitary:

$$\mathcal{D}_{1/N} \times \mathcal{D}_{1/N}^{\dagger} = \mathcal{C}$$
$$\equiv \int [Da]_M \int [D\bar{a}]_M \exp[\oint_M (\frac{iN}{4\pi} a da - \frac{iN}{4\pi} \bar{a} d\bar{a} + \frac{i}{2\pi}(a - \bar{a}) dA)]$$
$$\neq 1$$

- $\mathcal{C}$ is the condensation defect [Kong 2013, Kong-Wen 2014, Else-Nayak 2017, Gaiotto-JohnsonFreyd 2019, Choi-Cordova-Hsin-Lam-SHS 2022, Freed-Moore-Teleman 2022…] from higher gauging [Roumpedakis-Seifnashri-SHS 2022] of the $\mathbb{Z}_N$ subgroup of the $U(1)$ magnetic one-form symmetry.

Non-invertible chiral symmetry

Operator	Gauge-invariant?	Conserved (topological)?	Invertible?
$U_\alpha(M) = \exp(i\alpha \oint_M \star j^A)$	✓	✗	N/A
" $\hat{U}_\alpha(M) = \exp[i\alpha \oint_M (\star j^A - \frac{1}{4\pi^2} AdA)]$ "	✗	✓	✓
$\mathcal{D}_{\frac{1}{N}}(M) = \int [Da]_M \exp[\oint_M (\frac{2\pi i}{N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$	✓	✓	✗

Non-invertible chiral symmetry

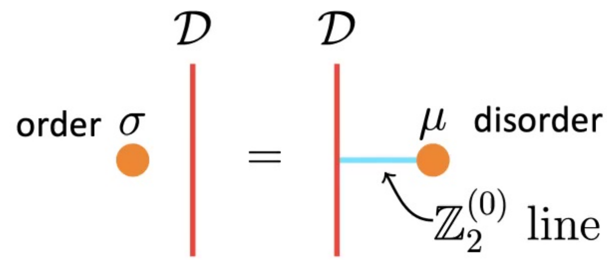
- It is easy to generalize this construction to an arbitrary rational chiral rotation $\alpha = 2\pi p/N$ with $\gcd(p, N) = 1$.

$$\mathcal{D}_{\frac{p}{N}}(M) \equiv \exp[\oint_M (\frac{2\pi i p}{N} \star j^A + \mathcal{A}^{N,p}[dA/N])]$$

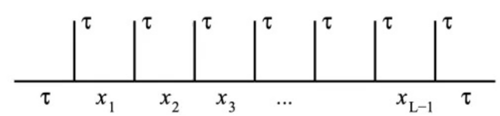
where $\mathcal{A}^{N,p}$ is the 2+1d minimal $\mathbb{Z}_N$ TQFT [Hsin-Lam-Seiberg 2018].

- Therefore, the continuous, invertible $U(1)_A$ chiral symmetry is broken by the ABJ anomaly to a discrete, non-invertible global symmetry labeled by the rational numbers $\frac{p}{N} \in \mathbb{Q}/\mathbb{Z}$.

Non-invertible symmetry



- In recent years, there have been rapid developments of non-invertible global symmetry [Bhardwaj-Tachikawa 2017, Chang-Lin-SHS-Yin-Wang 2018,…, Choi-Cordova-Hsin-Lam-SHS 2021, Kaidi-Ohmori-Zheng 2021,…].
- It is discovered in a variety of quantum systems, including Ising model (Kramers-Wannier duality defect), axions, QED, QCD, etc.
- Lattice realization: anyonic chains [Feiguin et al. 2006, Gils et al. 2013,…,].
- In 1+1d, these symmetries are described by the mathematical theory of fusion category. In higher dimensions, the mathematical language for non-invertible symmetries is still under development [Freed-Moore-Teleman 2022].



Why are they "symmetries"?

Why should we think of the non-invertible conserved operators as generalized global symmetries?

- They lead to conservation laws and selection rules [..., Choi-Lam-SHS 2022, Lin-Okada-Seifnashri-Tachikawa 2022,...].
- Some non-invertible symmetries can be gauged [Brunner-Carqueville-Plencner 2014].
- They can have generalized anomalies, which lead to generalized 't Hooft anomaly matching conditions. New constraints on renormalization group flows [Chang-Lin-SHS-Wang-Yin 2018, Komargodski et al. 2020].
- This inclusion consolidates conjectures in quantum gravity [Rudelius-SHS 2020, Heidenreich et al. 2021]:

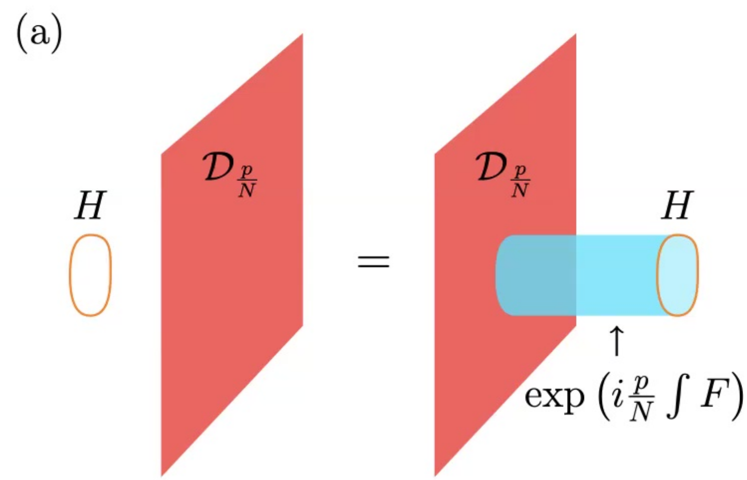
no **generalized** global symmetry $\Leftrightarrow$ completeness of gauge spectrum

Selection rule in QED

- The operator $\mathcal{D}_{p/N}$ acts invertibly on the fermions as a chiral rotation with $\alpha = 2\pi p/N$.
- It acts non-invertibly on the 't Hooft lines $H(\gamma)$ by the Witten effect:

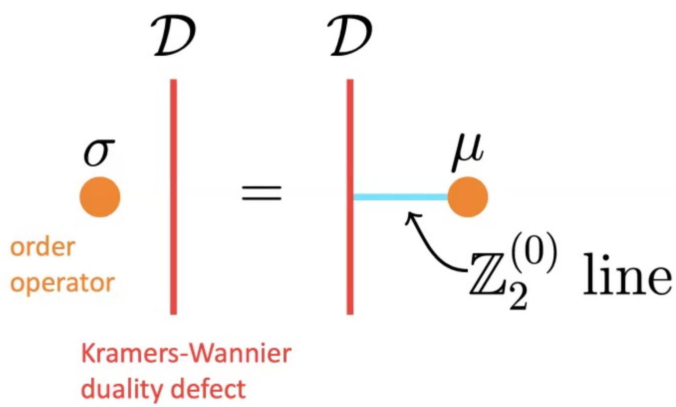
$$H(\gamma) \mapsto H(\gamma)\exp(\frac{ip}{N}\int F)$$

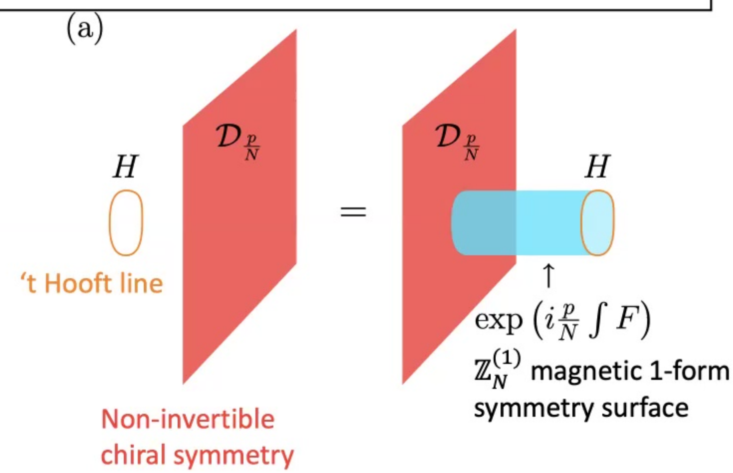
- The selection rule on the fermions on flat space amplitudes from $\mathcal{D}_{p/N}$ are the same as the naïve $U(1)_A$ symmetry.
- Note that there is no $U(1)$ instanton in flat space because $\pi_3\big(U(1)\big) = 0$.



1+1d Ising CFT vs. 3+1d QED

1+1d Ising CFT	3+1d QED
non-invertible Kramers-Wannier defect	non-invertible chiral symmetry
$\mathbb{Z}_2^{(0)}$ 0-form symmetry	$\mathbb{Z}_N^{(1)}$ magnetic 1-form symmetry
order operator σ	't Hooft line H
disorder operator μ	dyonic line





Electron mass

- Let us explore various consequences of the non-invertible symmetry in QED.
- Naturalness ['t Hooft 1980]: Impose a global symmetry group G. The Lagrangian should include all G-invariant terms with coefficients of order one with no fine-tuning.
- QED Lagrangian: $\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\left(\partial_\mu - iA_\mu\right)\gamma^\mu\Psi$
- The electron mass term $m\bar{\Psi}\Psi$ violates the non-invertible global symmetry.
- Therefore, electron is naturally massless in QED because of the non-invertible global symmetry.
- In contrast, scalar QED has no enhanced global symmetry at the massless point – Coleman-Weinberg mechanism.

‘t Hooft Naturalness

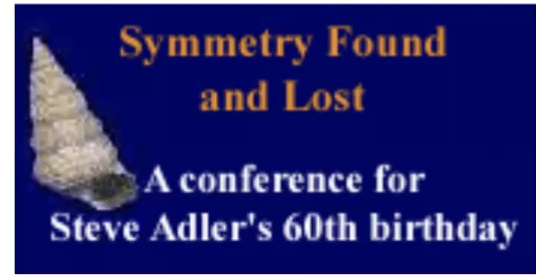
III2. NATURALNESS IN QUANTUM ELECTRODYNAMICS

Quantum Electrodynamics as a renormalizable model of electrons (and muons if desired) and photons is an example of a "natural" field theory. The parameters α, m_e (and m_μ) may be small independently. In particular m_e (and m_μ) are very small at large μ. The relevant symmetry here is chiral symmetry, for the electron and the muon separately. We need not be concerned about the Adler-Bell-Jackiw anomaly here because the photon field being Abelian cannot acquire non-trivial topological winding numbers[4].

‘t Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking* (1980)

Symmetry Lost and Found

[Choi-Lam-SHS 2205.05086 (PRL), Cordova-Ohmori 2022]



Year

$U(1)_A$ **IS** a (non-invertible) symmetry!

2022

Fractional Quantum Hall Effect 1982

$U(1)_A$ is **NOT** a symmetry

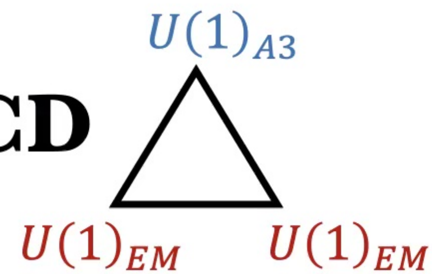
Adler-Bell-Jackiw 1969

$U(1)_A$ is a symmetry

23

Non-invertible symmetry in QCD

[Choi-Lam-SHS 2205.05086 (PRL)]



- Below the electroweak scale, the massless QCD Lagrangian for the up and down quarks has a chiral global symmetry (corresponding to π^0)

$$U(1)_{A3}: \begin{pmatrix} u \\ d \end{pmatrix} \to \exp(i\alpha\gamma_5\sigma_3) \begin{pmatrix} u \\ d \end{pmatrix}$$

- It suffers from the ABJ anomaly with the electromagnetic $U(1)_{EM}$ gauge symmetry.
- By the exact same construction, we conclude that there is an infinite non-invertible global symmetry $\mathcal{D}_{p/N}$ in the UV QCD from $U(1)_{A3}$.
- How does the IR pion Lagrangian capture this non-invertible global symmetry?

Pion

- The pion Lagrangian

$$\mathcal{L}_{IR} = \frac{1}{2}\left(\partial_\mu \pi^0\right)^2 + ig\, \pi^0 F \wedge F + \cdots$$

- The pion field is compact, $\pi^0 \sim \pi^0 + 2\pi f_\pi$, where $f_\pi \sim 92.4 MeV$.
- The non-invertible global symmetry $\mathcal{D}_{p/N}$ shifts the pion field,

$$\pi^0 \to \pi^0 - 2\pi \frac{p}{N} f_\pi .$$

- The equations of motion in the presence of the non-invertible global symmetry $\mathcal{D}_{p/N}$ fix the coefficient g for $\pi^0 F \wedge F$, which gives the dominant contribution to the neutral pion decay $\pi^0 \to \gamma\gamma$.

Pion

$$\mathcal{L}_{IR} = \frac{1}{2}(\partial_\mu \pi^0)^2 + ig\,\pi^0 F \wedge F$$

$$\mathcal{D}_{1/N}(M) = \int [Da]_M \exp[\oint_{x=0} (\frac{2\pi i}{N} \star j^{A3} + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$$

- Inserting $\mathcal{D}_{1/N}$ at $x = 0$ as a defect, the equations of motion are
  - π^0 EOM: $\pi^0|_{x=0^+} - \pi^0|_{x=0^-} = -\frac{2\pi}{N} f_\pi$
  - a EOM: $Nda + F = 0$
  - A EOM: $2ig(\pi^0|_{x=0^+} - \pi^0|_{x=0^-})F = \frac{i}{2\pi} da$
- Combining the above, it fixes $g = \frac{1}{8\pi^2 f_\pi}$.

Pion decay

- Conventionally, the pion decay $\pi^0 \to \gamma\gamma$ is explained by the ABJ anomaly. Of course, since the fine structure constant is small, we can treat the electromagnetic gauge field as background, then the $\pi^0 F \wedge F$ follows from the ‘t Hooft anomaly matching.
- We have provided an alternative explanation for the pion decay as a direct consequence from matching the non-invertible global symmetry in the UV QCD.
- The non-invertible global symmetry gives an invariant characterization of the ABJ anomaly in terms of the *existence* of a generalized global symmetry, rather than the *absence* thereof.

Generalizations?

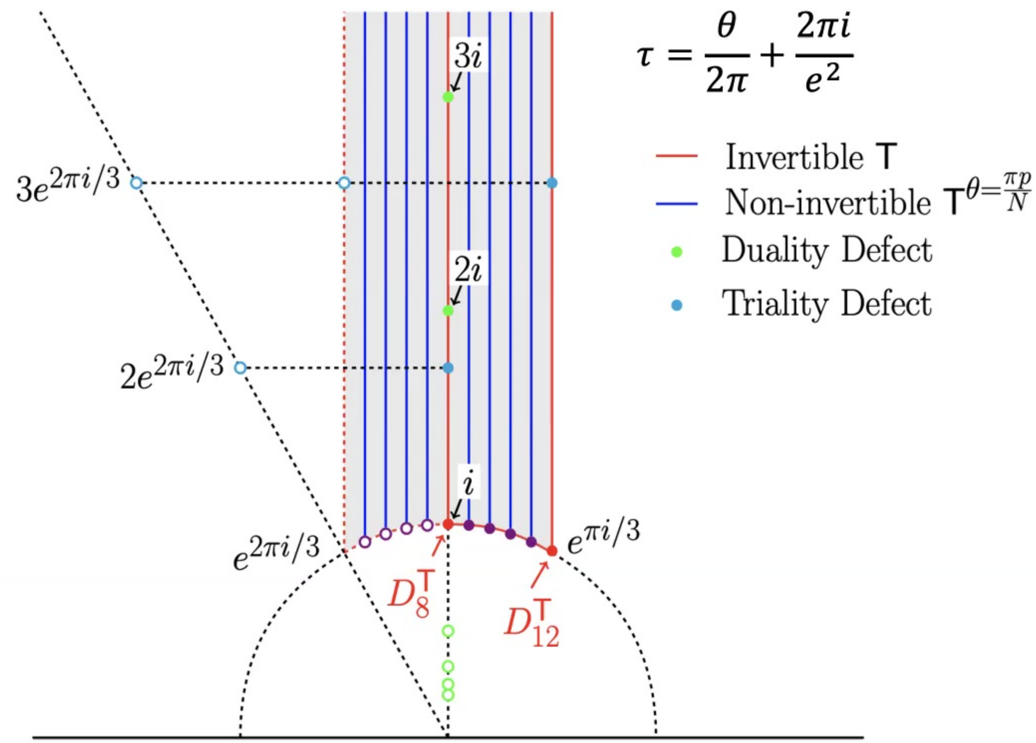
- Our construction **does not generalize straightforwardly** to the following cases. It's *not* to say that they are impossible.

1. Axial rotation with an irrational angle.
   - No irrational quantum Hall state.
   - See [Karasik 2022, Garcia Etxebarria-Iqbal 2022] for an alternative construction.
2. ABJ anomalies involving $SU(N)$ gauge groups.
   - No magnetic 1-form symmetry. No fractional $SU(N)$ charge.
   - Generalization for $PSU(N)$ gauge groups [Cordova-Ohmori 2022].
3. 1+1d QED.
   - No magnetic symmetry.

Non-invertible CP symmetry

[Choi-Lam-SHS 2208.04331 (PRL)]

- It is commonly stated that CP or T is violated whenever the θ-angle is neither 0 or π.
- $U(1)$ gauge theory is time-reversal invariant for every rational θ angle
$$\theta = \frac{\pi p}{N}$$
- Non-invertible CP and time-reversal symmetry.
- Strong CP problem?



Non-invertible symmetries of axions

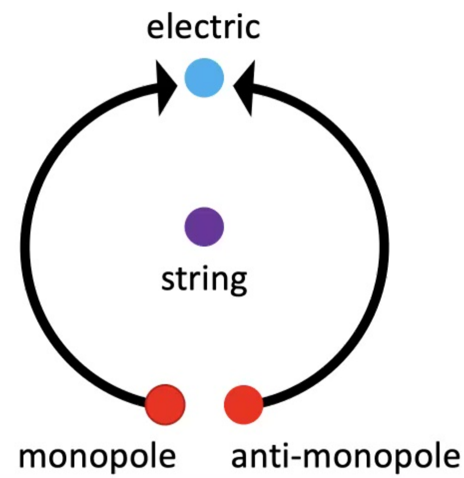
[Choi-Lam-SHS 2212.04499]

$$\frac{f^2}{2} d\theta \wedge\star\, d\theta + \frac{1}{2e^2} F \wedge\star\, F - \frac{iK}{8\pi^2} \theta F \wedge F$$

- $\theta(x) \sim \theta(x) + 2\pi$ is the dynamical axion field, a periodic scalar field.
- When the axion-photon coupling $|K| > 1$, there is a higher group symmetry [Hidaka-Nitta-Yokokura 2020x2, Brennan-Cordova 2020].
- Even at $|K| = 1$, there are non-invertible symmetries.
- The shift symmetry $\theta(x) \to \theta(x) + 2\pi p/N$ is a non-invertible 0-form symmetry $\mathcal{D}^{(0)}_{p/N}$.
- Furthermore, there is a non-invertible 1-form symmetry [Choi-Lam-SHS 2022, Yokokura 2022]. It gives a global symmetry interpretation of the classic results of [Callan-Harvey 1985, Naculich 1988,…].

Anomalous Gauss law

- In axion-Maxwell theory, Gauss law is anomalous
$$-\frac{i}{e^2} d \star F = \frac{1}{4\pi^2} d\theta \wedge F$$
- Hence there is **no** conserved and quantized electric charge that can be measured by the ordinary Gauss law.
- Indeed, monopoles can pair create and annihilate around an axion string to create an electric particle.



Non-invertible Gauss law

[Choi-Lam-SHS 2212.04499]

$$\hat{U}^{(1)}_{2\pi\frac{p}{N}}(\Sigma^{(2)}) = \exp\left(\frac{2\pi i p}{N} Q_{Page}\right) = \exp\left[\frac{2\pi i p}{N} \oint_{\Sigma^{(2)}} \left(-\frac{i}{e^2} \star F - \frac{1}{4\pi^2} \theta dA\right)\right]$$

- The naïve Gauss law operator $\hat{U}^{(1)}_{\alpha}(\Sigma^{(2)})$ is not gauge invariant because of the second term. The exponent is the "Page charge", which is not gauge invariant.
- Use a 1+1d $\mathbb{Z}_N$ gauge theory to "cure" it.

$$\mathcal{D}^{(1)}_{\frac{p}{N}}(\Sigma^{(2)}) = \int [D\phi Dc]_{\Sigma^{(2)}} \exp\left[\oint_{\Sigma^{(2)}} \left(\frac{2\pi p}{Ne^2} \star F + \frac{iN}{2\pi} \phi dc + \frac{ip}{2\pi} \theta dc + \frac{i}{2\pi} \phi dA\right)\right]$$

where ϕ is a compact scalar and c is a 1-form gauge field, both living on the Gauss surface $\Sigma^{(2)}$.

We choose a Euler counterterm to normalize the sphere expectation value of $\mathcal{D}^{(1)}_{\frac{p}{N}}$ without any insertion to be 1.

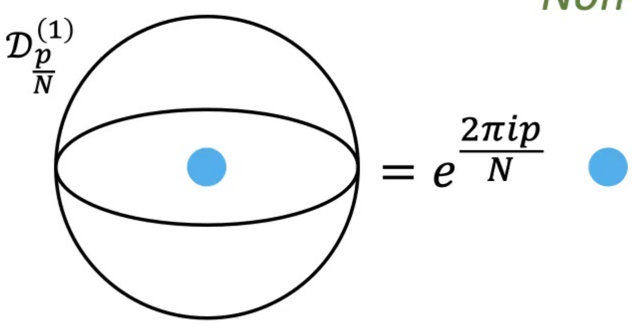
Non-invertible Gauss law

[Choi-Lam-SHS 2212.04499]

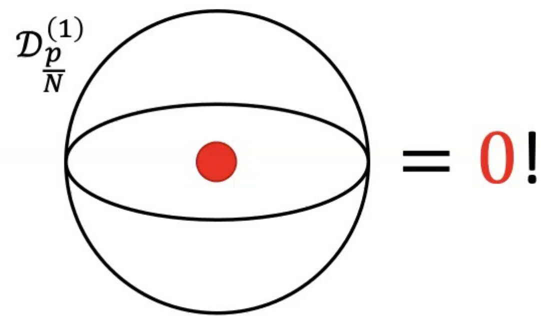
$$\mathcal{D}^{(1)}_{\frac{p}{N}}(\Sigma^{(2)}) = \int [D\phi Dc]_{\Sigma^{(2)}} \exp\left[\oint_{\Sigma^{(2)}} \left(\frac{2\pi p}{Ne^2}\star F + \frac{iN}{2\pi}\phi dc + \frac{ip}{2\pi}\theta dc + \frac{i}{2\pi}\phi dA\right)\right]$$

- $\mathcal{D}^{(1)}_{\frac{p}{N}}$ is gauge-invariant, topological (and in particular conserved), but non-invertible. It is a non-invertible 1-form global symmetry.

Non-invertible Gauss law



charge-1 electron



charge-1 monopole

Constraints on symmetry breaking scales

[Choi-Lam-SHS 2212.04499]

- The generalized global symmetries are typically emergent in an RG flow to the axion-Maxwell theory.
- The non-invertible symmetries lead to **universal constraints** on the symmetry breaking scales in any UV completion of the axion model, generalizing[Brennan-Cordova 2020]:

$$m_{electric} \lesssim \min(m_{monopole}, \sqrt{T})$$

Consistent with anomaly inflow
[Callan-Harvey 1985]

- T: axion string tension

Conclusion

- In massless QED and QCD, the continuous, invertible $U(1)_A$ symmetry is broken by the ABJ anomaly into a discrete, non-invertible symmetry $\mathcal{D}_{p/N}$ labeled by rational numbers.

$$\mathcal{D}_{\frac{1}{N}}(M) \equiv \int [Da]_M \exp[\oint_M (\frac{2\pi i}{2N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$$

- The non-invertible symmetry is a composition of the naïve axial rotation with a fractional quantum Hall state.
- To put it in the maximally offensive way, the neutral pion decays $\pi^0 \to \gamma\gamma$ because of the non-invertible global symmetry [Choi-Lam-SHS 2205.05086].
- Non-invertible Gauss law in axion-Maxwell theory [Choi-Lam-SHS 2212.04499].