Title: Causal Inference Lecture - 230329

Speakers: Robert Spekkens

Collection: Causal Inference: Classical and Quantum

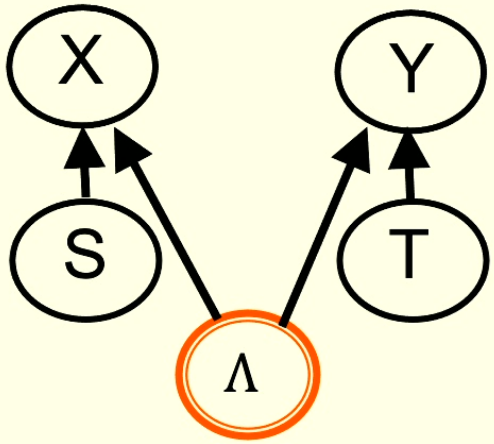
Date: March 29, 2023 - 10:00 AM

URL: https://pirsa.org/23030076

Abstract: zoom link: https://pitp.zoom.us/j/94143784665?pwd=VFJpajVIMEtvYmRabFYzYnNRSVAvZz09

Inflation ideas in the Bell scenario

Causal structure



Parameters

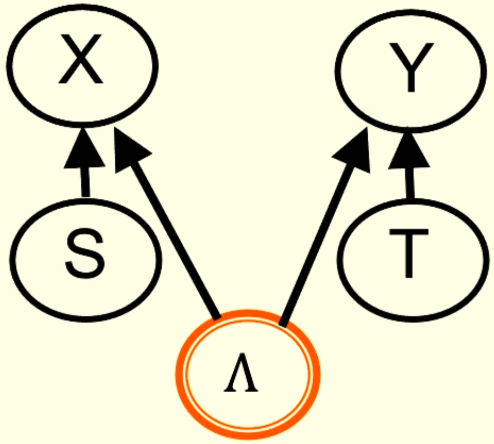
$P_{X|S\Lambda}$

$P_{Y|T\Lambda}$

P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda}$$

Causal structure   Parameters



$P_{X|S\Lambda}$
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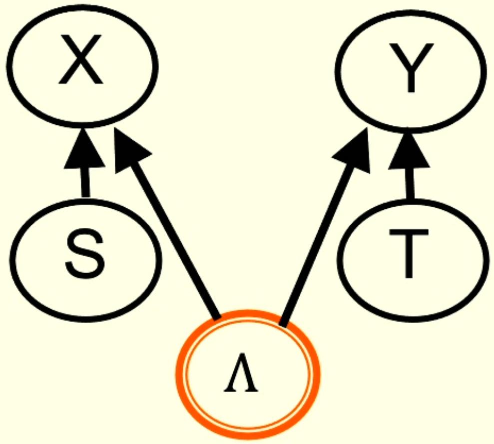
$$P_{XY|ST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda}$$

Examples of causal compatibility constraints:

$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$

Causal structure Parameters



$$P_{X|S\Lambda}$$
$$P_{Y|T\Lambda}$$
$$P_{\Lambda}$$

$$P_{XY|ST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda}$$

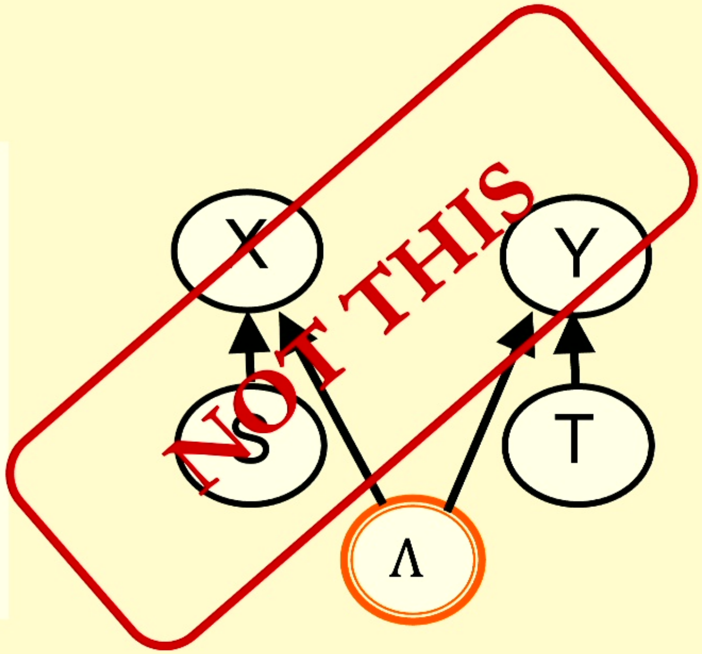
Examples of causal compatibility constraints:

$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

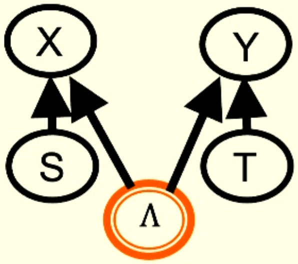
Clauser, Horne, Shimony and Holte, Phys. Rev. Lett.23, 880 (1967)

		(X,Y)			
		0 and 0	0 and 1	1 and 0	1 and 1
(S,T)	0 and 0	43%	7%	7%	43%
	0 and 1	43%	7%	7%	43%
	1 and 0	43%	7%	7%	43%
	1 and 1	7%	43%	43%	7%



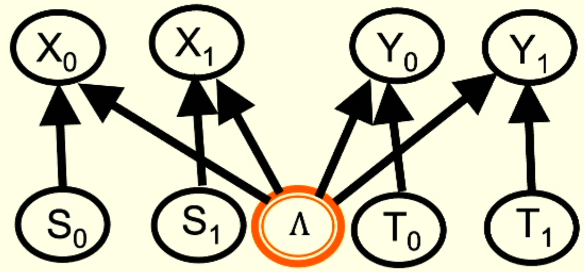
Violates the
CHSH Inequalities

Bell model M



$P_{X|S\Lambda}$
$P_{Y|T\Lambda}$
P_{Λ}

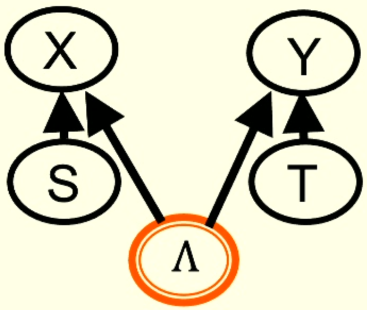
Inflated model M'



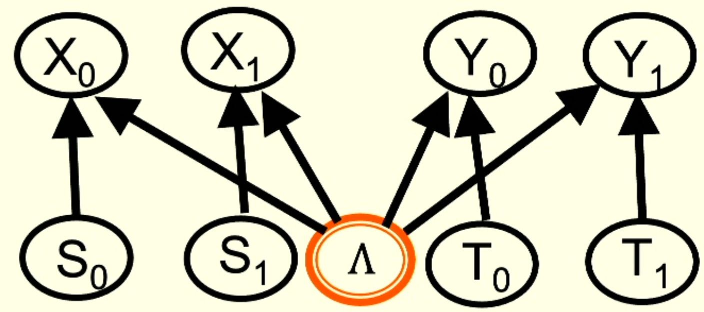
$P_{X_0|S_0\Lambda}$
$P_{X_1|S_1\Lambda}$
$P_{Y_0|T_0\Lambda}$
$P_{Y_1|T_1\Lambda}$
P_{Λ}

$P_{X_0|S_0\Lambda} = P_{X_1|S_1\Lambda}$
$P_{Y_0|T_0\Lambda} = P_{Y_1|T_1\Lambda}$

Bell model M



Inflated model M'



$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

Causal compatibility inequality in M

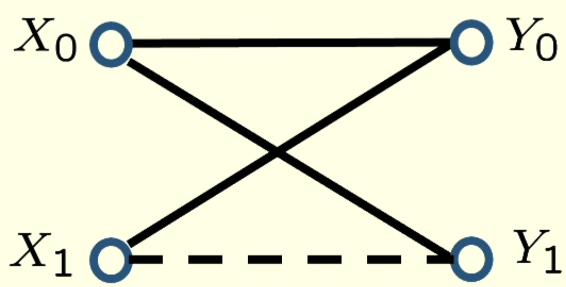
$$\frac{1}{4}\sum_{x=y} P_{X_0Y_0|S_0T_0}(xy|00) + \frac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0T_1}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{X_1Y_0|S_1T_0}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_1T_1}(xy|11) \leq \frac{3}{4}$$

Causal compatibility inequality in M'

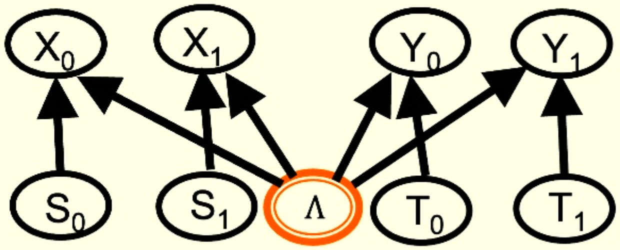
Consider any distribution over 4 variables $Q_{X_0 X_1 Y_0 Y_1}$

The marginals $Q_{X_0Y_0}, Q_{X_0Y_1}, Q_{X_1Y_0}, Q_{X_0Y_1}$ satisfy

$$\frac{1}{4}\sum_{x=y} Q_{X_0Y_0}(xy) + \frac{1}{4}\sum_{x=y} Q_{X_0Y_1}(xy)$$
$$\frac{1}{4}\sum_{x=y} Q_{X_1Y_0}(xy) + \frac{1}{4}\sum_{x\neq y} Q_{X_1Y_1}(xy) \leq \frac{3}{4}$$

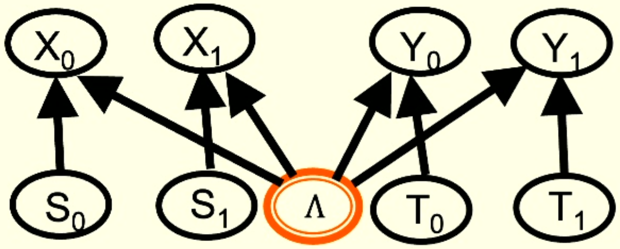


Inflated model M'



$$\text{Take } Q_{X_0X_1Y_0Y_1}(\cdot) := P_{X_0X_1Y_0Y_1|S_0S_1T_0T_1}(\cdot|0101)$$

Inflated model M'



$$X_0Y_0 \perp S_1T_1|S_0T_0 \implies P_{X_0Y_0|S_0T_0S_1T_1} = P_{X_0Y_0|S_0T_0}$$

$$X_0Y_1 \perp S_1T_0|S_0T_1 \implies P_{X_0Y_1|S_0T_0S_1T_1} = P_{X_0Y_1|S_0T_1}$$

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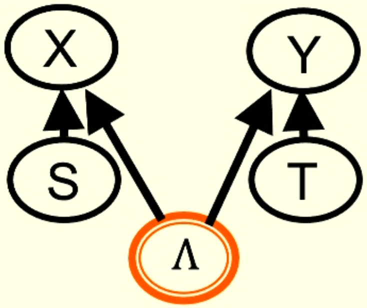
$$X_1Y_1 \perp S_0T_0|S_1T_1 \implies P_{X_1Y_1|S_0T_0S_1T_1} = P_{X_1Y_1|S_1T_1}$$

$$\frac{1}{4}\sum_{x=y} P_{X_0Y_0|S_0S_1T_0T_1}(xy|0101) + \frac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0S_1T_0T_1}(xy|0101)$$
$$\frac{1}{4}\sum_{x=y} P_{X_1Y_0|S_0S_1T_0T_1}(xy|0101) + \frac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_0S_1T_0T_1}(xy|0101) \leq \frac{3}{4}$$

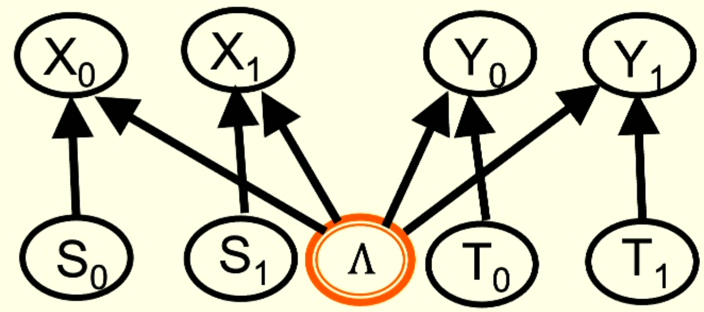
$\Rightarrow$

$$\frac{1}{4}\sum_{x=y} P_{X_0Y_0|S_0T_0}(xy|00) + \frac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0T_1}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{X_1Y_0|S_1T_0}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_1T_1}(xy|11) \leq \frac{3}{4}$$

Bell model M



Inflated model M'



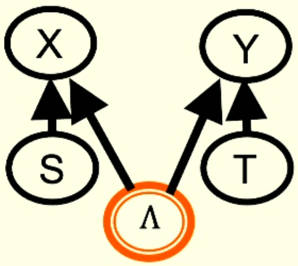
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{XY|ST}(xy|00) + \tfrac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{XY|ST}(xy|10) + \tfrac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \tfrac{3}{4}$$

Causal compatibility inequality in M

$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_0Y_0|S_0T_0}(xy|00) + \tfrac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0T_1}(xy|01)$$
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_1Y_0|S_1T_0}(xy|10) + \tfrac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_1T_1}(xy|11) \leq \tfrac{3}{4}$$

Causal compatibility inequality in M'

Bell model M



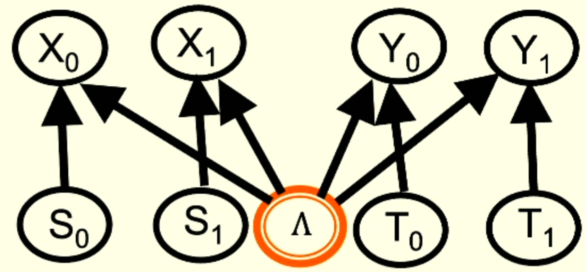
$P_{X|S\Lambda}$
$P_{Y|T\Lambda}$
P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda}$$

$P_{XY|ST}$

compatible with M

Inflated model M'



$P_{X_0|S_0\Lambda}$
$P_{X_1|S_1\Lambda}$
$P_{Y_0|T_0\Lambda}$
$P_{Y_1|T_1\Lambda}$
P_{Λ}

$P_{X_0|S_0\Lambda} = P_{X_1|S_1\Lambda}$
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$$P_{X_0Y_0|S_0T_0} = \sum_{\Lambda} P_{X_0|S_0\Lambda} P_{Y_0|T_0\Lambda} P_{\Lambda}$$
$$P_{X_0Y_1|S_0T_1} = \sum_{\Lambda} P_{X_0|S_0\Lambda} P_{Y_1|T_1\Lambda} P_{\Lambda}$$
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$P_{X_0Y_0|S_0T_0}, P_{X_0Y_1|S_0T_1}, P_{X_1Y_0|S_1T_0}, P_{X_1Y_1|S_1T_1}$

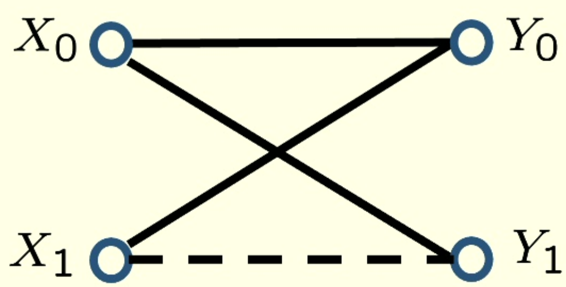
where $P_{X_iY_j|S_iT_j} = P_{XY|ST}$

compatible with M'

Consider any distribution over 4 variables $Q_{X_0 X_1 Y_0 Y_1}$

The marginals $Q_{X_0 Y_0}, Q_{X_0 Y_1}, Q_{X_1 Y_0}, Q_{X_0 Y_1}$ satisfy

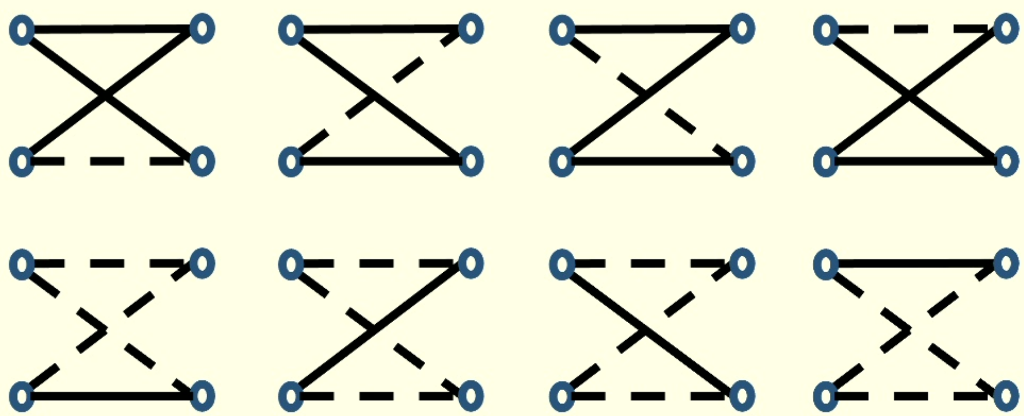
$$\frac{1}{4}\sum_{x=y} Q_{X_0 Y_0}(xy) + \frac{1}{4}\sum_{x=y} Q_{X_0 Y_1}(xy)$$
$$\frac{1}{4}\sum_{x=y} Q_{X_1 Y_0}(xy) + \frac{1}{4}\sum_{x\neq y} Q_{X_1 Y_1}(xy) \leq \frac{3}{4}$$



All the Bell inequalities for binary settings and outcomes

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

+ permutations corresponding to the 8 frustrated four-node networks

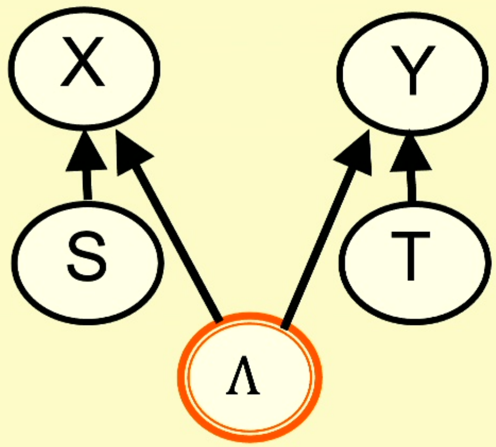


Hardy's version of Bell's theorem via the inflation technique

Hardy-type correlations

$$P_{XY|ST}(\cdot|00) = a_{00}[00] + a_{01}[01] + a_{10}[10] + a_{11}[11]$$
$$P_{XY|ST}(\cdot|01) = b_{00}[00] + b_{01}[01] + b_{11}[11]$$
$$P_{XY|ST}(\cdot|10) = c_{00}[00] + c_{10}[10] + c_{11}[11]$$
$$P_{XY|ST}(\cdot|11) = d_{00}[00] + d_{01}[01] + d_{10}[10]$$

where $a_{11} > 0$

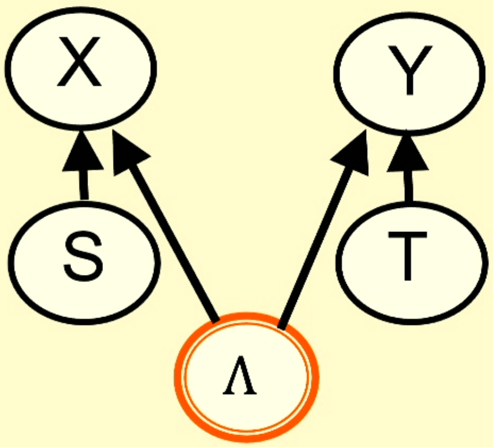


Compatible?

Hardy-type correlations

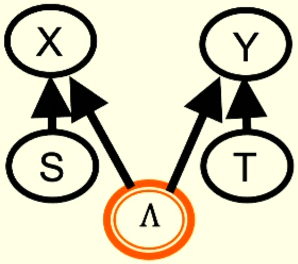
(X,Y)

(S,T)	0 and 0	0 and 1	1 and 0	1 and 1
0 and 0	.	.	.	>0
0 and 1	.	.	0	.
1 and 0	.	0	.	.
1 and 1	.	.	.	0



Compatible?

Bell model M



$P_{X|S\Lambda}$
$P_{Y|T\Lambda}$
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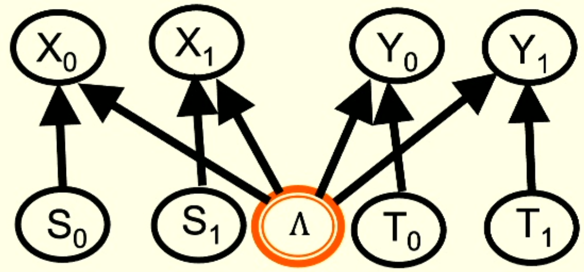
$$P_{XY|ST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda}$$

$P_{XY|ST}$
compatible with M

Inflated model M'



$P_{X_0|S_0\Lambda}$
$P_{X_1|S_1\Lambda}$
$P_{Y_0|T_0\Lambda}$
$P_{Y_1|T_1\Lambda}$
P_{Λ}

$P_{X_0|S_0\Lambda} = P_{X_1|S_1\Lambda}$
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$$P_{X_0Y_0|S_0T_0} = \sum_{\Lambda} P_{X_0|S_0\Lambda} P_{Y_0|T_0\Lambda} P_{\Lambda}$$
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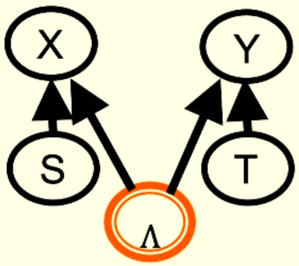
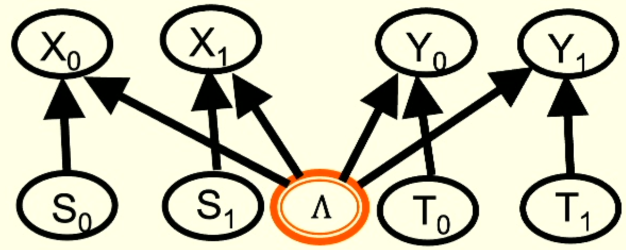
$P_{X_0Y_0|S_0T_0}, P_{X_0Y_1|S_0T_1}, P_{X_1Y_0|S_1T_0}, P_{X_1Y_1|S_1T_1}$
where $P_{X_iY_j|S_iT_j} = P_{XY|ST}$
compatible with M'

Bell model M



Inflated model M'



Take $Q_{X_0X_1Y_0Y_1}(\cdot) := P_{X_0X_1Y_0Y_1|S_0S_1T_0T_1}(\cdot|0101)$

$$P_{XY|ST}(\cdot|00) = a_{00}[00] + a_{01}[01] + a_{10}[10] + a_{11}[11]$$
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$$P_{XY|ST}(\cdot|11) = d_{00}[00] + d_{01}[01] + d_{10}[10]$$

where $a_{11} > 0$

compatible with M

$$Q_{X_0Y_0} = a_{00}[00] + a_{01}[01] + a_{10}[10] + a_{11}[11]$$
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where $a_{11} > 0$

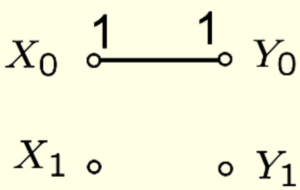
compatible with M'

Suppose there is a distribution $Q_{X_0 X_1 Y_0 Y_1}$
with marginals

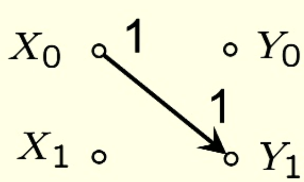
$$Q_{X_0 Y_0} = a_{00}[00] + a_{01}[01] + a_{10}[10] + a_{11}[11]$$
$$Q_{X_0 Y_1} = b_{00}[00] + b_{01}[01] + b_{11}[11]$$
$$Q_{X_1 Y_0} = c_{00}[00] + c_{10}[10] + c_{11}[11]$$
$$Q_{X_1 Y_1} = d_{00}[00] + d_{01}[01] + d_{10}[10]$$

where $a_{11} > 0$

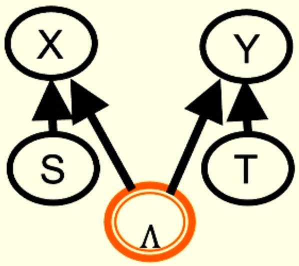
Sometimes



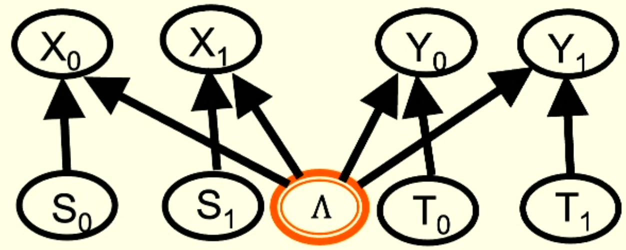
Always



Bell model M



Inflated model M'



Take $Q_{X_0X_1Y_0Y_1}(\cdot) := P_{X_0X_1Y_0Y_1|S_0S_1T_0T_1}(\cdot|0101)$

$$P_{XY|ST}(\cdot|00) = a_{00}[00] + a_{01}[01] + a_{10}[10] + a_{11}[11]$$
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where $a_{11} > 0$

incompatible with M

⇐

$$Q_{X_0Y_0} = a_{00}[00] + a_{01}[01] + a_{10}[10] + a_{11}[11]$$
$$Q_{X_0Y_1} = b_{00}[00] + b_{01}[01] + b_{11}[11]$$
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incompatible with M'

Suppose there is a distribution $Q_{X_0X_1Y_0Y_1}$ with marginals

$$Q_{X_0Y_0} = a_{00}[00] + a_{01}[01] + a_{10}[10] + a_{11}[11]$$
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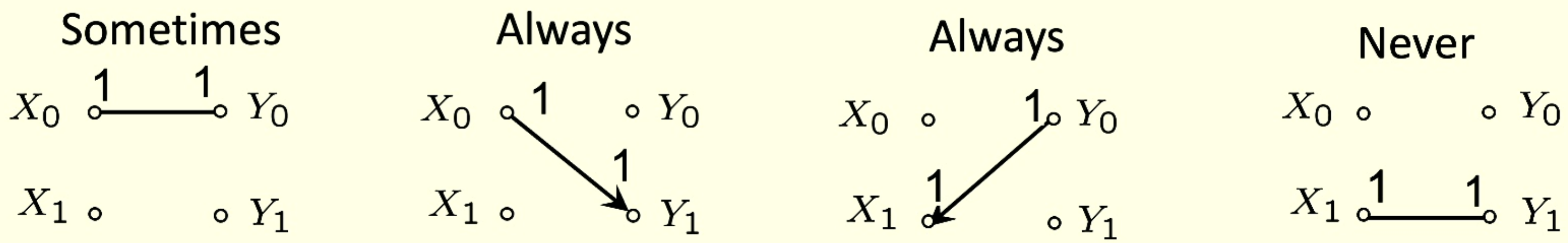
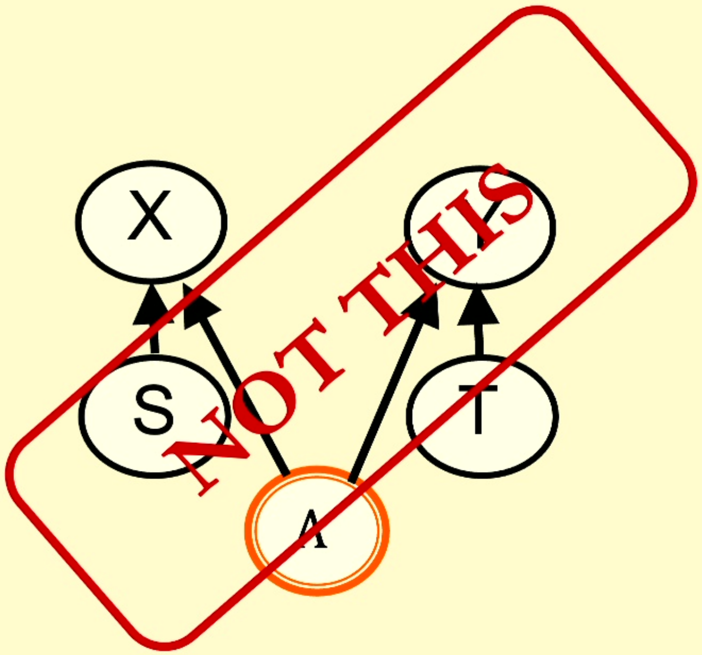


Hardy-type correlations

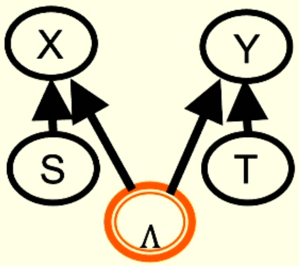
(X,Y)

(S,T)	0 and 0	0 and 1	1 and 0	1 and 1
0 and 0	.	.	.	>0
0 and 1	.	.	0	.
1 and 0	.	0	.	.
1 and 1	.	.	.	0

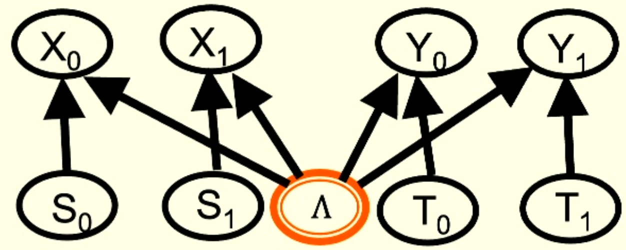


Hardy, PRL 71, 1665 (1993)

Bell model M



Inflated model M'



Take $Q_{X_0 X_1 Y_0 Y_1}(\cdot) := P_{X_0 X_1 Y_0 Y_1 | S_0 S_1 T_0 T_1}(\cdot|0101)$

$$P_{XY|ST}(\cdot|00) = a_{00}[00] + a_{01}[01] + a_{10}[10] + a_{11}[11]$$
$$P_{XY|ST}(\cdot|01) = b_{00}[00] + b_{01}[01] + b_{11}[11]$$
$$P_{XY|ST}(\cdot|10) = c_{00}[00] + c_{10}[10] + c_{11}[11]$$
$$P_{XY|ST}(\cdot|11) = d_{00}[00] + d_{01}[01] + d_{10}[10]$$

where $a_{11} > 0$

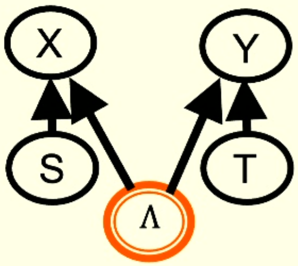
incompatible with M

$$Q_{X_0 Y_0} = a_{00}[00] + a_{01}[01] + a_{10}[10] + a_{11}[11]$$
$$Q_{X_0 Y_1} = b_{00}[00] + b_{01}[01] + b_{11}[11]$$
$$Q_{X_1 Y_0} = c_{00}[00] + c_{10}[10] + c_{11}[11]$$
$$Q_{X_1 Y_1} = d_{00}[00] + d_{01}[01] + d_{10}[10]$$

where $a_{11} > 0$

incompatible with M'

Bell model M



$P_{X|S\Lambda}$
$P_{Y|T\Lambda}$
P_{Λ}

$P_{X|ST} := \sum_Y P_{XY|ST}$
$P_{Y|ST} := \sum_X P_{XY|ST}$

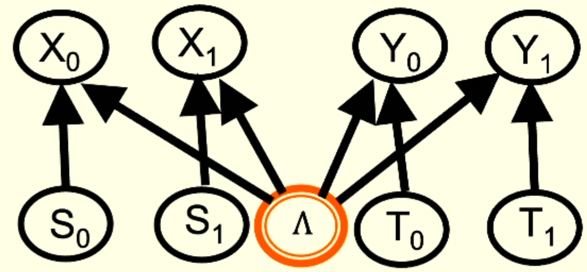
causal structure implies

$P_{X|ST} = P_{X|S}$
$P_{Y|ST} = P_{Y|T}$

$(P_{XY|ST}, P_{X|S}, P_{Y|T})$

compatible with M

Inflated model M'



$P_{X_0|S_0\Lambda}$
$P_{X_1|S_1\Lambda}$
$P_{Y_0|T_0\Lambda}$
$P_{Y_1|T_1\Lambda}$
P_{Λ}

$P_{X_0|S_0\Lambda} = P_{X_1|S_1\Lambda}$
$P_{Y_0|T_0\Lambda} = P_{Y_1|T_1\Lambda}$

$P_{X_i|S_iT_j} := \sum_{Y_j} P_{X_iY_j|S_iT_j}$
$P_{Y_j|S_iT_j} := \sum_{X_i} P_{X_iY_j|S_iT_j}$

causal structure implies

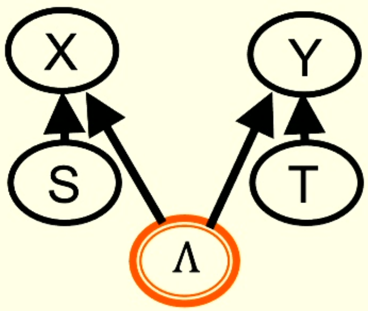
$P_{X_i|S_iT_j} = P_{X_i|S_i}$
$P_{Y_j|S_iT_j} = P_{Y_j|T_j}$

$(P_{X_iY_j|S_iT_j}, P_{X_i|S_i}, P_{Y_j|T_j})\ \forall i, j$

where $P_{X_iY_j|S_iT_j} = P_{XY|ST}$
$P_{X_i|S_i} = P_{X|S}$
$P_{Y_j|T_j} = P_{Y|T}$

compatible with M'

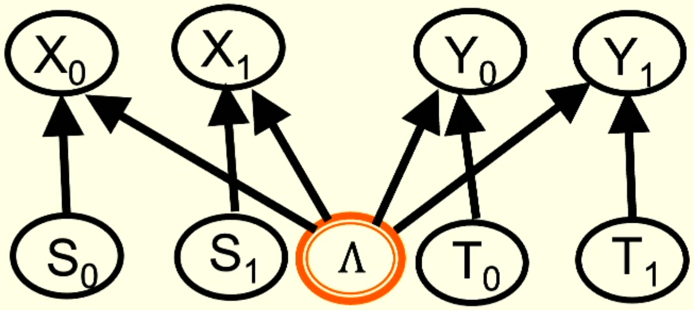
Bell model M



$I_{st}(X:Y)$ defined by $P_{XY|ST}(\cdot|st)$
$H_s(X)$ defined by $P_{X|S}(\cdot|s)$
$H_t(Y)$ defined by $P_{Y|T}(\cdot|t)$

Entropic
causal compatibility inequality in M

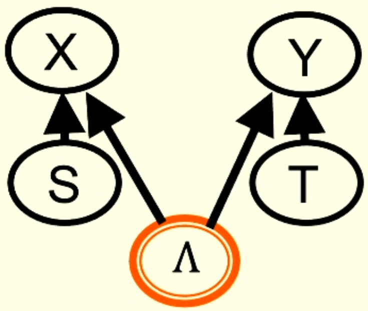
Inflated model M'



$I_{st}(X_i:Y_j)$ defined by $P_{X_iY_j|S_iT_j}(\cdot|st)$
$H_s(X_i)$ defined by $P_{X_i|S_i}(\cdot|s)$
$H_t(Y_i)$ defined by $P_{Y_i|T_i}(\cdot|t)$

Entropic
causal compatibility inequality in M'

Bell model M

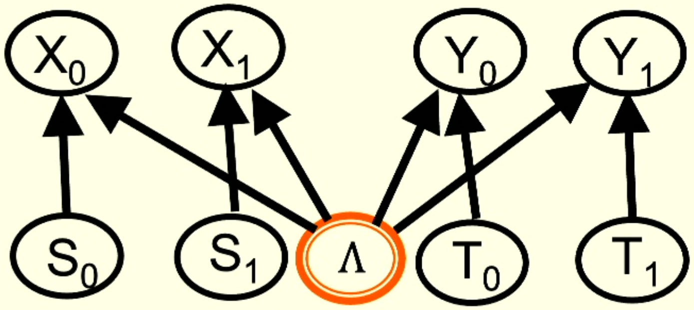


$I_{st}(X:Y)$ defined by $P_{XY|ST}(\cdot|st)$
$H_s(X)$ defined by $P_{X|S}(\cdot|s)$
$H_t(Y)$ defined by $P_{Y|T}(\cdot|t)$

$$I_{00}(X:Y)+I_{01}(X:Y)+I_{10}(X:Y)-I_{11}(X:Y) \leq H_0(X)+H_0(Y)$$

Causal compatibility inequality in M

Inflated model M'



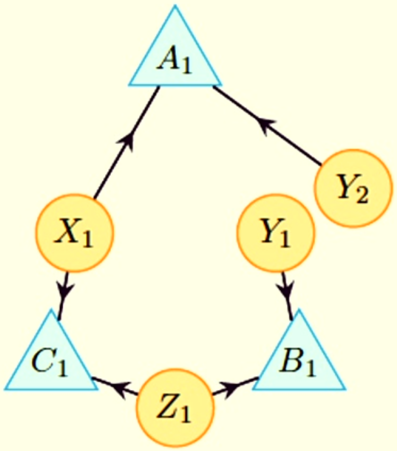
$I_{st}(X_i:Y_j)$ defined by $P_{X_iY_j|S_iT_j}(\cdot|st)$
$H_s(X_i)$ defined by $P_{X_i|S_i}(\cdot|s)$
$H_t(Y_i)$ defined by $P_{Y_i|T_i}(\cdot|t)$

$$I_{00}(X_0:Y_0)+I_{01}(X_0:Y_1)+I_{10}(X_1:Y_0)-I_{11}(X_1:Y_1) \leq H_0(X_0)+H_0(Y_0)$$

Causal compatibility inequality in M'

Recall example of entropic inequality for the triangle scenario

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is a valid set of marginals $\implies$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy $I(A_1 : C_1) + I(C_1 : B_1) - I(A_1 : B_1) \leq H(C_1)$



$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M'

$\implies A_1 \perp B_1 \implies I(A_1 : B_1) = 0$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M' $\implies$ $I(A_1 : C_1) + I(C_1 : B_1) \leq H(C_1)$

$I(A_1 : C_1) + I(C_1 : B_1) \leq H(C_1)$ is a causal compatibility inequality for M'

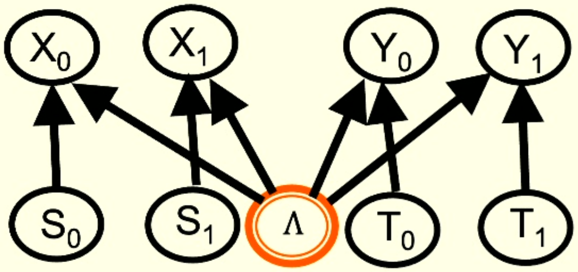
Consider the joint distribution

$$P_{X_0 X_1 Y_0 Y_1 S_0 S_1 T_0 T_1}$$

There are marginal constraints on:

$$P_{X_0 Y_0|S_0 T_0}, P_{X_0 Y_1|S_0 T_1}, P_{X_1 Y_0|S_1 T_0}, P_{X_1 Y_1|S_1 T_1},$$
$$P_{X_0|S_0}, P_{X_1|S_1}, P_{Y_0|T_0}, P_{Y_1|T_1}$$

Inflated model M'



Focussing on the entropies

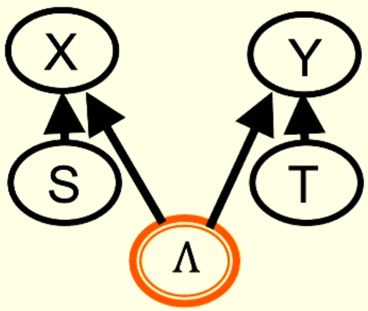
$$(H_{00}(X_0Y_0), H_{01}(X_0Y_1), H_{10}(X_1Y_0), H_{11}(X_1Y_1))$$

and making use of conditional independences implied by the causal structure

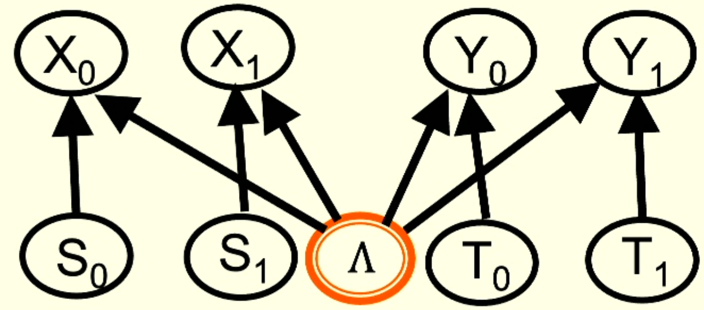
yields the inequality

$$I_{00}(X_0 : Y_0) + I_{01}(X_0 : Y_1) + I_{10}(X_1 : Y_0) - I_{11}(X_1 : Y_1)$$
$$\leq H_0(X_0) + H_0(Y_0)$$

Bell model M



Inflated model M'



$I_{st}(X:Y)$ defined by $P_{XY|ST}(\cdot|st)$
$H_s(X)$ defined by $P_{X|S}(\cdot|s)$
$H_t(Y)$ defined by $P_{Y|T}(\cdot|t)$

$$I_{00}(X:Y) + I_{01}(X:Y) + I_{10}(X:Y) - I_{11}(X:Y) \leq H_0(X) + H_0(Y)$$

Causal compatibility inequality in M

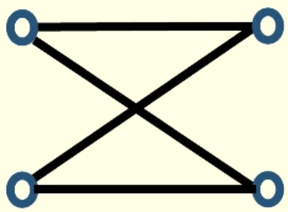
$I_{st}(X_i:Y_j)$ defined by $P_{X_iY_j|S_iT_j}(\cdot|st)$
$H_s(X_i)$ defined by $P_{X_i|S_i}(\cdot|s)$
$H_t(Y_i)$ defined by $P_{Y_i|T_i}(\cdot|t)$

$$I_{00}(X_0:Y_0) + I_{01}(X_0:Y_1) + I_{10}(X_1:Y_0) - I_{11}(X_1:Y_1) \leq H_0(X_0) + H_0(Y_0)$$

Causal compatibility inequality in M'

Braunstein & Caves. *Phys. Rev. Lett.* **61**, 662–665 (1988)

Local correlations



$$P_{XY|ST}(\cdot|00) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|01) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|10) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|11) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$

These are consistent with marginals of a single dist'n

$$P_{X_0X_1Y_0Y_1|S_0S_1T_0T_1}(\cdot|0101) = [0000] + [1111]$$

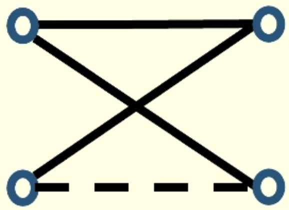
so must satisfy all the CHSH inequalities

$$I_{00}(X:Y) + I_{01}(X:Y) + I_{10}(X:Y) - I_{11}(X:Y) \leq H_0(X) + H_0(Y)$$

$\text{RHS} = 1 + 1 + 1 - 1 = 2$ $\qquad$ $\text{LHS} = 1 + 1 = 2$

Entropic inequality is satisfied

PR-box correlations



$$P_{XY|ST}(\cdot|00) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|01) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|10) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|11) = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$

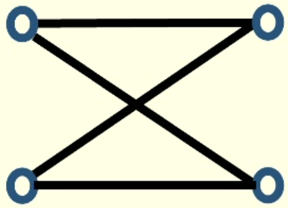
These marginals are incompatible with Bell model (known to violate CHSH inequalities maximally)

$$I_{00}(X:Y) + I_{01}(X:Y) + I_{10}(X:Y) - I_{11}(X:Y) \le H_0(X) + H_0(Y)$$

$\text{RHS} = 1 + 1 + 1 - 1 = 2$ $\qquad$ $\text{LHS} = 1 + 1 = 2$

Entropic inequality is still satisfied

Local correlations



$$P_{XY|ST}(\cdot|00) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|01) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|10) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|11) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$

These are consistent with marginals of a single dist'n

$$P_{X_0X_1Y_0Y_1|S_0S_1T_0T_1}(\cdot|0101) = [0000] + [1111]$$

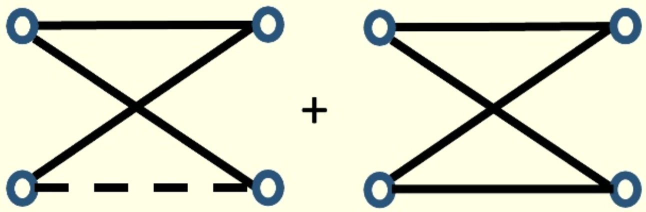
so must satisfy all the CHSH inequalities

$$I_{00}(X:Y) + I_{01}(X:Y) + I_{10}(X:Y) - I_{11}(X:Y) \leq H_0(X) + H_0(Y)$$

$\text{RHS} = 1 + 1 + 1 - 1 = 2$ $\qquad$ $\text{LHS} = 1 + 1 = 2$

Entropic inequality is satisfied

Mixture



$$P_{XY|ST}(\cdot|00) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|01) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|10) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|11) = (\tfrac{1}{2}[0] + \tfrac{1}{2}[1])(\tfrac{1}{2}[0] + \tfrac{1}{2}[1])$$

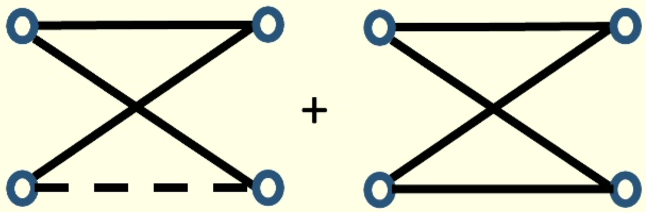
These marginals are incompatible

$$I_{00}(X:Y) + I_{01}(X:Y) + I_{10}(X:Y) - I_{11}(X:Y) \leq H_0(X) + H_0(Y)$$

$\mathrm{RHS} = 1 + 1 + 1 - 0 = 3$ $\quad\quad$ $\mathrm{LHS} = 1 + 1 = 2$

Entropic inequality is violated

Mixture



$$P_{XY|ST}(\cdot|00) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|01) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|10) = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XY|ST}(\cdot|11) = (\tfrac{1}{2}[0] + \tfrac{1}{2}[1])(\tfrac{1}{2}[0] + \tfrac{1}{2}[1])$$

These marginals are incompatible

$$I_{00}(X:Y) + I_{01}(X:Y) + I_{10}(X:Y) - I_{11}(X:Y)$$
$$\leq H_0(X) + H_0(Y)$$

$\text{RHS} = 1 + 1 + 1 - 0 = 3$ $\qquad$ $\text{LHS} = 1 + 1 = 2$

Entropic inequality is violated

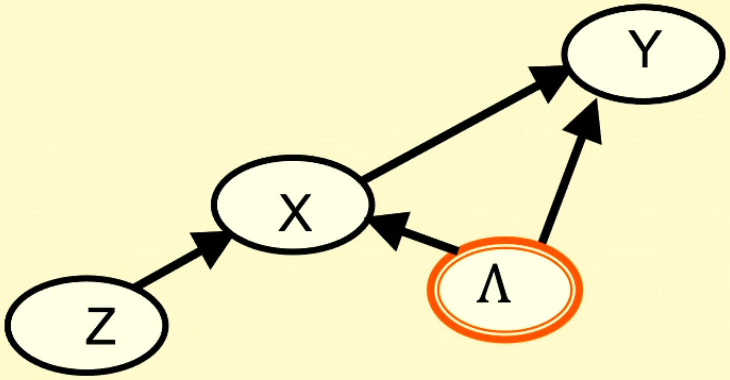
Chaves & Fritz *Phys. Rev. A* **85**, 032113 (2012)

Approaches to Bell arguments that follow essentially
the logic of the inflation technique:

Fine's proof of Bell inequalities
Hardy's proof of Bell's theorem
The Greenberger-Horne-Zeilinger proof of Bell's theorem
Bell's theorem by way of Kochen-Specker noncontextuality
Braunstein-Caves entropic inequalities
Symmetric extensions / shareability of local correlations
Etcetera

Deriving inequality constraints for the instrumental scenario

Instrumental model



$P_{X|\Lambda Z}$

$P_{Y|\Lambda X}$

P_{Λ}

$$P_{XY|Z} = \sum_{\Lambda} P_{Y|X\Lambda} P_{X|Z\Lambda} P_{\Lambda}$$

|Z|=|X|=|Y|=2

$$P_{XY|Z}(00|0) + P_{XY|Z}(01|1) \leq 1$$
$$P_{XY|Z}(10|0) + P_{XY|Z}(11|1) \leq 1$$
$$P_{XY|Z}(00|1) + P_{XY|Z}(01|0) \leq 1$$
$$P_{XY|Z}(10|1) + P_{XY|Z}(11|0) \leq 1$$

These can be derived by brute-force quantifier elimination on the distribution over the latent variable.

It suffices to note that the instrumental DAG is gearable

And the fact that there is only a single latent variable means that the quantifier elimination problem is linear

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It suffices to note that the instrumental DAG is gearable

And the fact that there is only a single latent variable means that the quantifier elimination problem is linear

$$P_{XY|Z} = \sum_{f,g} \delta_{Y|g(X)} \delta_{X|f(Z)} P_{FG}(fg)$$

If X,Y,Z are binary, Λ can have cardinality 16

$p_{xy|z} := P_{XY|Z}(xy|z)$

$x, y, z \in \{0, 1\}$

$q_{fg} := P_{F,G}(f, g)$

$f, g \in \{\mathrm{id}, \mathrm{fp}, \mathrm{r}_0, \mathrm{r}_1\}$

$$p_{00|0} = q_{\mathrm{r}_0,\mathrm{r}_0} + q_{\mathrm{r}_0,\mathrm{id}} + q_{\mathrm{id},\mathrm{r}_0} + q_{\mathrm{id},\mathrm{id}}$$

$$p_{01|0} = q_{\mathrm{r}_0,\mathrm{r}_1} + q_{\mathrm{r}_0,\mathrm{fp}} + q_{\mathrm{id},\mathrm{r}_1} + q_{\mathrm{id},\mathrm{fp}}$$

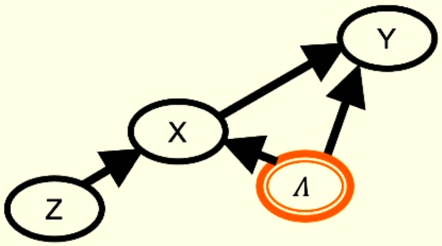
$$\vdots$$

$$0 \leq q_{fg} \leq 1 \; \forall f, g$$

16 linear equalities + inequalities

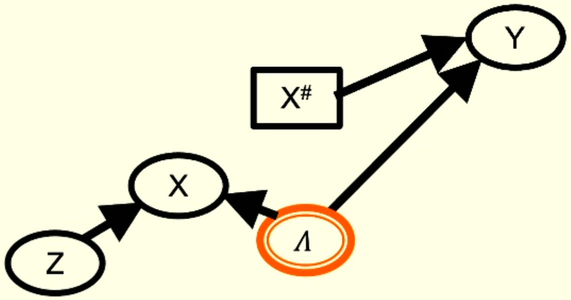
Do linear quantifier elimination on the 16 q's.

Instrumental model M



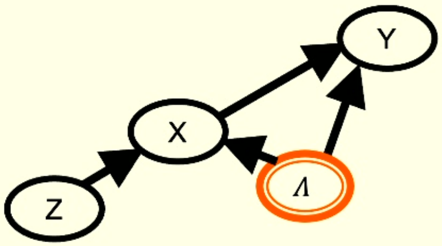
$P_{X|\Lambda Z}$
$P_{Y|\Lambda X}$
P_{Λ}

Interrupted version M'



$P_{X|\Lambda Z}$
$X^{\#} = x$
$P_{Y|\Lambda X^{\#}}$
P_{Λ}

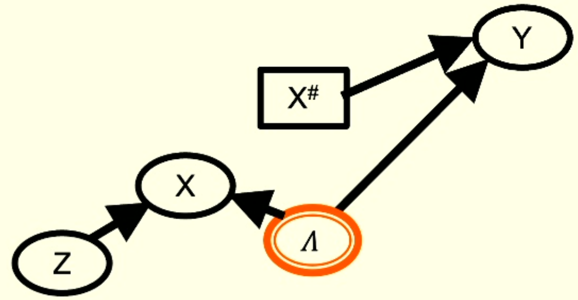
Instrumental model M



$$P_{X|\Lambda Z}$$
$$P_{Y|\Lambda X}$$
$$P_{\Lambda}$$

$$P_{XY|Z}(x \cdot |\cdot) = \sum_{\Lambda} P_{Y|X\Lambda}(\cdot|x\cdot) P_{X|Z\Lambda}(x|\cdot\cdot) P_{\Lambda}(\cdot)$$

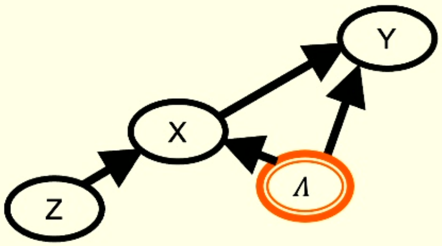
Interrupted version M'



$$P_{X|\Lambda Z}$$
$$X^{\#} = x$$
$$P_{Y|\Lambda X^{\#}}$$
$$P_{\Lambda}$$

$$P_{XY|ZX^{\#}}(x \cdot | \cdot x) = \sum_{\Lambda} P_{Y|X^{\#}\Lambda}(\cdot|x\cdot) P_{X|Z\Lambda}(x|\cdot\cdot) P_{\Lambda}(\cdot)$$

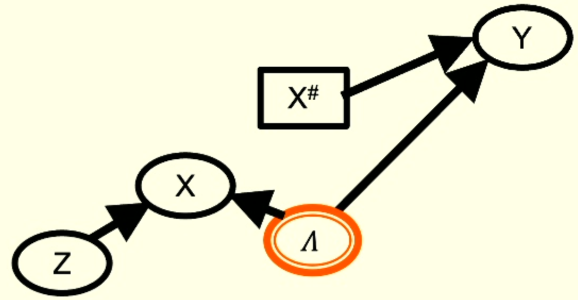
Instrumental model M



$P_{X|\Lambda Z}$
$P_{Y|\Lambda X}$
P_{Λ}

$$P_{XY|Z}(x \cdot |\cdot) = \sum_{\Lambda} P_{Y|X\Lambda}(\cdot|x\cdot) P_{X|Z\Lambda}(x|\cdot\cdot) P_{\Lambda}(\cdot)$$

Interrupted version M’

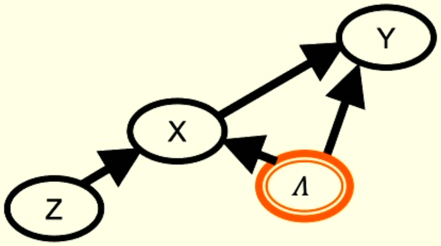


$P_{X|\Lambda Z}$
$X^{\#} = x$
$P_{Y|\Lambda X^{\#}}$
P_{Λ}

$$P_{XY|ZX^{\#}}(x \cdot | \cdot x) = \sum_{\Lambda} P_{Y|X^{\#}\Lambda}(\cdot|x\cdot) P_{X|Z\Lambda}(x|\cdot\cdot) P_{\Lambda}(\cdot)$$

$P_{XY|Z}$ is compatible with M ⇒ some $P_{XY|ZX^{\#}}$ where $P_{XY|ZX^{\#}}(x \cdot | \cdot x) = P_{XY|Z}(x \cdot |\cdot)$ is compatible with M'

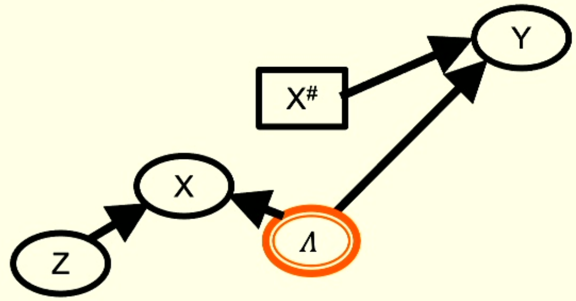
Instrumental model M



$P_{XY|Z}$

is compatible with M

Interrupted version M'

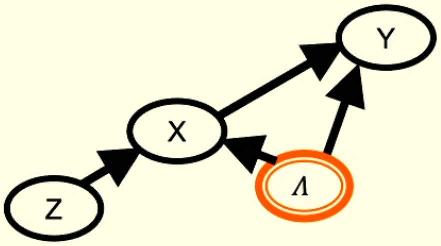


some $P_{XY|ZX\#}$ where

$$P_{XY|ZX\#}(xy|zx) = P_{XY|Z}(xy|z) \; \forall x, y, z$$

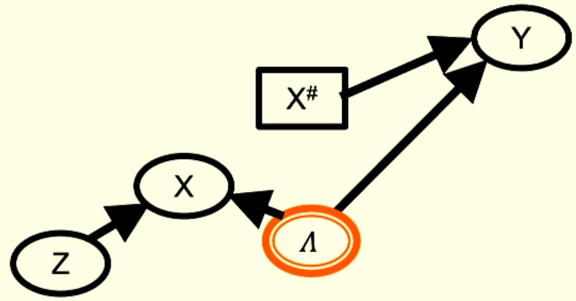
is compatible with M'

Instrumental model M



causal compatibility inequality on
$\{P_{XY|Z}(xy|z)\}_{x,y,z}$
in model M

Interrupted version M'



causal compatibility inequality on
$\{P_{XY|ZX\#}(xy|zx)\}_{x,y,z}$
in model M'

To obtain

causal compatibility inequality on

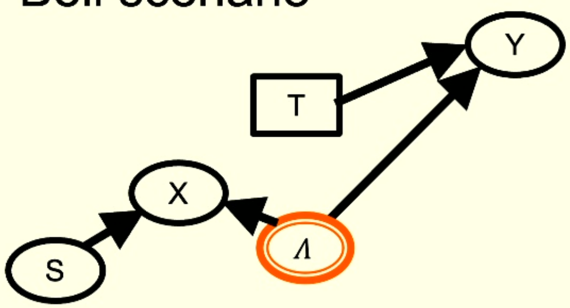
$$\{P_{XY|ZX^{\#}}(xy|zx)\}_{x,y,z}$$

in model M'

Eliminate parameters

$P_{XY|ZX^{\#}}(xy|zx')$ where $x \neq x'$

Bell scenario



$$Y \perp_d S|T \implies P_{Y|ST} = P_{Y|T}$$

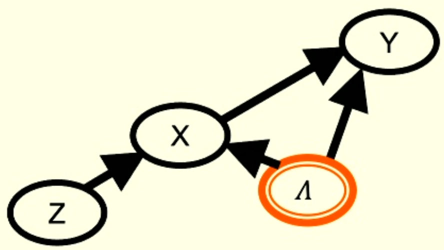
$$P_{XY|ST}(00|00) + P_{XY|ST}(10|00) = P_{Y|T}(0|0)$$
$$P_{XY|ST}(01|00) + P_{XY|ST}(11|00) = P_{Y|T}(1|0)$$
$$P_{XY|ST}(00|10) + P_{XY|ST}(10|10) = P_{Y|T}(0|0)$$
$$P_{XY|ST}(01|10) + P_{XY|ST}(11|10) = P_{Y|T}(1|0)$$

$$P_{XY|ST}(11|10) \geq 0 \implies P_{Y|T}(1|0) - P_{XY|ST}(01|10) \geq 0$$
$$P_{XY|ST}(10|00) \geq 0 \implies P_{Y|T}(0|0) - P_{XY|ST}(00|00) \geq 0$$

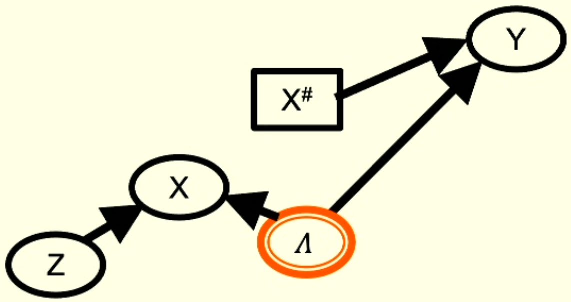
$$\implies P_{Y|T}(0|0) - P_{XY|ST}(00|00) + P_{Y|T}(1|0) - P_{XY|ST}(01|10) \geq 0$$

$$\implies P_{XY|ST}(00|00) + P_{XY|ST}(01|10) \leq 1$$

Instrumental model M



Interrupted version M'



$$P_{XY|Z}(00|0) + P_{XY|Z}(01|1) \leq 1$$

is a causal compatibility inequality in model M

$$P_{XY|ZX^{\#}}(00|00) + P_{XY|ZX^{\#}}(01|10) \leq 1$$

is a causal compatibility inequality in model M'

These can be derived by brute-force quantifier elimination on the distribution over the latent variable.

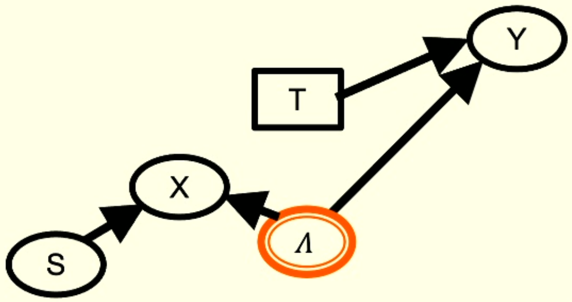
It suffices to note that the instrumental DAG is gearable

And the fact that there is only a single latent variable means that the quantifier elimination problem is linear

$$P_{XY|Z} = \sum_{f,g} \delta_{Y|g(X)} \delta_{X|f(Z)} P_{FG}(fg)$$

If |Z|=3, |X|=|Y|=2, Λ can have cardinality $|X|^{|Z|}|Y|^{|X|} = 2^3 2^2 = 32$

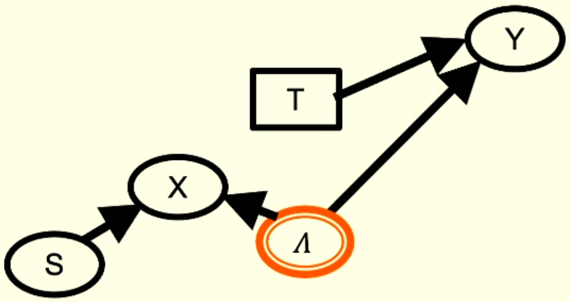
Bell scenario $S \in \{0,1,2\}$



A facet inequality has the form:

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \; - \frac{1}{2} P_{XY|ST}(11|20) \leq \frac{3}{4}$$

Bell scenario $S \in \{0, 1, 2\}$



A facet inequality has the form:

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \; -\frac{1}{2}P_{XY|ST}(11|20) \leq \frac{3}{4}$$

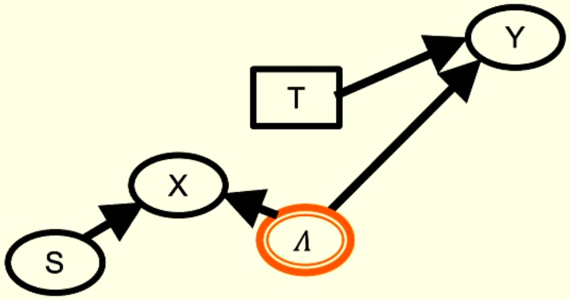
⇨

$$P_{XY|ST}(00|00) + P_{XY|ST}(11|01)$$
$$+P_{XY|ST}(00|10) + P_{XY|ST}(10|11)$$
$$+P_{XY|ST}(01|20) \leq 1$$

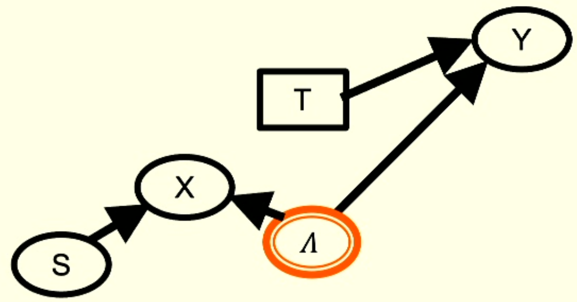
Significance:

Quantum violations of CHSH inequalities in the Bell scenario imply quantum violations of the Bonet inequality in the instrumental scenario

Bell scenario $S \in \{0, 1, 2\}$



Bell scenario $S \in \{0, 1\}$



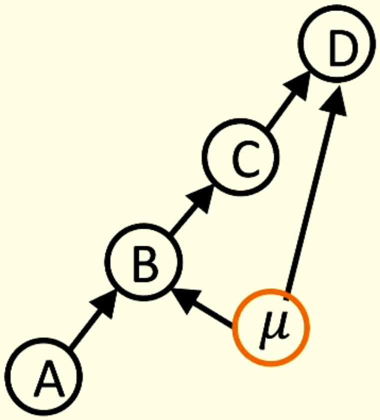
$$
\begin{aligned}
P'_{XY|ST} &= P_{XY|ST} \qquad S \in \{0,1\} \\
&= \delta_{X,0} P_{Y|T} \quad S = 2
\end{aligned}
$$

compatible

$P_{XY|ST}$

compatible

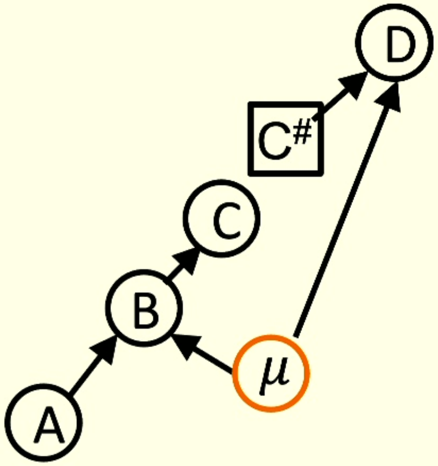
Verma model M



$$\mathrm{P}_{D|\mu C}$$
$$\mathrm{P}_{C|B}$$
$$\mathrm{P}_{B|A\mu}$$
$$\mathrm{P}_{A}$$
$$\mathrm{P}_{\mu}$$

$$P_{ABCD} = \left(\sum_{\mu} \mathrm{P}_{D|\mu C}\mathrm{P}_{B|A\mu}\mathrm{P}_{\mu}\right)\mathrm{P}_{C|B}\mathrm{P}_{A}$$

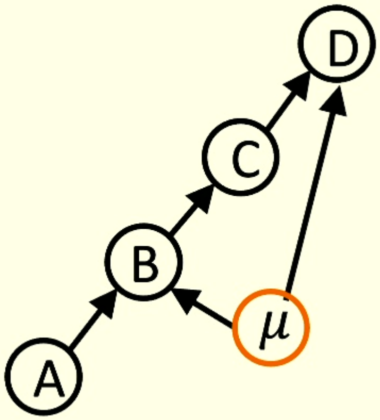
Interrupted version M'



$$\mathrm{P}_{D|\mu C\#}$$
$$C^{\#} = c$$
$$\mathrm{P}_{C|B}$$
$$\mathrm{P}_{B|A\mu}$$
$$\mathrm{P}_{A}$$
$$\mathrm{P}_{\mu}$$

$$\mathrm{P}_{BD|AC\#} = \sum_{\mu} \mathrm{P}_{D|\mu C\#}\mathrm{P}_{B|A\mu}\mathrm{P}_{\mu}$$

Verma model M



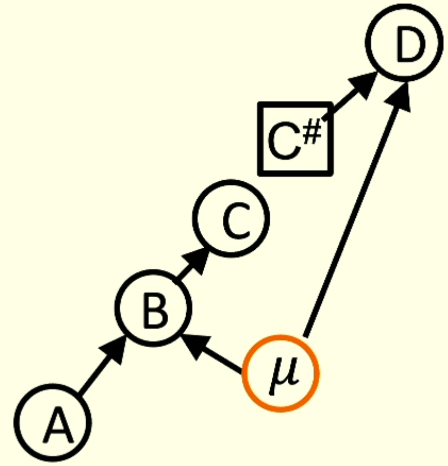
$$\mathrm{P}_{D|\mu C}$$
$$\mathrm{P}_{C|B}$$
$$\mathrm{P}_{B|A\mu}$$
$$\mathrm{P}_{A}$$
$$\mathrm{P}_{\mu}$$

$$P_{ABCD} = \left(\underbrace{\sum_{\mu} \mathrm{P}_{D|\mu C} \mathrm{P}_{B|A\mu} \mathrm{P}_{\mu}}_{= Q_{BD|AC}} \right) \mathrm{P}_{C|B} \mathrm{P}_{A}$$

$$Q_{BD|AC} = \frac{\mathrm{P}_{ABCD}}{\mathrm{P}_{C|B}\mathrm{P}_{A}}$$

P_{ABCD}
is compatible with M

Interrupted version M'



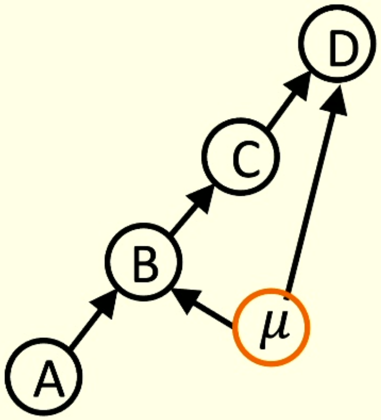
$$\mathrm{P}_{D|\mu C\#}$$
$$C^{\#} = c$$
$$\mathrm{P}_{C|B}$$
$$\mathrm{P}_{B|A\mu}$$
$$\mathrm{P}_{A}$$
$$\mathrm{P}_{\mu}$$

$$\mathrm{P}_{BD|AC\#} = \sum_{\mu} \mathrm{P}_{D|\mu C\#} \mathrm{P}_{B|A\mu} \mathrm{P}_{\mu}$$

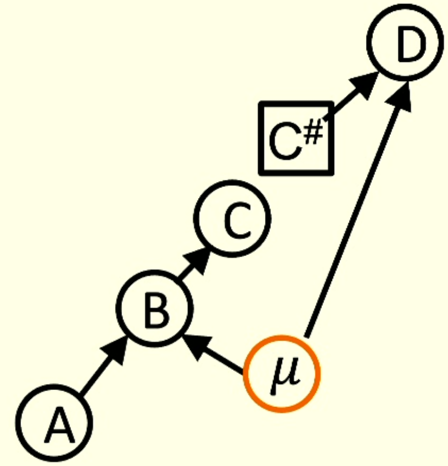
$$\mathrm{P}_{BD|AC\#} = \frac{\mathrm{P}_{ABCD}}{\mathrm{P}_{C|B}\mathrm{P}_{A}}$$

is compatible with M'

Verma model M



Interrupted version M'



$$Q_{BD|AC} = \frac{\mathrm{P}_{ABCD}}{\mathrm{P}_{C|B}\mathrm{P}_A}$$

$$\sum_d Q_{BD|AC}(bd|ac) = \sum_d Q_{BD|AC}(bd|ac') \ \forall c, c'$$

$$\sum_b Q_{BD|AC}(bd|ac) = \sum_b Q_{BD|AC}(bd|ac') \ \forall c, c'$$

in model M

Verma constraint

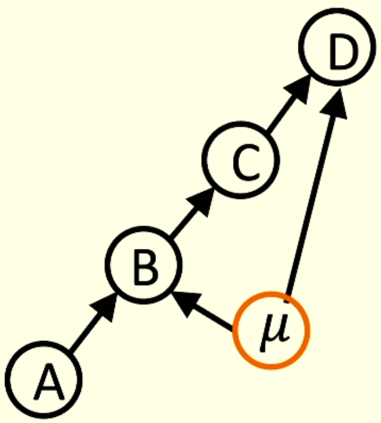
⇐

$$\sum_d P_{BD|AC\#}(bd|ac) = \sum_d P_{BD|AC\#}(bd|ac') \ \forall c, c'$$

$$\sum_b P_{BD|AC\#}(bd|ac) = \sum_b P_{BD|AC\#}(bd|ac') \ \forall c, c'$$

in model M'

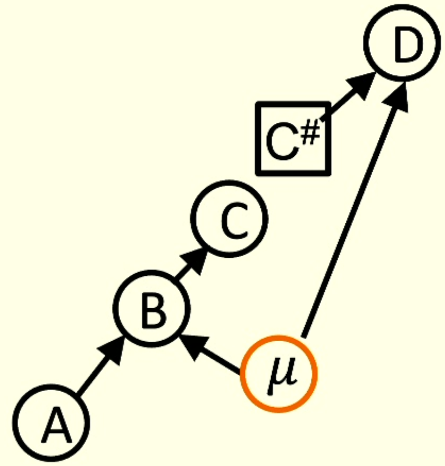
Verma model M



$$\frac{\mathrm{P}_{ABCD}}{\mathrm{P}_{C|B}\mathrm{P}_{A}}(bdac)$$

satisfies Bell inequalities in model M

Interrupted version M’

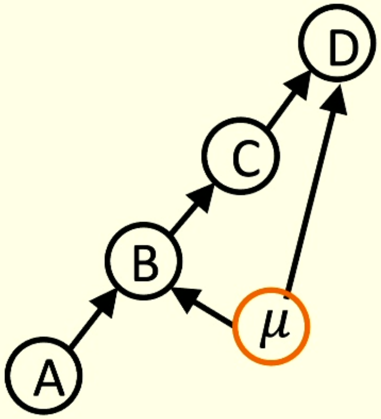


$$P_{BD|AC\#}(bd|ac)$$

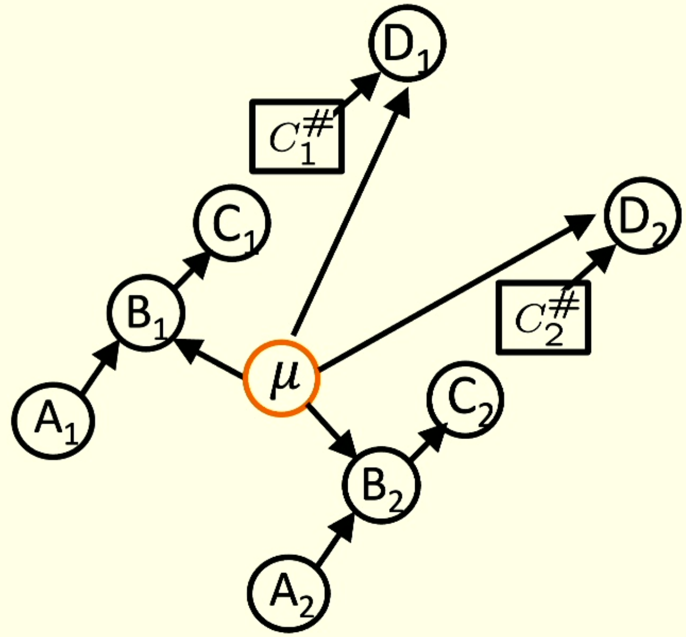
satisfies Bell inequalities in model M'

Combining interruption and inflation

Verma model M



Interrupted and inflated version M'



$$\frac{\mathrm{P}_{ABCD}}{\mathrm{P}_{C|B}\mathrm{P}_{A}}(bdac)$$

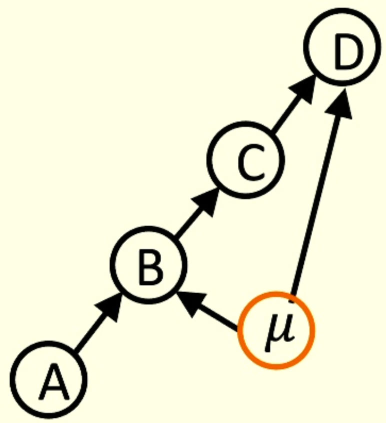
satisfies Bell inequalities in model M

⇐

equality constraints in model M'

Combining interruption and inflation

Verma model M

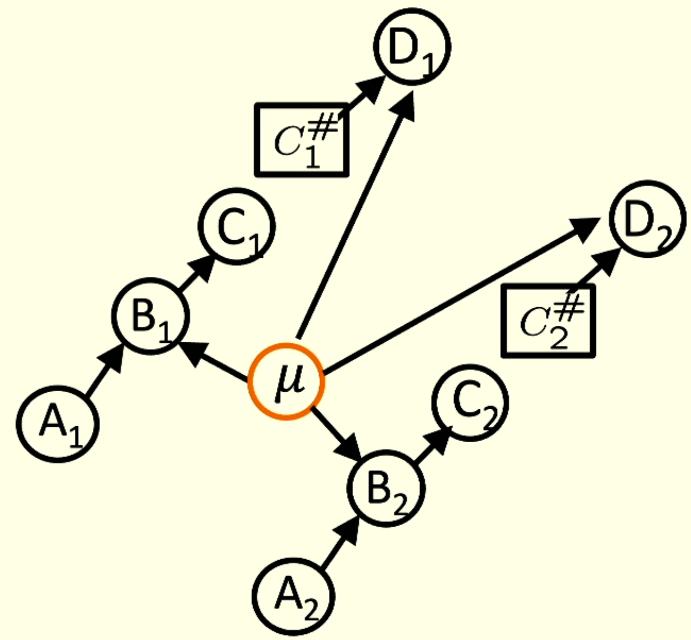


$$\frac{\mathrm{P}_{ABCD}}{\mathrm{P}_{C|B}\mathrm{P}_{A}}(bdac)$$

satisfies Bell inequalities in model M

⇐

Interrupted and inflated version M'



equality constraints in model M'