Title: Causal Inference Lecture - 230322

Speakers: Robert Spekkens

Collection: Causal Inference: Classical and Quantum

Date: March 22, 2023 - 10:00 AM

URL: https://pirsa.org/23030074

Abstract: zoom link: https://pitp.zoom.us/j/94143784665?pwd=VFJpajVIMEtvYmRabFYzYnNRSVAvZz09

The inflation technique for causal inference

The marginal problem

Example 1:

$$P_{XY}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$

Question: $\exists P_{XYZ}$

such that
$$P_{XY} = P_{XY}^{\text{target}}$$
$$P_{YZ} = P_{YZ}^{\text{target}}$$
$$P_{XZ} = P_{XZ}^{\text{target}}$$

where
$$P_{XY} := \sum_Z P_{XYZ}$$
$$P_{YZ} := \sum_X P_{XYZ}$$
$$P_{XZ} := \sum_Y P_{XYZ}$$

?

Example 1:

$$P_{XY}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$

Question: $\exists P_{XYZ}$

such that

$$P_{XY} = P_{XY}^{\text{target}}$$
$$P_{YZ} = P_{YZ}^{\text{target}}$$
$$P_{XZ} = P_{XZ}^{\text{target}}$$

where

$$P_{XY} := \textstyle\sum_Z P_{XYZ}$$
$$P_{YZ} := \textstyle\sum_X P_{XYZ}$$
$$P_{XZ} := \textstyle\sum_Y P_{XYZ}$$

?

Answer: yes!

$$P_{XYZ} = \tfrac{1}{2}[001] + \tfrac{1}{2}[110]$$

Example 2:

$$P_{XY}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11] = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$

Question: $\exists P_{XYZ}$

such that
$$P_{XY} = P_{XY}^{\text{target}}$$
$$P_{YZ} = P_{YZ}^{\text{target}}$$
$$P_{XZ} = P_{XZ}^{\text{target}}$$

where
$$P_{XY} := \sum_Z P_{XYZ}$$
$$P_{YZ} := \sum_X P_{XYZ}$$
$$P_{XZ} := \sum_Y P_{XYZ}$$

?

Answer: no!

consider $[000], [001], [010], [011], [100], [101], [110], [111]$

Example 3:

$$P_{XY}^{\text{target}} = (\tfrac{1}{2}[0] + \tfrac{1}{2}[1])(\tfrac{1}{2}[0] + \tfrac{1}{2}[1])$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$

Question: $\exists P_{XYZ}$

such that
$$P_{XY} = P_{XY}^{\text{target}}, \quad P_{YZ} = P_{YZ}^{\text{target}}, \quad P_{XZ} = P_{XZ}^{\text{target}}$$

where
$$P_{XY} := \sum_Z P_{XYZ}, \quad P_{YZ} := \sum_X P_{XYZ}, \quad P_{XZ} := \sum_Y P_{XYZ}$$

?

Example 3:

$$P_{XY}^{\text{target}} = (\tfrac{1}{2}[0] + \tfrac{1}{2}[1])(\tfrac{1}{2}[0] + \tfrac{1}{2}[1])$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$

Question: $\exists P_{XYZ}$

such that
$$P_{XY} = P_{XY}^{\text{target}}, \quad P_{YZ} = P_{YZ}^{\text{target}}, \quad P_{XZ} = P_{XZ}^{\text{target}}$$
where
$$P_{XY} := \sum_Z P_{XYZ}, \quad P_{YZ} := \sum_X P_{XYZ}, \quad P_{XZ} := \sum_Y P_{XYZ}$$
?

Answer: no! $P_{YZ}^{\text{target}} = \frac{1}{2}[00] + \frac{1}{2}[11]$ excludes $[010], [110], [001], [101]$

Example 3:

$$P_{XY}^{\text{target}} = (\tfrac{1}{2}[0] + \tfrac{1}{2}[1])(\tfrac{1}{2}[0] + \tfrac{1}{2}[1])$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$

Question: $\exists P_{XYZ}$

such that

$$P_{XY} = P_{XY}^{\text{target}}$$
$$P_{YZ} = P_{YZ}^{\text{target}}$$
$$P_{XZ} = P_{XZ}^{\text{target}}$$

where

$$P_{XY} := \sum_Z P_{XYZ}$$
$$P_{YZ} := \sum_X P_{XYZ}$$
$$P_{XZ} := \sum_Y P_{XYZ}$$

?

Answer: no!

$P_{YZ}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$ excludes $[010], [110], [001], [101]$

$P_{XZ}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$ excludes $[001], [011], [100], [110]$

Example 3:

$$P_{XY}^{\text{target}} = (\tfrac{1}{2}[0] + \tfrac{1}{2}[1])(\tfrac{1}{2}[0] + \tfrac{1}{2}[1])$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$

Question: $\exists P_{XYZ}$

such that $P_{XY} = P_{XY}^{\text{target}}$, $P_{YZ} = P_{YZ}^{\text{target}}$, $P_{XZ} = P_{XZ}^{\text{target}}$

where $P_{XY} := \sum_Z P_{XYZ}$, $P_{YZ} := \sum_X P_{XYZ}$, $P_{XZ} := \sum_Y P_{XYZ}$?

Answer: no!

$P_{YZ}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$ excludes $[010], [110], [001], [101]$

$P_{XZ}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$ excludes $[001], [011], [100], [110]$

Therefore only $[000], [111]$ X and Y must be correlated

Consider binary X, Y and Z

$$\exists P_{XYZ} \text{ with } \quad P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ} \text{ as marginals}$$

Marginal compatibility constraints

$$P_X = \sum_Y P_{XY} = \sum_Z P_{XZ}$$

$$P_Y = \sum_X P_{XY} = \sum_Z P_{YZ}$$

$$P_Z = \sum_X P_{XZ} = \sum_Y P_{YZ}$$

$$P_X + P_Y + P_Z - P_{XY} - P_{YZ} - P_{XZ} \leq 1$$

How this is proven:

$$\exists Q_{XYZ} \text{ with } \quad P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ} \text{ as marginals}$$

$$\forall x,y : P_{XY}(xy) = \sum_z Q_{XYZ}(xyz)$$
$$\forall y,z : P_{YZ}(yz) = \sum_x Q_{XYZ}(xyz)$$
$$\forall x,z : P_{XZ}(xz) = \sum_y Q_{XYZ}(xyz)$$
$$\forall x : P_X(x) = \sum_{y,z} Q_{XYZ}(xyz)$$
$$\forall y : P_Y(y) = \sum_{x,z} Q_{XYZ}(xyz)$$
$$\forall z : P_Z(z) = \sum_{x,y} Q_{XYZ}(xyz)$$
$$\forall x,y,z : 0 \leq Q_{XYZ}(xyz) \leq 1$$

Linear quantifier elimination

$$\forall x : P_X(x) = \sum_y P_{XY}(xy) = \sum_z P_{XZ}(xz)$$
$$\forall y : P_Y(y) = \sum_x P_{XY}(xy) = \sum_z P_{YZ}(yz)$$
$$\forall z : P_Z(z) = \sum_x P_{XZ}(xz) = \sum_y P_{YZ}(yz)$$
$$\forall x,y,z : P_X(x) + P_Y(y) + P_Z(z) - P_{XY}(xy) - P_{YZ}(yz) - P_{XZ}(xz) \leq 1$$

Example 2 revisited:

$$P_{XY}^{\text{target}} = \frac{1}{2}[01] + \frac{1}{2}[10] \qquad P_X^{\text{target}} = \frac{1}{2}[0] + \frac{1}{2}[1]$$
$$P_{YZ}^{\text{target}} = \frac{1}{2}[01] + \frac{1}{2}[10] \qquad P_Y^{\text{target}} = \frac{1}{2}[0] + \frac{1}{2}[1]$$
$$P_{XZ}^{\text{target}} = \frac{1}{2}[01] + \frac{1}{2}[10] \qquad P_Z^{\text{target}} = \frac{1}{2}[0] + \frac{1}{2}[1]$$

The equality constraints are satisfied

Consider the inequality constraint for (x,y,z) =(0,0,0):

$$P_X(0) + P_Y(0) + P_Z(0) - P_{XY}(00) - P_{YZ}(00) - P_{XZ}(00) \leq 1$$

$$LHS = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - 0 - 0 - 0 = \frac{3}{2}$$

$$\frac{3}{2} \not\leq 1$$

Example 3 revisited:

$$
\begin{aligned}
P_{XY}^{\text{target}} &= (\tfrac{1}{2}[0] + \tfrac{1}{2}[1])(\tfrac{1}{2}[0] + \tfrac{1}{2}[1]) \\
P_{YZ}^{\text{target}} &= \tfrac{1}{2}[00] + \tfrac{1}{2}[11] \\
P_{XZ}^{\text{target}} &= \tfrac{1}{2}[00] + \tfrac{1}{2}[11]
\end{aligned}
$$

Consider the inequality constraint for (x,y,z) =(0,0,1):

$$P_X(0) + P_Y(0) + P_Z(1) - P_{XY}(00) - P_{YZ}(01) - P_{XZ}(01) \leq 1$$

$$LHS = \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{4} - 0 - 0 = \tfrac{5}{4}$$

$$\frac{5}{4} \not\leq 1$$

Example 4 (example 1 with symmetry constraint):

$$P_{XY}^{\text{target}} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$$
$$P_{YZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$
$$P_{XZ}^{\text{target}} = \tfrac{1}{2}[01] + \tfrac{1}{2}[10]$$

Question: $\exists P_{XYZ}$ symmetric under exchange of Y and Z

such that
$$P_{XY} = P_{XY}^{\text{target}} \quad P_{YZ} = P_{YZ}^{\text{target}} \quad P_{XZ} = P_{XZ}^{\text{target}}$$
where
$$P_{XY} := \sum_Z P_{XYZ} \quad P_{YZ} := \sum_X P_{XYZ} \quad P_{XZ} := \sum_Y P_{XYZ}$$
?

Answer: No!

$P_{XYZ} = \tfrac{1}{2}[001] + \tfrac{1}{2}[110]$ is not symmetric under exchange of Y and Z

Incorporating symmetry constraints

$\exists Q_{XYZ}$ with $P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ}$ as marginals

$$\forall x, y : P_{XY}(xy) = \sum_z Q_{XYZ}(xyz)$$
$$\forall y, z : P_{YZ}(yz) = \sum_x Q_{XYZ}(xyz)$$
$$\forall x, z : P_{XZ}(xz) = \sum_y Q_{XYZ}(xyz)$$
$$\forall x : P_X(x) = \sum_{y,z} Q_{XYZ}(xyz)$$
$$\forall y : P_Y(y) = \sum_{x,z} Q_{XYZ}(xyz)$$
$$\forall z : P_Z(z) = \sum_{x,y} Q_{XYZ}(xyz)$$
$$\forall x, y, z : 0 \leq Q_{XYZ}(xyz) \leq 1$$
$$\forall a, a' : Q_{XYZ}(xaa') = Q_{XYZ}(xa'a)$$

Still linear quantifier elimination!

Finding **all constraints** for a set of distributions on specified subsets of variables to be the set of marginals of some joint distribution

vs.

For a **particular** set of distributions on specified subsets of variables, testing whether or not it is obtainable as the set of marginals of a joint distribution

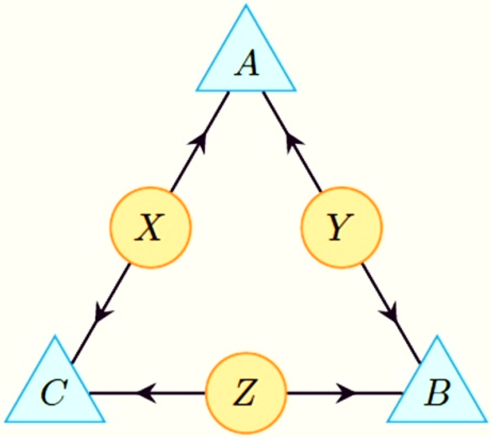
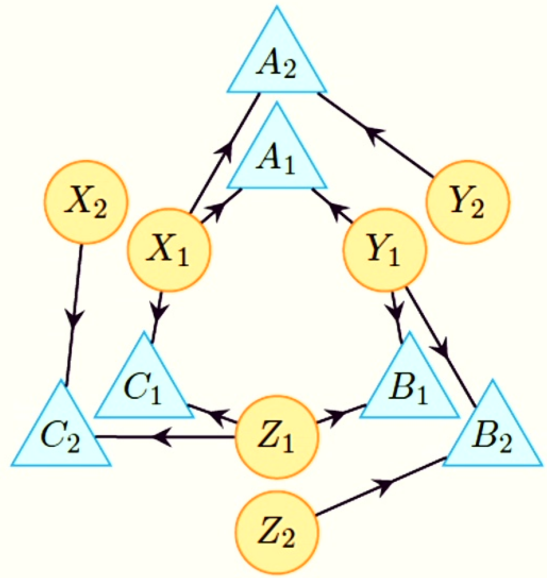
$\exists Q_{XYZ}$ with $P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ}$ as marginals ?

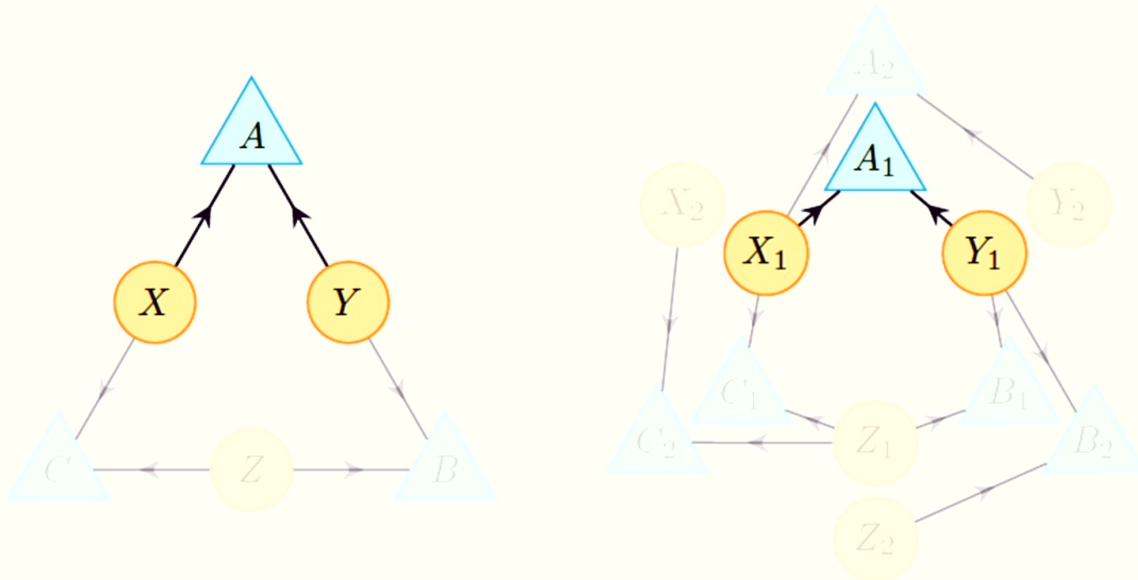
$$\forall x,y : P_{XY}(xy) = \sum_z Q_{XYZ}(xyz)$$
$$\forall y,z : P_{YZ}(yz) = \sum_x Q_{XYZ}(xyz)$$
$$\forall x,z : P_{XZ}(xz) = \sum_y Q_{XYZ}(xyz)$$
$$\forall x : P_X(x) = \sum_{y,z} Q_{XYZ}(xyz)$$
$$\forall y : P_Y(y) = \sum_{x,z} Q_{XYZ}(xyz)$$
$$\forall z : P_Z(z) = \sum_{x,y} Q_{XYZ}(xyz)$$
$$\forall x,y,z : 0 \leq Q_{XYZ}(xyz) \leq 1$$

A linear program can determine if there are any solutions to this system of linear equations

If not, the linear program can return a witness of infeasibility---a single inequality that is violated by the set of dist'ns

Inflation DAGs


A
X
Y
C
Z
B

A_2
A_1
X_2
X_1
Y_1
Y_2
C_1
B_1
C_2
Z_1
B_2
Z_2


A
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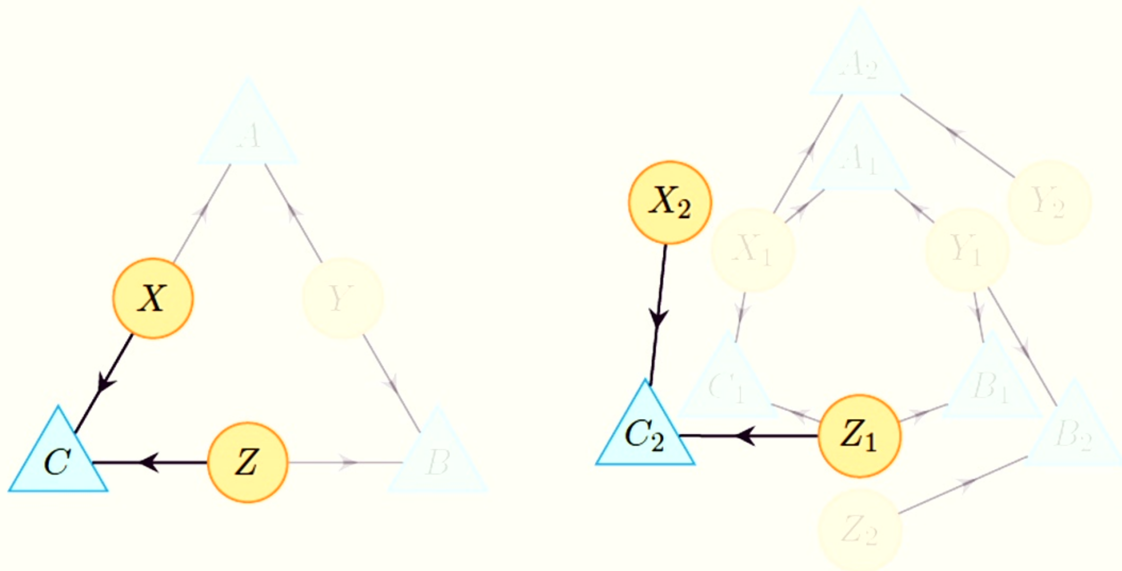
B2
Z2


A
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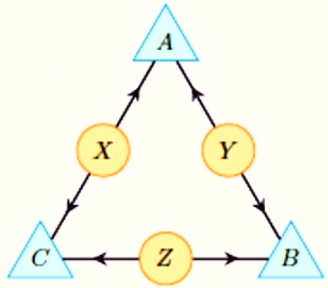
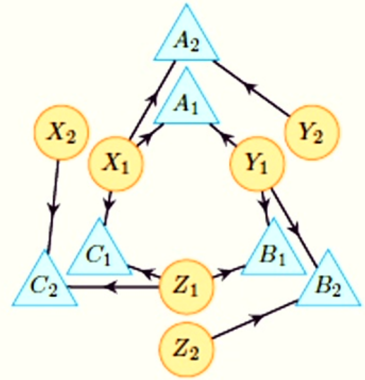
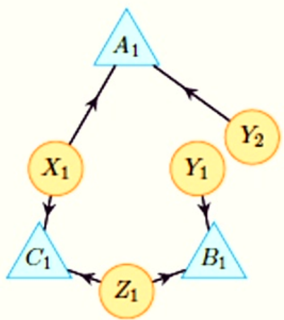
Z2



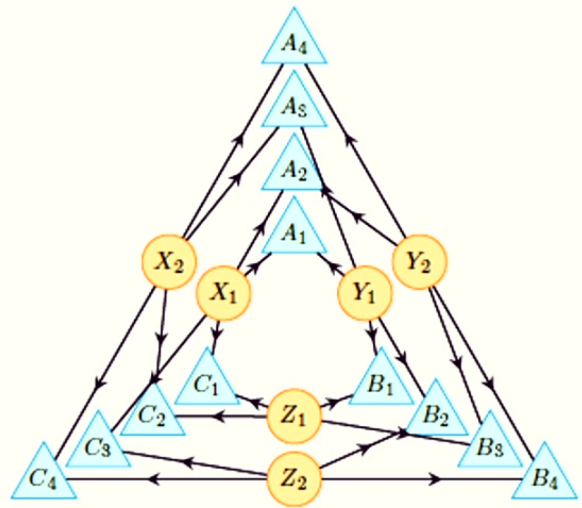
The Triangle Scenario



Spiral Inflation



Cut Inflation



Large Inflation

Sameness up to copy-indices

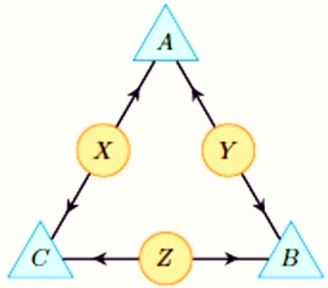
Let X ⊆ Nodes(G), X' ⊆ Nodes(G')

X ~ X'

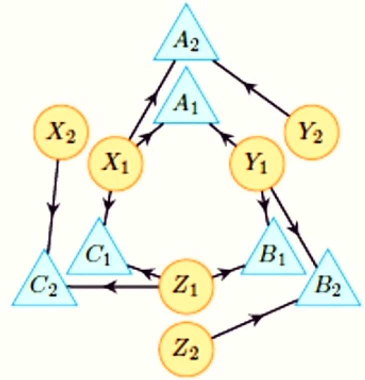
denotes that X' contains exactly one copy of every node in X

SubDAG_G' (X') ~ SubDAG_G (X)

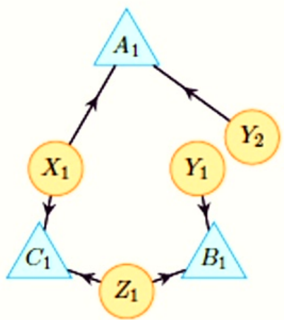
Denotes that X ~ X' and that an edge is present between two nodes in X' iff it is present between the two associated nodes in X.



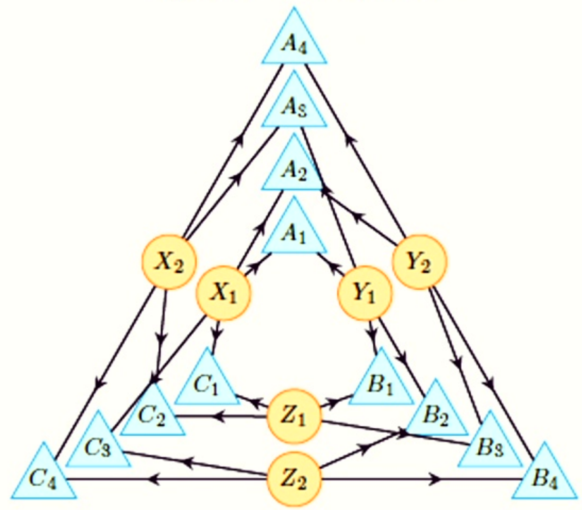
The Triangle Scenario



Spiral Inflation

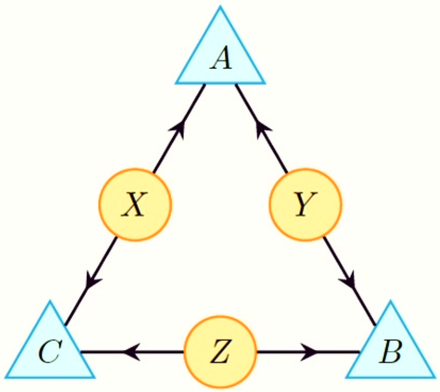


Cut Inflation



Large Inflation

model M on DAG G



$\mathrm{P}_{A|XY}$

$\mathrm{P}_{B|XZ}$

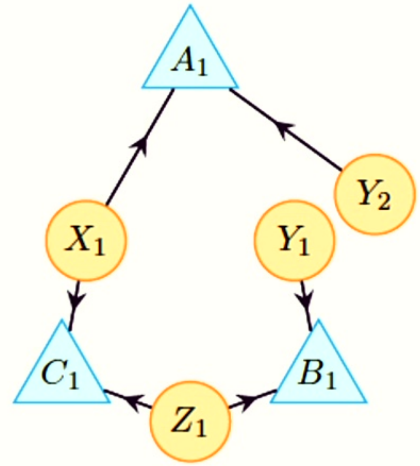
$\mathrm{P}_{C|YZ}$

P_{X}

P_{Y}

P_{Z}

M' = G $\to$ G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|X_1Z_1}$

$\mathrm{P}_{C_1|Y_1Z_1}$

P_{X_1}

P_{Y_1}

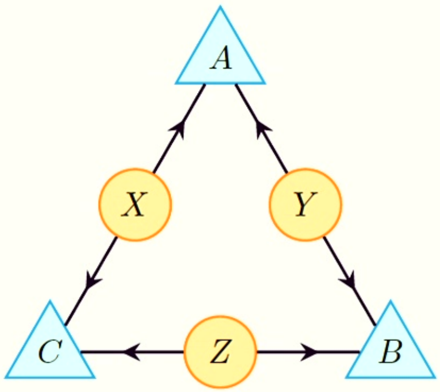
P_{Y_2}

P_{Z_1}

with symmetry constraint:

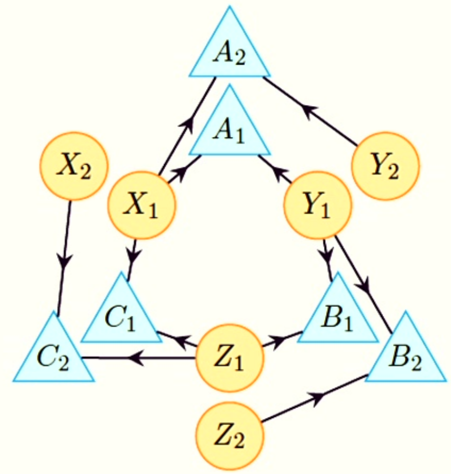
$$\mathrm{P}_{Y_1} = \mathrm{P}_{Y_2}$$

model M on DAG G



$$\mathrm{P}_{A|XY}$$
$$\mathrm{P}_{B|XZ}$$
$$\mathrm{P}_{C|YZ}$$
$$\mathrm{P}_{X}$$
$$\mathrm{P}_{Y}$$
$$\mathrm{P}_{Z}$$

M’ = G → G’ Inflation of M



$$\mathrm{P}_{A_1|X_1Y_1} \quad \mathrm{P}_{A_2|X_1Y_2}$$
$$\mathrm{P}_{B_1|Y_1Z_1} \quad \mathrm{P}_{B_2|Y_1Z_2}$$
$$\mathrm{P}_{C_1|X_1Z_1} \quad \mathrm{P}_{C_2|X_2Z_1}$$
$$\mathrm{P}_{X_1} \quad \mathrm{P}_{X_2}$$
$$\mathrm{P}_{Y_1} \quad \mathrm{P}_{Y_2}$$
$$\mathrm{P}_{Z_1} \quad \mathrm{P}_{Z_2}$$

with symmetry constraints:

$$\mathrm{P}_{A_1|X_1Y_1} = \mathrm{P}_{A_2|X_1Y_2}$$
$$\mathrm{P}_{B_1|Y_1Z_1} = \mathrm{P}_{B_2|Y_1Z_2}$$
$$\mathrm{P}_{C_1|X_1Z_1} = \mathrm{P}_{C_2|X_2Z_1}$$
$$\mathrm{P}_{X_1} = \mathrm{P}_{X_2}$$
$$\mathrm{P}_{Y_1} = \mathrm{P}_{Y_2}$$
$$\mathrm{P}_{Z_1} = \mathrm{P}_{Z_2}$$

Suppose $G' \in \text{inflations}(G)$

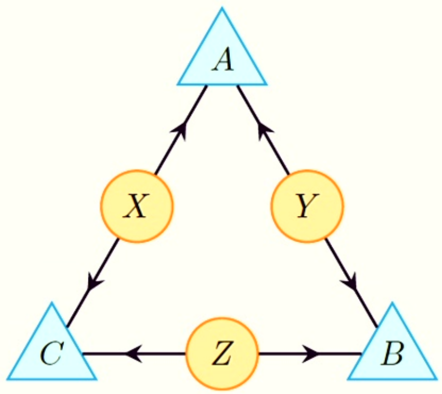
then

$$M' = \text{Inflation}_{G \to G'}(M)$$

if and only if

$$\forall A_i, A_j \in \text{nodes}(G') : \; \mathrm{P}_{A_i|\mathrm{Pa}_{G'}(A_i)} = \mathrm{P}_{A_j|\mathrm{Pa}_{G'}(A_j)}.$$

Recall: A dist'n P on observed variables is **compatible** with a DAG G if there exists parameter choices that yield P

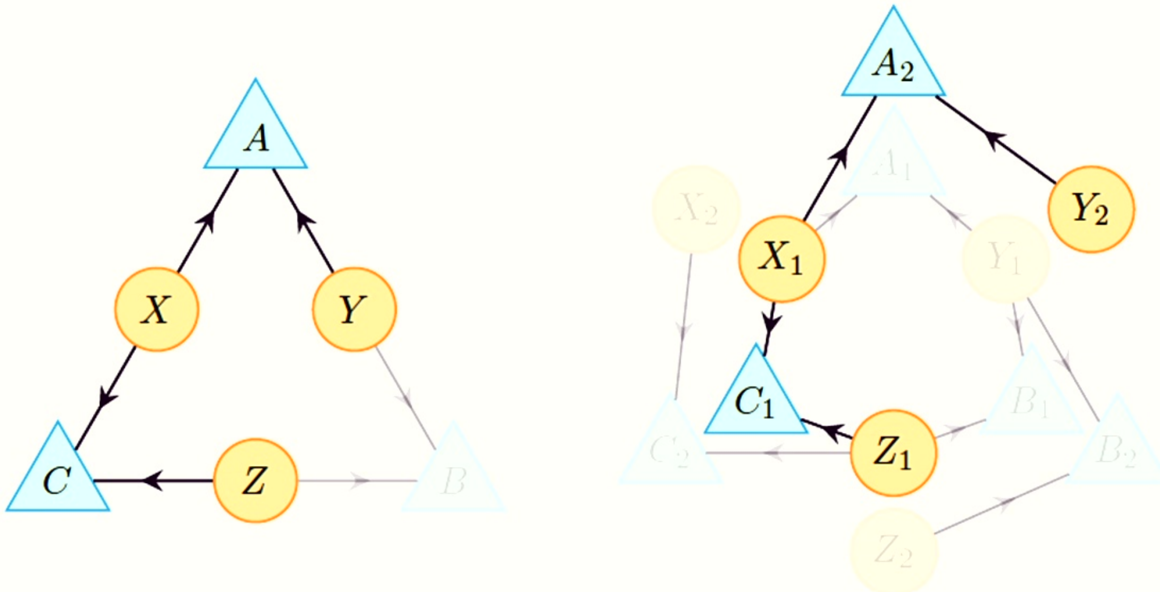


$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|XZ}$
$\mathrm{P}_{C|YZ}$
P_{X}
P_{Y}
P_{Z}

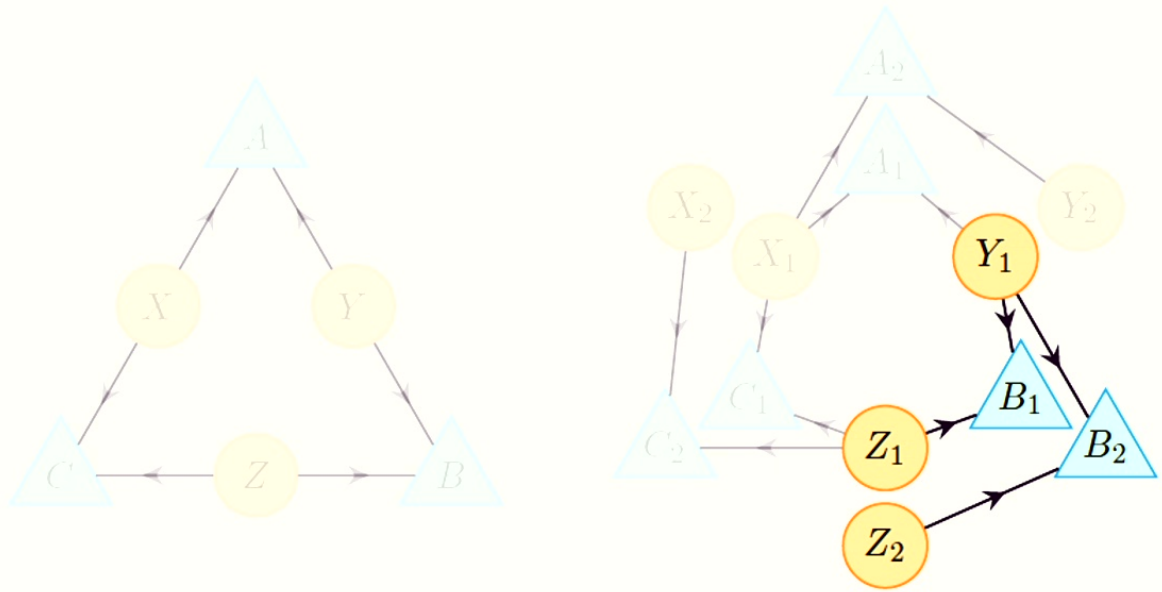
$$\mathrm{P}_{ABC} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{B|YZ}\mathrm{P}_{C|XZ}\mathrm{P}_X\mathrm{P}_Y\mathrm{P}_Z$$

$$\mathrm{P}_{AC} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{B|YZ}\mathrm{P}_X\mathrm{P}_Y\mathrm{P}_Z$$

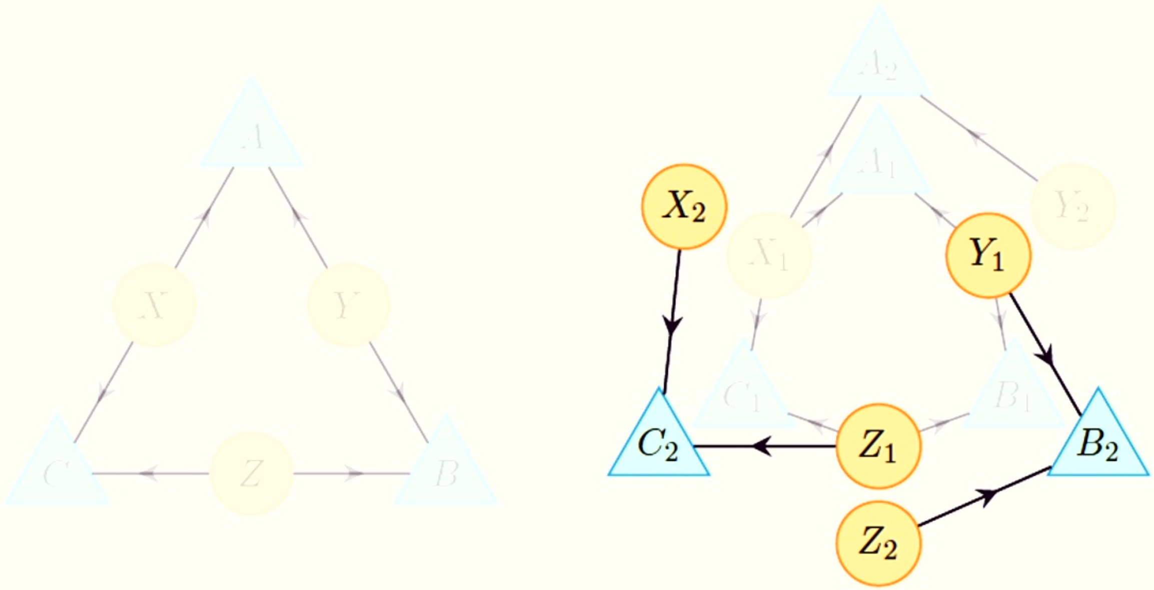
Injectable sets of observed variables
in the inflation DAG



$\{A_2 C_1\}$ is an injectable set

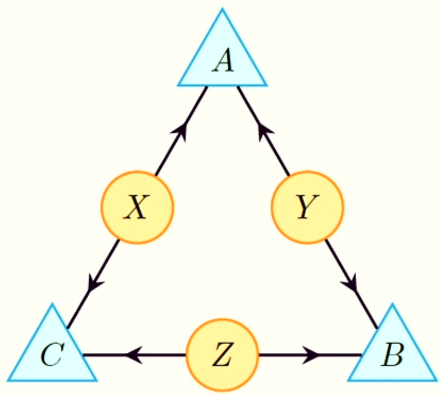


$\{B_1 B_2\}$ is *not* an injectable set



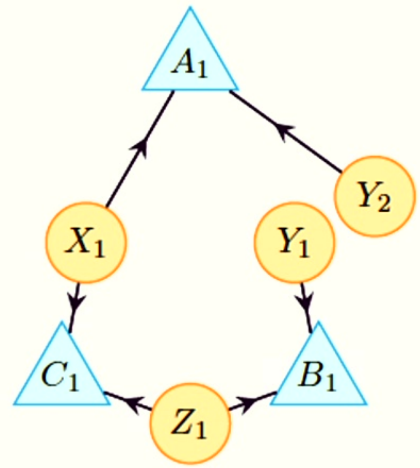
$\{B_2C_2\}$ is *not* an injectable set

model M on DAG G



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|XZ}$
$\mathrm{P}_{C|YZ}$
P_{X}
P_{Y}
P_{Z}

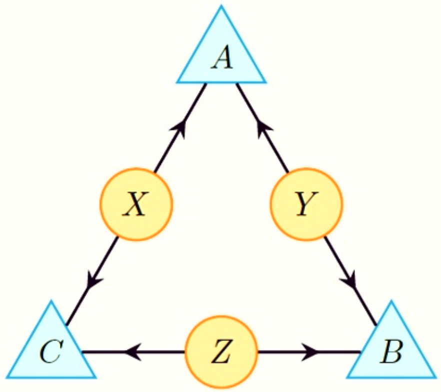
M' = G $\rightarrow$ G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|X_1Z_1}$
$\mathrm{P}_{C_1|Y_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

$\{A_1C_1\}$ is an injectable set

model M on DAG G



$\mathrm{P}_{A|XY}$

$\mathrm{P}_{B|XZ}$

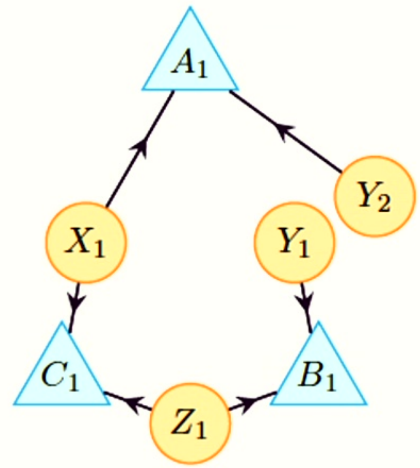
$\mathrm{P}_{C|YZ}$

P_{X}

P_{Y}

P_{Z}

M' = G → G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|X_1Z_1}$

$\mathrm{P}_{C_1|Y_1Z_1}$

P_{X_1}

P_{Y_1}

P_{Y_2}

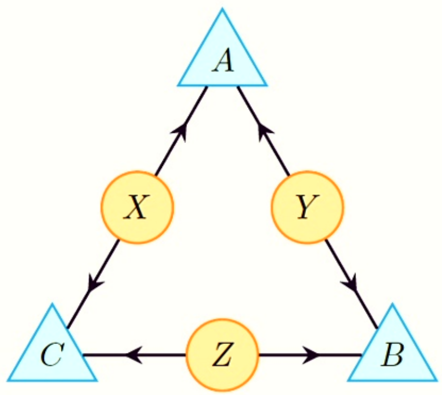
P_{Z_1}

$\{A_1C_1\}$ is an injectable set

$$\mathrm{P}_{A_1C_1} = \sum_{X_1Y_2Z_1} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{C_1|X_1Z_1}\mathrm{P}_{X_1}\mathrm{P}_{Y_2}\mathrm{P}_{Z_1}$$

$$\mathrm{P}_{AC} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{C|XZ}\mathrm{P}_{X}\mathrm{P}_{Y}\mathrm{P}_{Z}$$

model M on DAG G



$\mathrm{P}_{A|XY}$

$\mathrm{P}_{B|XZ}$

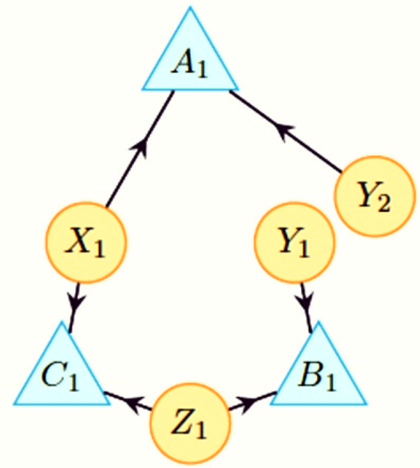
$\mathrm{P}_{C|YZ}$

P_{X}

P_{Y}

P_{Z}

M' = G → G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|X_1Z_1}$

$\mathrm{P}_{C_1|Y_1Z_1}$

P_{X_1}

P_{Y_1}

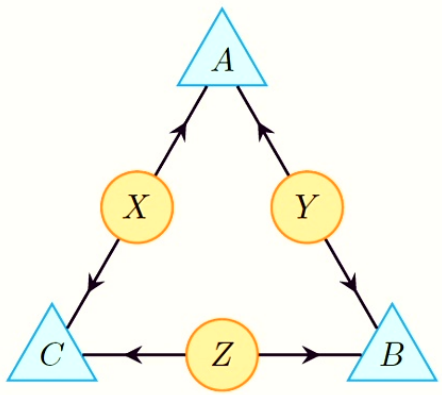
P_{Y_2}

P_{Z_1}

$\{A_1B_1\}$ is *not* an injectable set

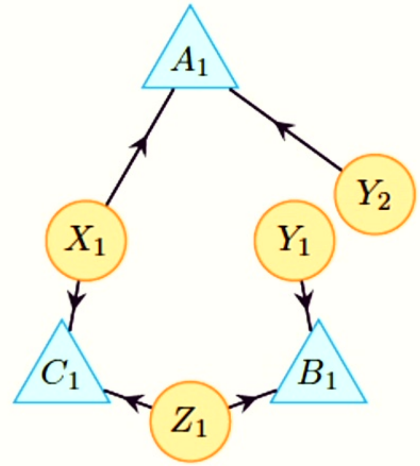
$$\mathrm{P}_{A_1B_1} = \left(\sum_{X_1Y_2} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{Y_2}\mathrm{P}_{X_1}\right)\left(\sum_{Z_1Y_1} \mathrm{P}_{B_1|Y_1Z_1}\mathrm{P}_{Y_1}\mathrm{P}_{Z_1}\right)$$

model M on DAG G



$P_{A|XY}$
$P_{B|XZ}$
$P_{C|YZ}$
P_X
P_Y
P_Z

M' = G → G' Inflation of M



$P_{A_1|X_1Y_2}$
$P_{B_1|X_1Z_1}$
$P_{C_1|Y_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

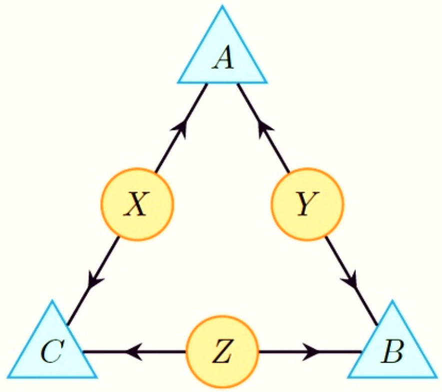
Injectable sets: $\{A_1\},\{B_1\},\{C_1\},\{A_1C_1\},\{B_1C_1\}$

$(P_A, P_B, P_C, P_{AC}, P_{BC})$ compatible with M $\Longrightarrow$ $(P_{A_1}, P_{B_1}, P_{C_1}, P_{A_1C_1}, P_{B_1C_1})$ compatible with M'

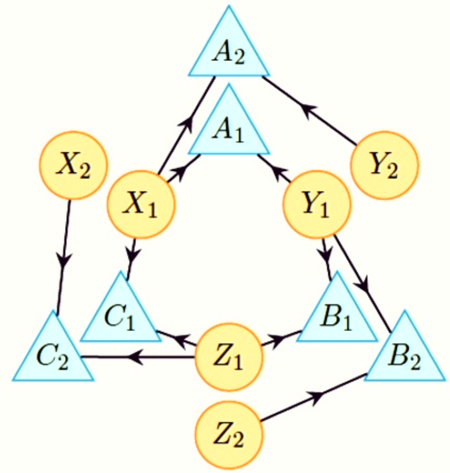
where

$P_{A_1} = P_A$ $\quad P_{A_1C_1} = P_{AC}$
$P_{B_1} = P_B$ $\quad P_{B_1C_1} = P_{BC}$
$P_{C_1} = P_C$

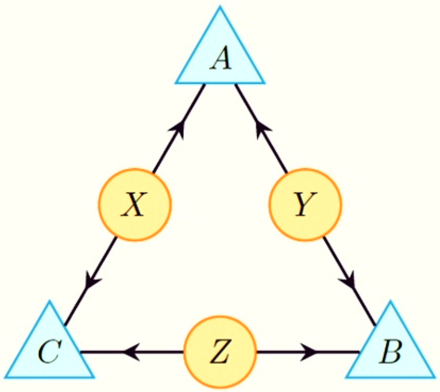
model M on DAG G



M’ = G $\rightarrow$ G’ Inflation of M



Injectable sets: $\{A_1\}, \{B_1\}, \{C_1\}, \{A_2\}, \{B_2\}, \{C_2\},$
$\{A_1B_1\}, \{A_1, B_2\}, \{B_1C_1\}, \{B_1, C_2\}, \{C_1, A_1\}, \{C_1, A_2\},$
$\{A_1B_1C_1\}$

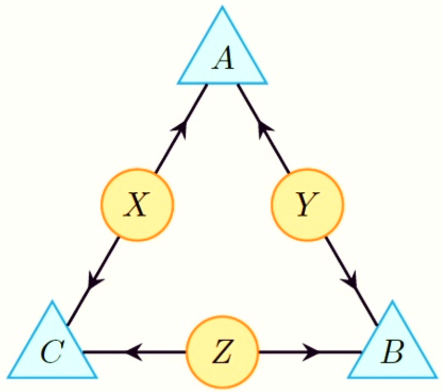


is incompatible with

$$P_{ABC} = \frac{1}{2}[000] + \frac{1}{2}[111]$$

Strategy: assume compatibility and derive a contradiction

Causal model M



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|XZ}$
$\mathrm{P}_{C|YZ}$
P_X
P_Y
P_Z

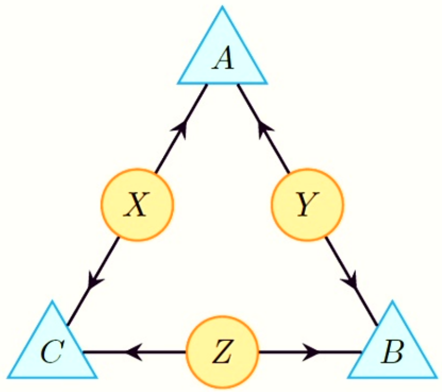
$$(P_{AC}, P_{BC})$$

where $P_{AC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

$P_{BC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

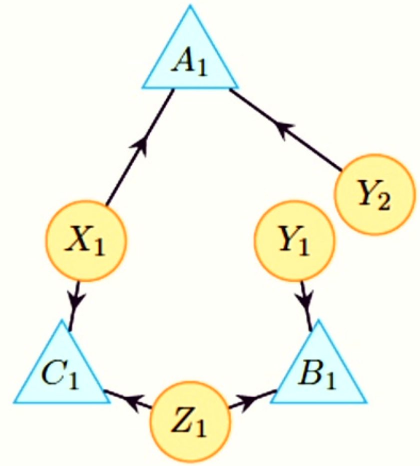
is compatible with M

Causal model M



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|XZ}$
$\mathrm{P}_{C|YZ}$
P_X
P_Y
P_Z

M' = G → G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|X_1Z_1}$
$\mathrm{P}_{C_1|Y_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

(P_{AC}, P_{BC})

where $P_{AC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

$P_{BC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is compatible with M

$\Longrightarrow$

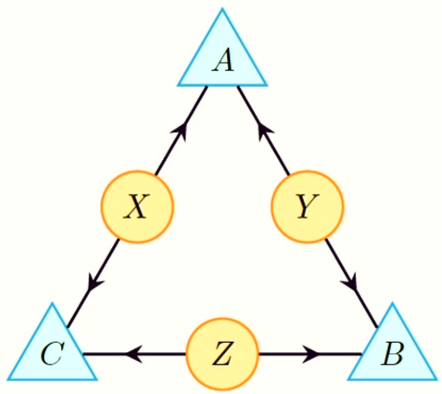
$(P_{A_1C_1}, P_{B_1C_1})$

where $\mathrm{P}_{A_1C_1} = \mathrm{P}_{AC}$

$\mathrm{P}_{B_1C_1} = \mathrm{P}_{BC}$

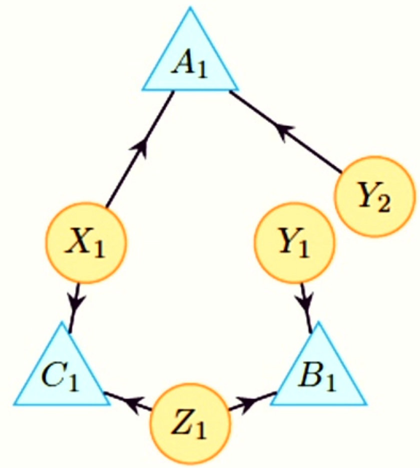
is compatible with M'

Causal model M



$P_{A|XY}$
$P_{B|XZ}$
$P_{C|YZ}$
P_X
P_Y
P_Z

M' = G $\to$ G' Inflation of M



$P_{A_1|X_1Y_2}$
$P_{B_1|X_1Z_1}$
$P_{C_1|Y_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

(P_{AC}, P_{BC})

where $P_{AC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

$P_{BC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is compatible with M

$\Longrightarrow$

$(P_{A_1C_1}, P_{B_1C_1})$

where $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is compatible with M'

Proof:

If $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

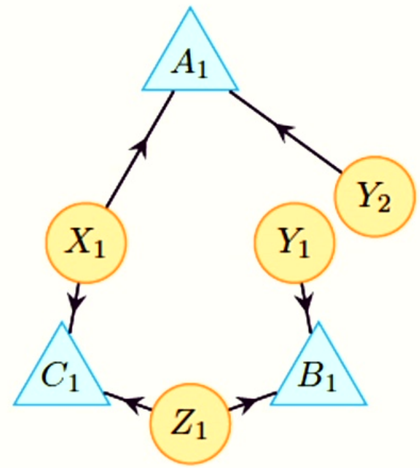
$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

then

$$P_{A_1B_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$$

(recall example 3 of marginal problem)

M' = G → G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|X_1Z_1}$

$\mathrm{P}_{C_1|Y_1Z_1}$

P_{X_1}

P_{Y_1}

P_{Y_2}

P_{Z_1}

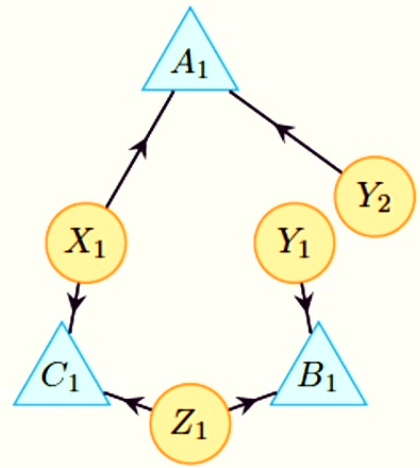
$$(P_{A_1C_1}, P_{B_1C_1})$$

where $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is not compatible with M'

M' = G → G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|X_1Z_1}$
$\mathrm{P}_{C_1|Y_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

Proof:

If $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$
$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

then

$$P_{A_1B_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$$

(recall example 3 of marginal problem)

But this violates $A_1 \perp B_1$

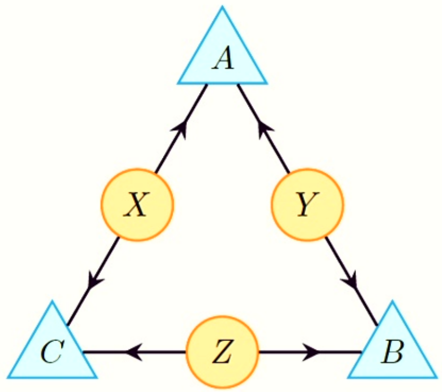
which is required by the d-separation relation $A_1 \perp_d B_1$

$$(P_{A_1C_1}, P_{B_1C_1})$$

where $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$
$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

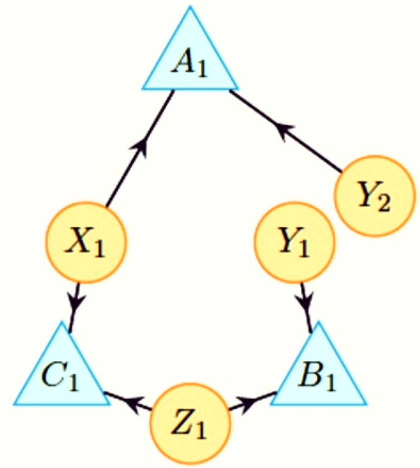
is not compatible with M'

Causal model M



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|XZ}$
$\mathrm{P}_{C|YZ}$
P_{X}
P_{Y}
P_{Z}

M' = G $\rightarrow$ G' Inflation of M



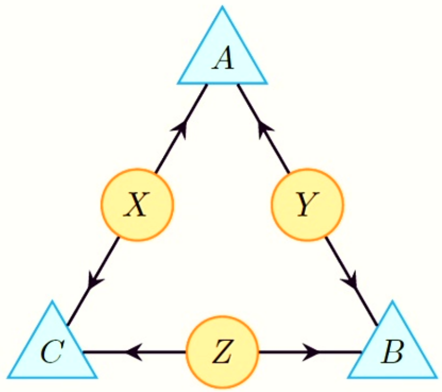
$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|X_1Z_1}$
$\mathrm{P}_{C_1|Y_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

$$(P_{AC}, P_{BC})$$
where $P_{AC} = \frac{1}{2}[00] + \frac{1}{2}[11]$
$P_{BC} = \frac{1}{2}[00] + \frac{1}{2}[11]$
is compatible with M

$$\Longrightarrow$$

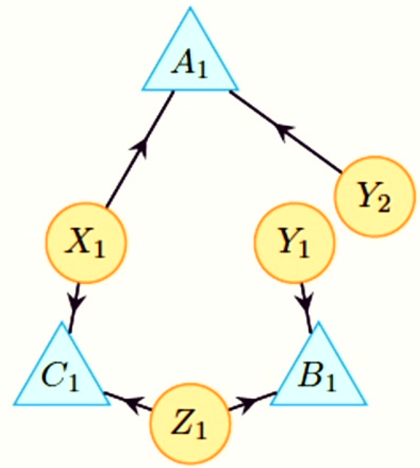
$$(P_{A_1C_1}, P_{B_1C_1})$$
where $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$
$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$
is compatible with M'

Causal model M



$P_{A|XY}$
$P_{B|XZ}$
$P_{C|YZ}$
P_X
P_Y
P_Z

M’ = G $\rightarrow$ G’ Inflation of M



$P_{A_1|X_1Y_2}$
$P_{B_1|X_1Z_1}$
$P_{C_1|Y_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

(P_{AC}, P_{BC})

where $P_{AC} = \frac{1}{2}[00] + \frac{1}{2}[11]$
$P_{BC} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is not compatible with M

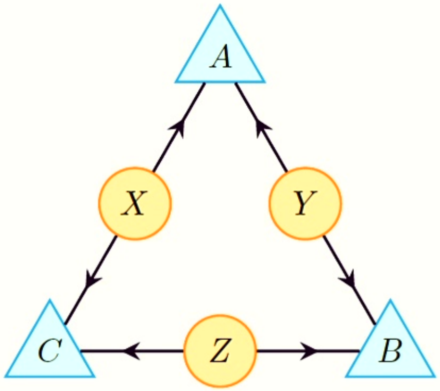
$\Longleftarrow$

$(P_{A_1C_1}, P_{B_1C_1})$

where $P_{A_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$
$P_{B_1C_1} = \frac{1}{2}[00] + \frac{1}{2}[11]$

is not compatible with M'

Causal model M



$$\mathrm{P}_{A|XY}$$
$$\mathrm{P}_{B|XZ}$$
$$\mathrm{P}_{C|YZ}$$
$$\mathrm{P}_{X}$$
$$\mathrm{P}_{Y}$$
$$\mathrm{P}_{Z}$$

$$P_{ABC} = \tfrac{1}{2}[000] + \tfrac{1}{2}[111]$$

is not compatible with M

$\Leftarrow$

$$(P_{AC}, P_{BC})$$

where $P_{AC} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$

$P_{BC} = \tfrac{1}{2}[00] + \tfrac{1}{2}[11]$

is not compatible with M

M’ = G $\rightarrow$ G’ Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$

$\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$$
is compatible with M

$$\Longrightarrow$$

$$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is compatible with M'

M' = G → G' Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$

$\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$
is not compatible with M

$\Longleftarrow$

$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is not compatible with M'

Let $I_{\mathcal{S}}$ be an inequality that acts on the family of distributions $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

Whenever

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$ is compatible with M $\Longrightarrow$ $I_{\mathcal{S}}$ is **satisfied** for $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

Let $I_{\mathcal{S}}$ be an inequality that acts on the family of distributions $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

Whenever

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$ is compatible with M $\Longrightarrow$ $I_{\mathcal{S}}$ is **satisfied** for $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

we say that

$I_{\mathcal{S}}$ is a **causal compatibility inequality** for model M

M′ = G → G′ Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$ $\qquad$ $\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$ is compatible with M $\Longrightarrow$ $\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$ where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$ is compatible with M'

$\Downarrow$

$I_{\mathcal{S}}$ is **satisfied** for $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$ $\Longleftarrow$ $I_{\mathcal{S}'}$ is **satisfied** for $\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$ where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$

M′ = G $\rightarrow$ G′ Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$ $\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

is compatible with M

$$\Downarrow$$

$I_{\mathcal{S}}$ is **satisfied** for
$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

$I_{\mathcal{S}}$ is a **causal compatibility inequality** for model M

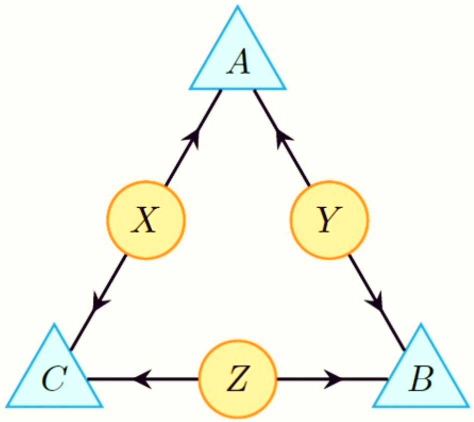
M’ = G $\rightarrow$ G’ Inflation of M

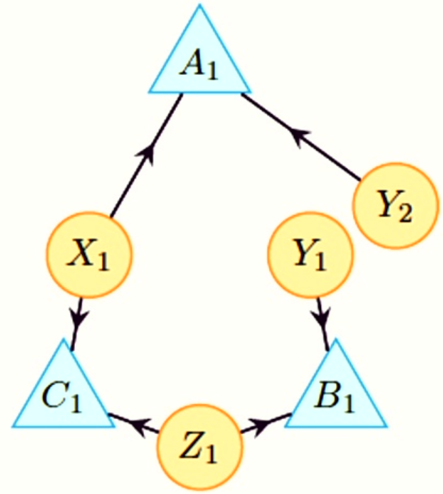
$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$

$\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$I_{\mathcal{S}}$ is a **causal compatibility inequality** for model M $\Longleftarrow$ $I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

binary A, B and C





$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

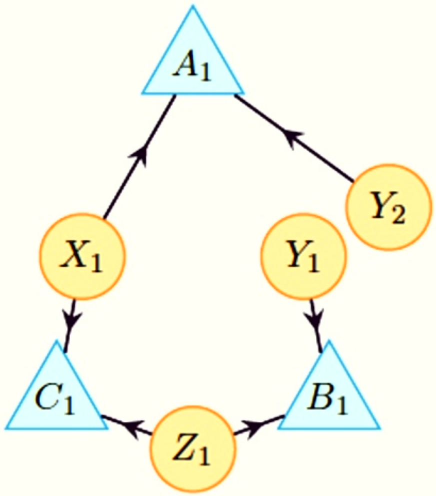
is a causal compatibility inequality for M

⇐

$$P_{A_1} + P_{B_1} + P_{C_1} - P_{A_1} P_{B_1} - P_{B_1 C_1} - P_{A_1 C_1} \leq 1$$

is a causal compatibility inequality for M'

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is a valid set of marginals $\implies$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

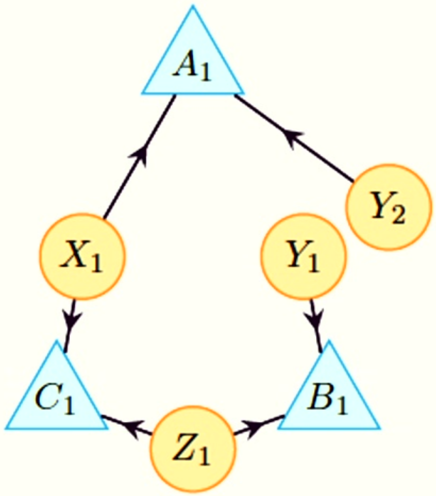


$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M'

$$\implies A_1 \perp B_1 \implies P_{A_1B_1} = P_{A_1}P_{B_1}$$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M' $\implies$ $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1}) \text{ is a valid set of marginals} \implies (P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1}) \text{ satisfy } P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$$



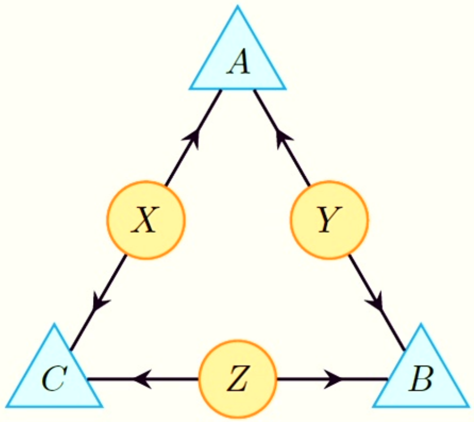
$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M'

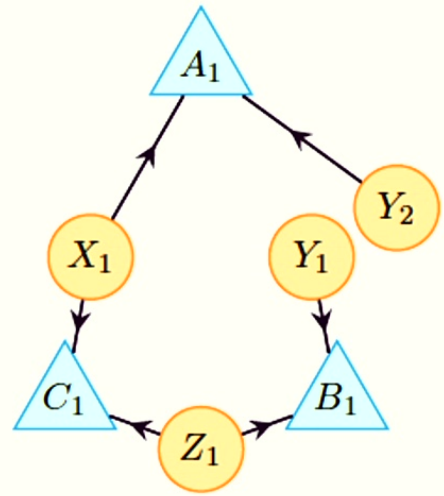
$$\implies A_1 \perp B_1 \implies P_{A_1B_1} = P_{A_1}P_{B_1}$$

$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1}) \text{ is compatible with } M' \implies P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$$

$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$ is a causal compatibility inequality for M'

binary A, B and C



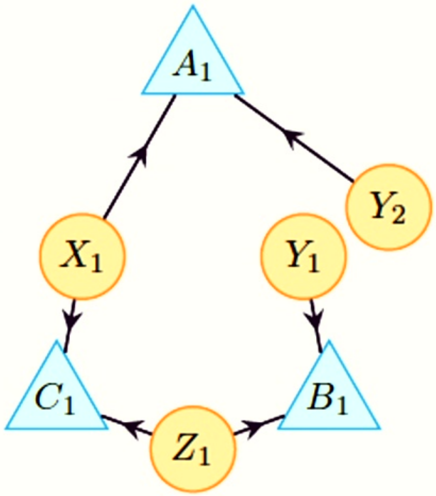


$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$
is a causal compatibility
inequality for M

$P_{A_1} + P_{B_1} + P_{C_1} - P_{A_1} P_{B_1} - P_{B_1 C_1} - P_{A_1 C_1} \leq 1$
is a causal compatibility
inequality for M'

$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$$
is a valid set of marginals $\Longrightarrow$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy

$$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$$

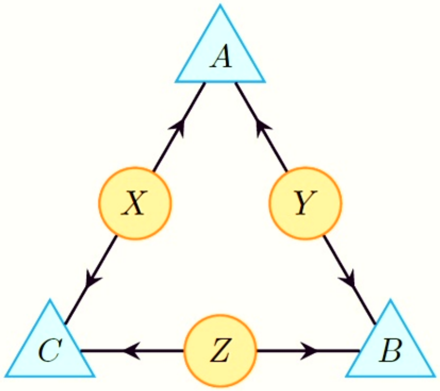


$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is compatible with M'

$$\Longrightarrow A_1 \perp B_1 \Longrightarrow P_{A_1B_1} = P_{A_1}P_{B_1}$$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is compatible with M' $\Longrightarrow$ $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$ is a causal compatibility inequality for M'

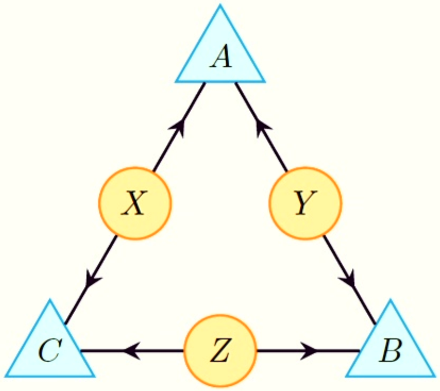


is incompatible with

$$P_{ABC} = \frac{1}{2}[000] + \frac{1}{2}[111]$$

Test causal compatibility inequality for (a,b,c) = (0,0,1)

$$P_A(0)+P_B(0)+P_C(1)-P_A(0)P_B(0)-P_{BC}(01)-P_{AC}(01) \leq 1$$



is incompatible with

$$P_{ABC} = \tfrac{1}{2}[000] + \tfrac{1}{2}[111]$$

Test causal compatibility inequality for (a,b,c) = (0,0,1)

$$P_A(0)+P_B(0)+P_C(1)-P_A(0)P_B(0)-P_{BC}(01)-P_{AC}(01) \leq 1$$

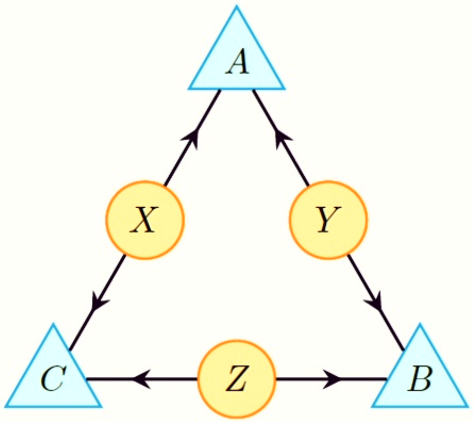
$$LHS = \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{4} - 0 - 0 = \tfrac{5}{4}$$

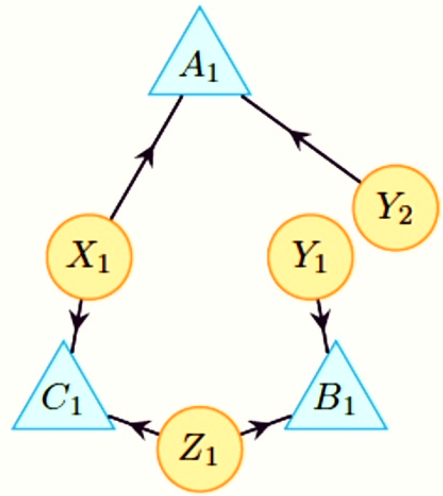
$$\frac{5}{4} \not\leq 1$$

violated!

Correlator-based causal compatibility inequalities

{+.-}-valued A, B and C





$$\langle AC\rangle + \langle BC\rangle \leq 1 + \langle A\rangle\langle B\rangle$$

is a causal compatibility inequality for M

⇐

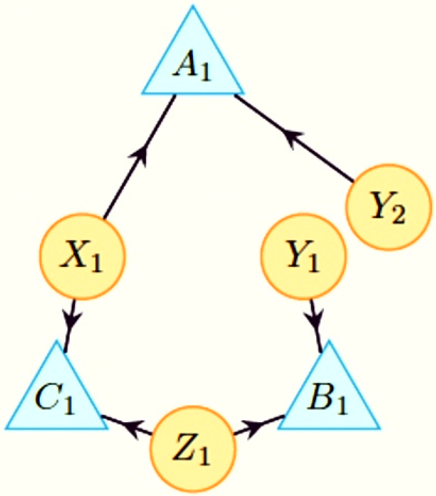
$$\langle A_1C_1\rangle + \langle B_1C_1\rangle \leq 1 + \langle A_1\rangle\langle B_1\rangle$$

is a causal compatibility inequality for M'

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is a valid set of marginals

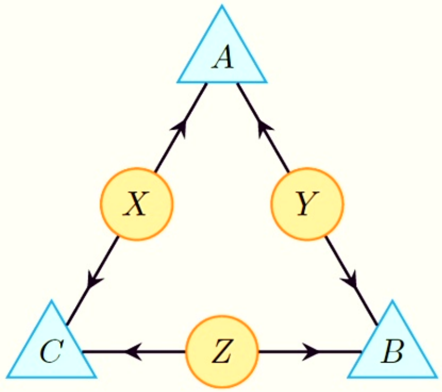
$\Longrightarrow$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy
$\langle A_1C_1\rangle + \langle B_1C_1\rangle - \langle A_1B_1\rangle \leq 1$



$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is compatible with M'

$$\Longrightarrow \; A_1 \perp B_1 \; \Longrightarrow \; \langle A_1B_1\rangle = \langle A_1\rangle\langle B_1\rangle$$

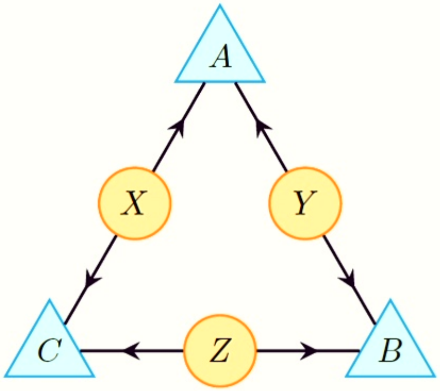


is incompatible with $P_{ABC} = \frac{1}{2}[+++] + \frac{1}{2}[---]$

Test causal compatibility inequality:

$$\langle AC \rangle + \langle BC \rangle \leq 1 + \langle A \rangle \langle B \rangle$$

$$LHS = (+1) + (+1) = 2 \qquad RHS = 1 + 0 = 1$$



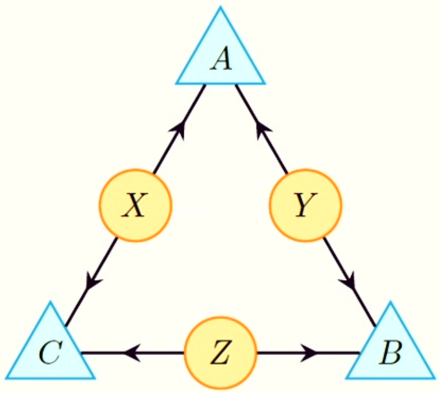
is incompatible with

$$P_{ABC} = \frac{1}{2}[000] + \frac{1}{2}[111]$$

Test entropic causal compatibility inequality

$$I(A:C) + I(C:B) \leq H(C)$$

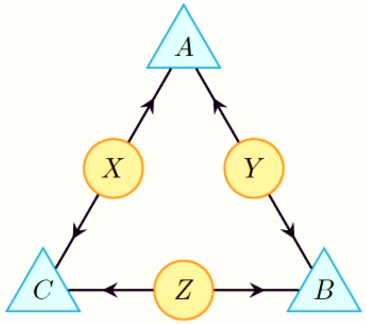
A harder case

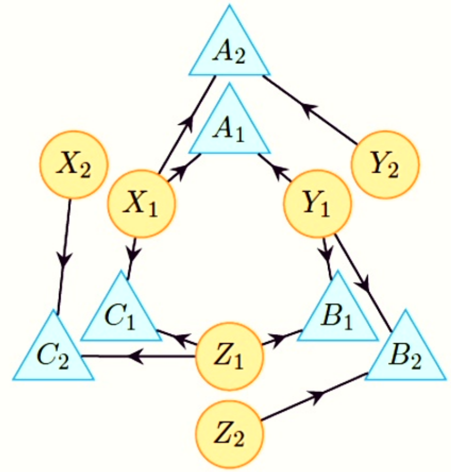


is incompatible with

$$P_{ABC} = \frac{1}{3}[001] + \frac{1}{3}[010] + +\frac{1}{3}[100]$$

Spiral inflation

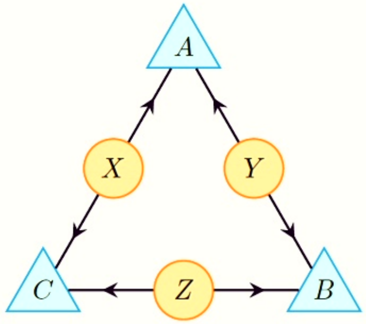




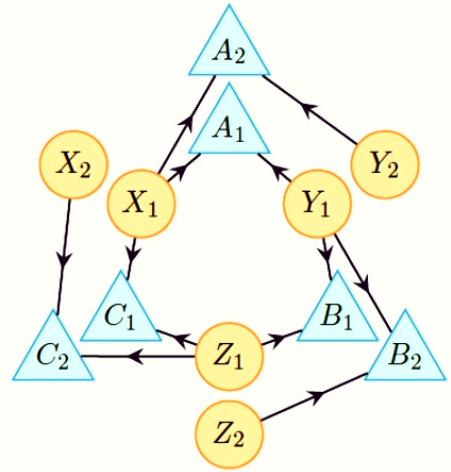
$$P_A(1)P_B(1)P_C(1) \leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1) + P_{AC}(11)P_B(1) + P_{ABC}(000)$$

$\Leftarrow$

$$P_{A_2}(1)P_{B_2}(1)P_{C_2}(1) \leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1) + P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)$$



Spiral inflation



$$P_A(1)P_B(1)P_C(1) \leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1) + P_{AC}(11)P_B(1) + P_{ABC}(000)$$

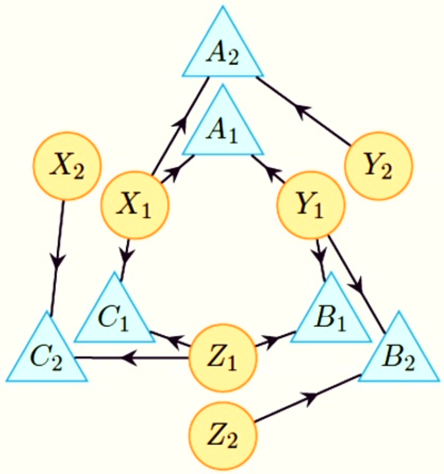
$$P_{A_2}(1)P_{B_2}(1)P_{C_2}(1) \leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1) + P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)$$

rules out

$$P_{ABC} = \frac{1}{3}[001] + \frac{1}{3}[010] + +\frac{1}{3}[100]$$

$$P_{A_2B_2C_2}(111) \leq P_{A_1B_2C_2}(111) + P_{B_1C_2A_2}(111) + P_{A_2C_1B_2}(111) + P_{A_1B_1C_1}(000)$$

Holds for marginals of any distribution over $A_1B_1C_1A_2B_2C_2$



Causal structure

$$A_1B_2 \perp_d C_2 \implies P_{A_1B_2C_2} = P_{A_1B_2}P_{C_2},$$
$$B_1C_2 \perp_d A_2 \implies P_{B_1C_2A_2} = P_{B_1C_2}P_{A_2},$$
$$A_2C_1 \perp_d B_2 \implies P_{A_2C_1B_2} = P_{A_2C_1}P_{B_2},$$
$$A_2 \perp_d B_2 \perp_d C_2 \implies P_{A_2B_2C_2} = P_{A_2}P_{B_2}P_{C_2}$$

$$P_{A_2}(1)P_{B_2}(1)P_{C_2}(1)$$
$$\leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1) + P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)$$

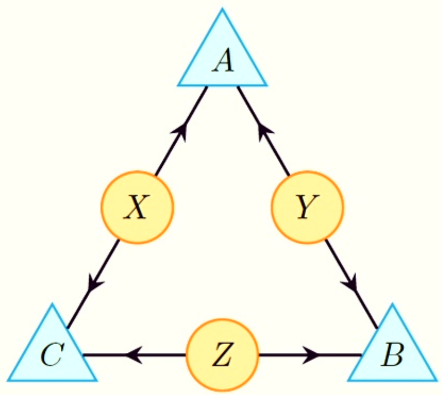
Holds for marginals of any distribution compatible with this structure

Expressible sets

of observed variables

in the inflation DAG

Causal model M



$P_{A|XY}$

$P_{B|XZ}$

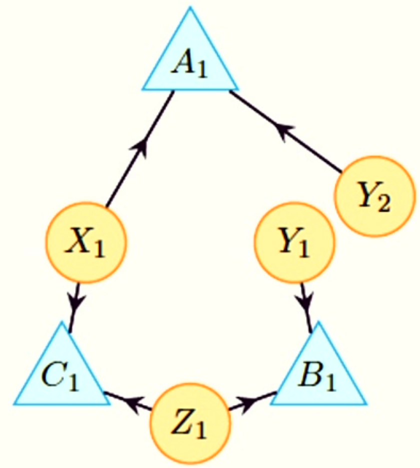
$P_{C|YZ}$

P_X

P_Y

P_Z

M' = G → G' Inflation of M



$P_{A_1|X_1Y_2}$

$P_{B_1|X_1Z_1}$

$P_{C_1|Y_1Z_1}$

P_{X_1}

P_{Y_1}

P_{Y_2}

P_{Z_1}

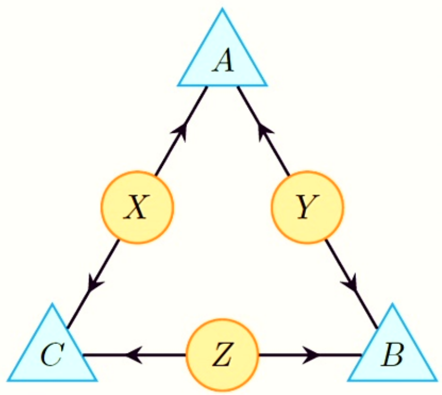
$\{A_1B_1\}$ is *not* an injectable set but it *is* an expressible set

$$P_{A_1B_1} = \left(\sum_{X_1Y_2} P_{A_1|X_1Y_2}P_{Y_2}P_{X_1}\right)\left(\sum_{Z_1Y_1} P_{B_1|Y_1Z_1}P_{Y_1}P_{Z_1}\right)$$

$$P_A = \sum_{XY} P_{A|XY}P_XP_Y \qquad P_B = \sum_{YZ} P_{B|YZ}P_YP_Z$$

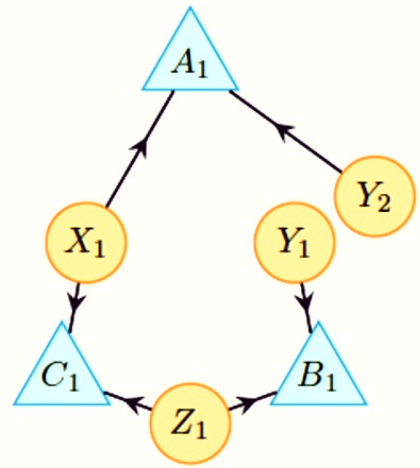
P_{AB} compatible with M $\Longrightarrow$ $P_{A_1B_1} = P_AP_B$ compatible with M'

Causal model M



$P_{A|XY}$
$P_{B|XZ}$
$P_{C|YZ}$
P_X
P_Y
P_Z

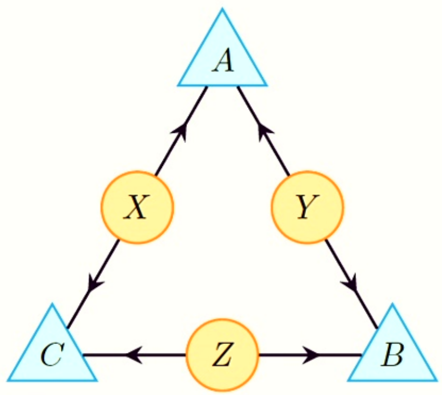
M' = G → G' Inflation of M



$P_{A_1|X_1Y_2}$
$P_{B_1|X_1Z_1}$
$P_{C_1|Y_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

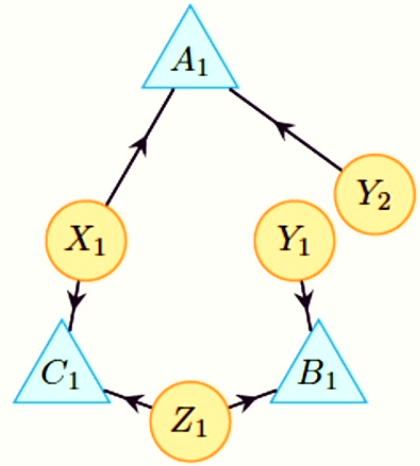
$\{A_1B_1C_1\}$ is *neither* an injectable set *nor* an expressible set

Causal model M



$P_{A|XY}$
$P_{B|XZ}$
$P_{C|YZ}$
P_X
P_Y
P_Z

M' = G → G' Inflation of M



$P_{A_1|X_1Y_2}$
$P_{B_1|X_1Z_1}$
$P_{C_1|Y_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

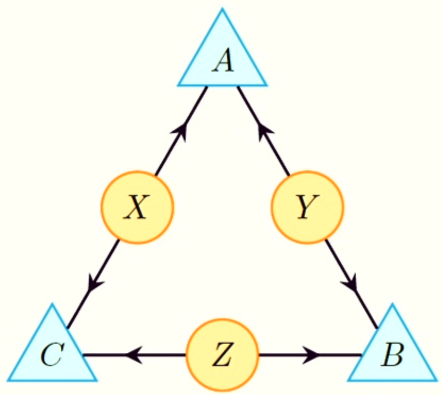
images of injectable sets

$\{A\}, \{B\}, \{C\}, \{AC\}, \{BC\}$

expressible sets:

$\{A_1\}, \{B_1\}, \{C_1\}, \{A_1C_1\}, \{B_1C_1\}, \{A_1, B_1\}$

Causal model M



$\mathrm{P}_{A|XY}$

$\mathrm{P}_{B|XZ}$

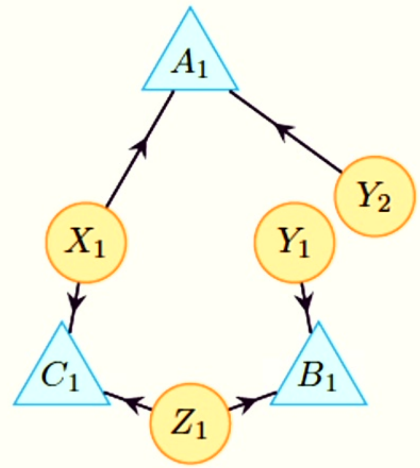
$\mathrm{P}_{C|YZ}$

P_{X}

P_{Y}

P_{Z}

M' = G → G' Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|X_1Z_1}$

$\mathrm{P}_{C_1|Y_1Z_1}$

P_{X_1}

P_{Y_1}

P_{Y_2}

P_{Z_1}

images of injectable sets

$\{A\},\{B\},\{C\},\{AC\},\{BC\}$

$(P_A, P_B, P_C, P_{AC}, P_{BC})$ compatible with M

$$\Longrightarrow$$

expressible sets:

$\{A_1\},\{B_1\},\{C_1\},\{A_1C_1\},\{B_1C_1\},\{A_1,B_1\}$

$(P_{A_1}, P_{B_1}, P_{C_1}, P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ compatible with M'

where

$\mathrm{P}_{A_1} = \mathrm{P}_A \qquad \mathrm{P}_{A_1C_1} = \mathrm{P}_{AC}$

$\mathrm{P}_{B_1} = \mathrm{P}_B \qquad \mathrm{P}_{B_1C_1} = \mathrm{P}_{BC}$

$\mathrm{P}_{C_1} = \mathrm{P}_C \qquad \mathrm{P}_{A_1B_1} = \mathrm{P}_A\mathrm{P}_B$

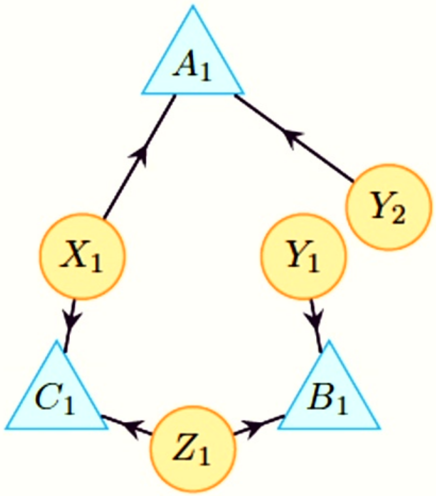
Note: the scheme described for deriving causal compatibility inequalities requires marginal compatibility inequalities that refer **only** to expressible sets

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is a valid set of marginals

$\Longrightarrow$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy
$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

expressible sets



$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is compatible with M'

$\Longrightarrow A_1 \perp B_1 \Longrightarrow P_{A_1B_1} = P_{A_1}P_{B_1}$

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$
is compatible with M'

$\Longrightarrow$

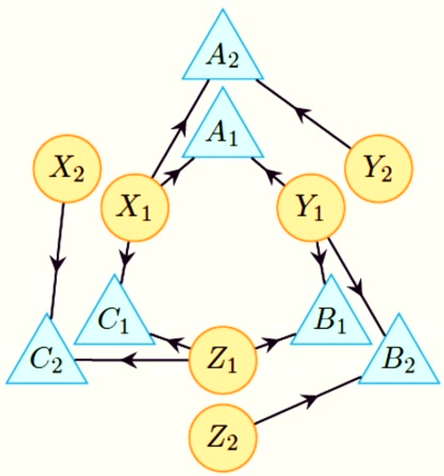
$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$ is a causal compatibility inequality for M'

$$P_{A_2B_2C_2}(111) \leq P_{A_1B_2C_2}(111) + P_{B_1C_2A_2}(111) + P_{A_2C_1B_2}(111) + P_{A_1B_1C_1}(000)$$

Holds for marginals of any distribution over $A_1B_1C_1A_2B_2C_2$

expressible sets



Causal structure

$$A_1B_2 \perp_d C_2 \implies P_{A_1B_2C_2} = P_{A_1B_2}P_{C_2},$$
$$B_1C_2 \perp_d A_2 \implies P_{B_1C_2A_2} = P_{B_1C_2}P_{A_2},$$
$$A_2C_1 \perp_d B_2 \implies P_{A_2C_1B_2} = P_{A_2C_1}P_{B_2},$$
$$A_2 \perp_d B_2 \perp_d C_2 \implies P_{A_2B_2C_2} = P_{A_2}P_{B_2}P_{C_2}$$

$$P_{A_2}(1)P_{B_2}(1)P_{C_2}(1)$$
$$\leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1) + P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)$$

Holds for marginals of any distribution compatible with this structure

Recall the derivation of linear inequalities for marginal compatibility

$\exists Q_{XYZ}$ with $P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ}$ as marginals

$$\forall x,y : P_{XY}(xy) = \sum_z Q_{XYZ}(xyz)$$
$$\forall y,z : P_{YZ}(yz) = \sum_x Q_{XYZ}(xyz)$$
$$\forall x,z : P_{XZ}(xz) = \sum_y Q_{XYZ}(xyz)$$
$$\forall x : P_X(x) = \sum_{y,z} Q_{XYZ}(xyz)$$
$$\forall y : P_Y(y) = \sum_{x,z} Q_{XYZ}(xyz)$$
$$\forall z : P_Z(z) = \sum_{x,y} Q_{XYZ}(xyz)$$
$$\forall x,y,z : 0 \leq Q_{XYZ}(xyz) \leq 1$$

We can always eliminate more parameters

Linear quantifier elimination

$$\forall x : P_X(x) = \sum_y P_{XY}(xy) = \sum_z P_{XZ}(xz)$$
$$\forall y : P_Y(y) = \sum_x P_{XY}(xy) = \sum_z P_{YZ}(yz)$$
$$\forall z : P_Z(z) = \sum_x P_{XZ}(xz) = \sum_y P_{YZ}(yz)$$
$$\forall x,y,z : P_X(x) + P_Y(y) + P_Z(z) - P_{XY}(xy) - P_{YZ}(yz) - P_{XZ}(xz) \leq 1$$

Polynomial inequality constraints for causal compatibility with the original DAG

⇐

Polynomial equality constraints for causal compatibility with the inflated DAG
(from d-separation relations)

+

Linear inequality constraints for marginal compatibility
(from linear quantifier elimination)

Technique for deriving causal compatibility inequalities:

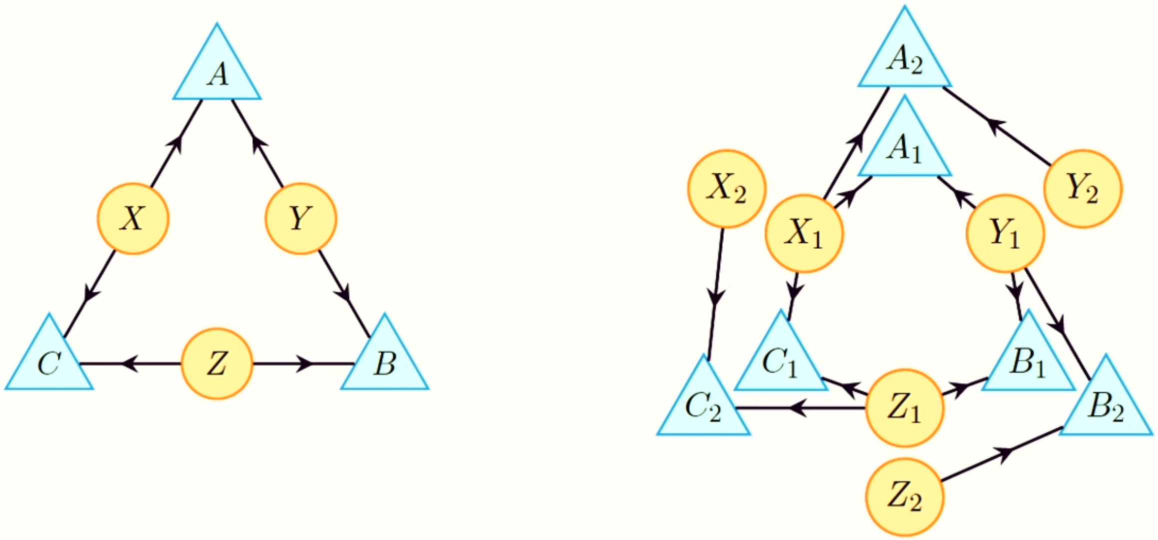
- Choose an inflation of the DAG and find the expressible sets based on its structure

- Derive **linear inequalities for marginal compatibility** of the distributions over the expressible sets by eliminating the parameters in the distributions over every nonexpressible set. This is achievable by **linear quantifier elimination**

- For each expressible set, rewrite the distribution thereon as a function of the distributions on the injectable sets that make up the expressible set---a **polynomial equality**

- Substitute these polynomial equalities into the marginal compatibility inequalities

- Drop the copy-indices to obtain causal compatibility inequalities for the original DAG



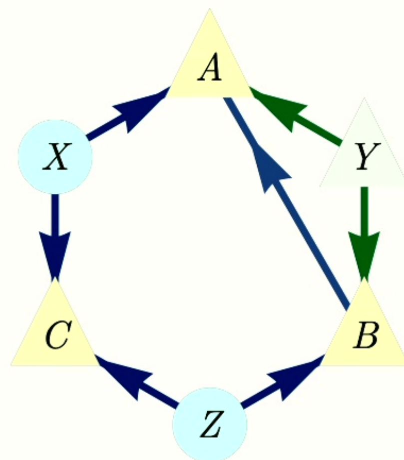
$$0 \leq 1 - \langle AC\rangle - \langle BC\rangle + \langle A\rangle\langle B\rangle \qquad (\times 12)$$

$$\begin{aligned}0 \leq\ & 3 - \langle A\rangle - \langle B\rangle - \langle C\rangle + 2\langle AB\rangle + 2\langle AC\rangle + 2\langle BC\rangle + \langle ABC\rangle \\ & + \langle A\rangle\langle B\rangle + \langle A\rangle\langle C\rangle + \langle B\rangle\langle C\rangle - \langle A\rangle\langle BC\rangle - \langle B\rangle\langle AC\rangle - \langle C\rangle\langle AB\rangle + \langle A\rangle\langle B\rangle\langle C\rangle\end{aligned} \qquad (\times 8)$$

$$\begin{aligned}0 \leq\ & 4 + 2\langle C\rangle - 2\langle AB\rangle - 3\langle AC\rangle - 2\langle BC\rangle - \langle ABC\rangle \\ & + 2\langle A\rangle\langle B\rangle + \langle A\rangle\langle C\rangle - \langle A\rangle\langle BC\rangle - \langle C\rangle\langle AB\rangle + \langle A\rangle\langle B\rangle\langle C\rangle\end{aligned} \qquad (\times 24)$$

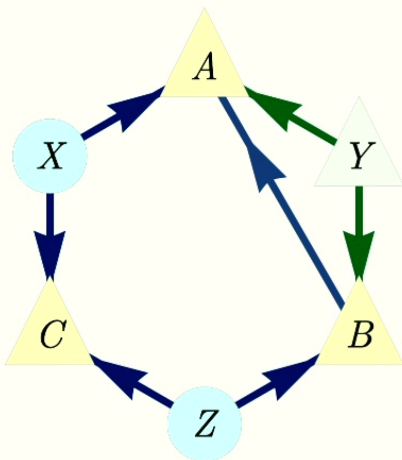
$$\begin{aligned}0 \leq\ & 4 - 2\langle AB\rangle - 2\langle AC\rangle - 2\langle BC\rangle - \langle ABC\rangle \\ & + 2\langle A\rangle\langle B\rangle + 2\langle A\rangle\langle C\rangle + 2\langle B\rangle\langle C\rangle - \langle A\rangle\langle BC\rangle - \langle B\rangle\langle AC\rangle - \langle C\rangle\langle AB\rangle\end{aligned} \qquad (\times 8)$$

Dag #16 from Henson Lal and Pusey



Shannon-type entropic inequalities could not prove that this was interesting

It was proven interesting in J. Pienaar, arXiv:1606.07798



is incompatible with

$$P_{ABCY} := \frac{[0000] + [0110] + [0001] + [1011]}{4}$$

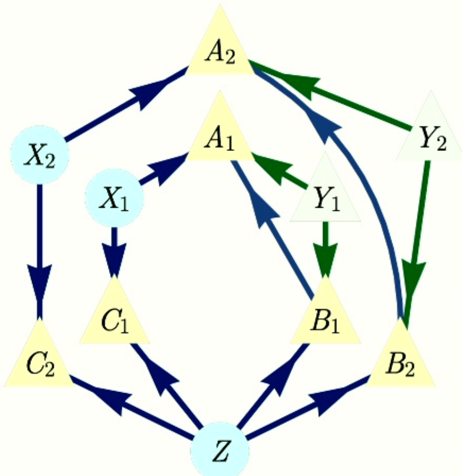
despite it having all the CI relations implied by d-separations

$$P_{B_2C_1Y_1Y_2}(0010) \leq P_{B_2C_2Y_1Y_2}(011) + P_{A_1C_1Y_1Y_2}(1010) + P_{A_1C_2Y_1Y_2}(0010)$$

Holds for marginals of any distribution over $A_1B_2C_1C_2Y_1Y_2$

$$P_{B_2C_1Y_1Y_2}(0010) \leq P_{B_2C_2Y_1Y_2}(011) + P_{A_1C_1Y_1Y_2}(1010) + P_{A_1C_2Y_1Y_2}(0010)$$

Holds for marginals of any distribution over $A_1B_2C_1C_2Y_1Y_2$



Causal structure

$$B_2C_1Y_2 \perp\!\!\!\perp_d Y_1 \implies P_{B_2C_1Y_2Y_1} = P_{B_2C_1Y_2}P_{Y_1}$$

$$B_2C_2Y_2 \perp\!\!\!\perp_d Y_1 \implies P_{B_2C_2Y_2Y_1} = P_{B_2C_2Y_2}P_{Y_1}$$

$$A_1C_1Y_1 \perp\!\!\!\perp_d Y_2 \implies P_{A_1C_1Y_1Y_2} = P_{A_1C_1Y_1}P_{Y_2}$$

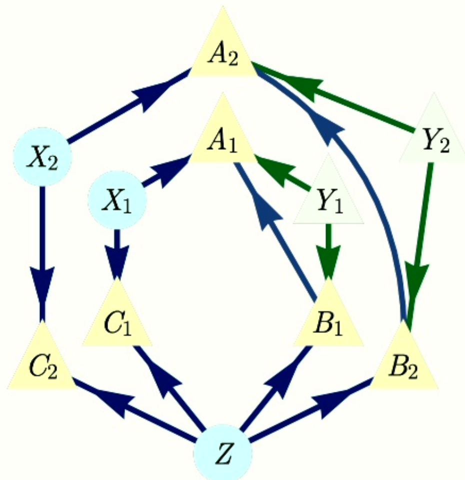
$$A_1C_2Y_1 \perp\!\!\!\perp_d Y_2 \implies P_{A_1C_2Y_1Y_2} = P_{A_1C_2Y_1}P_{Y_2}$$

$$P_{B_2C_1Y_2}(000)P_{Y_1}(1)$$
$$\leq P_{B_2C_2Y_2}(01)P_{Y_1}(1) + P_{A_1C_1Y_1}(101)P_{Y_2}(0) + P_{A_1C_2Y_1}(00)P_{Y_2}(0)$$

Holds for marginals of any distribution compatible with this structure

Not injectable

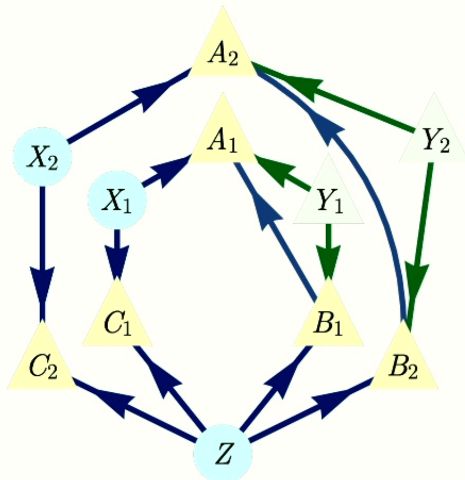
$$P_{B_2C_1Y_2}(000)P_{Y_1}(1)$$
$$\leq P_{B_2C_2Y_2}(010)P_{Y_1}(1) + P_{A_1C_1Y_1}(101)P_{Y_2}(0) + P_{A_1C_2Y_1}(001)P_{Y_2}(0)$$



$$P_{A_1C_2Y_1}(acy) = \sum_b P_{A_1C_2B_1Y_1}(abcy)$$

$$A_1 \perp_d C_2 | B_1 Y_1 \implies P_{A_1C_2B_1Y_1} = \frac{P_{A_1B_1Y_1}P_{C_2B_1Y_1}}{P_{B_1Y_1}}$$

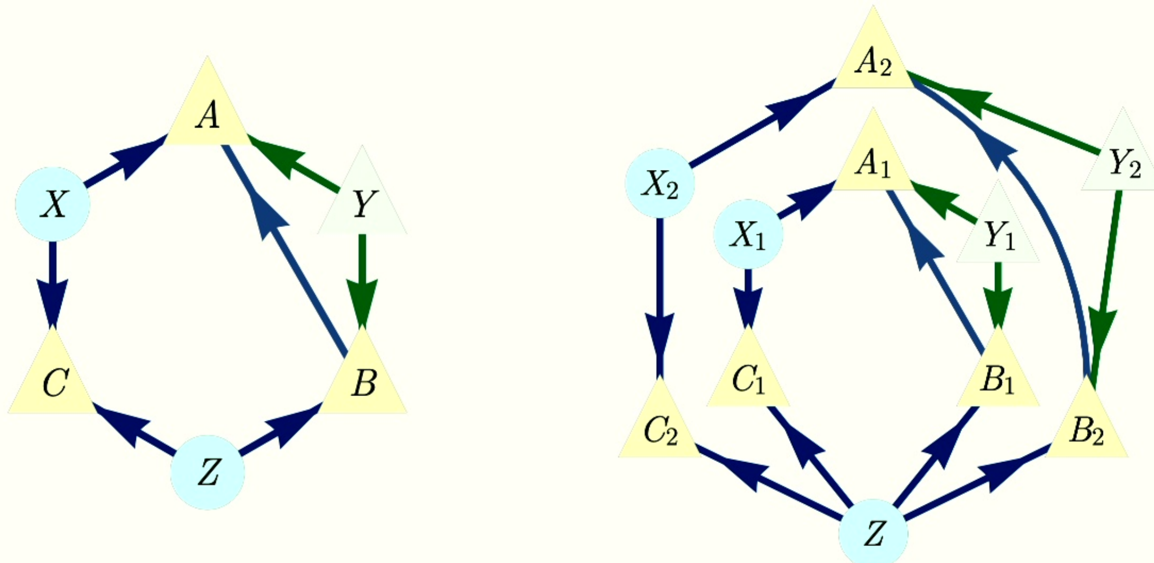
$$P_{B_2C_1Y_2}(000)P_{Y_1}(1)$$
$$\leq P_{B_2C_2Y_2}(010)P_{Y_1}(1) + P_{A_1C_1Y_1}(101)P_{Y_2}(0) + P_{A_1C_2Y_1}(001)P_{Y_2}(0)$$



$$P_{A_1C_2Y_1}(acy) = \sum_b P_{A_1C_2B_1Y_1}(abcy)$$

$$A_1 \perp_d C_2 | B_1Y_1 \implies P_{A_1C_2B_1Y_1} = \frac{P_{A_1B_1Y_1}P_{C_2B_1Y_1}}{P_{B_1Y_1}}$$

$$P_{B_2C_1Y_2}(00)P_{Y_1}(1)$$
$$\leq P_{B_2C_2Y_2}(010)P_{Y_1}(1) + P_{A_1C_1Y_1}(101)P_{Y_2}(0) + \sum_b \frac{P_{A_1B_1Y_1}(0b1)P_{B_1C_2Y_1}(0b1)}{P_{B_1Y_1}(b1)} P_{Y_2}(0)$$



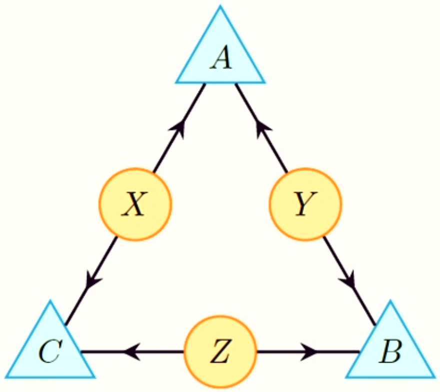
$$P_{BCY}(00)P_Y(1)$$

$$\leq P_{BCY}(010)P_Y(1) + P_{ACY}(10)P_Y(0) + \sum_b \frac{P_{ABY}(0b1)P_{BCY}(0b1)}{P_{BY}(b1)} P_Y(0)$$

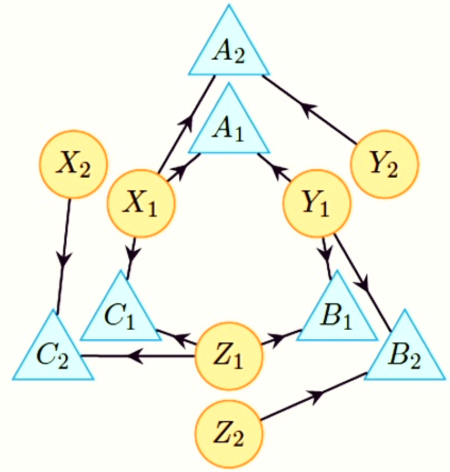
violated by

$$P_{ABCY} := \frac{[0000] + [0110] + [0001] + [1011]}{4}$$

Causal model M



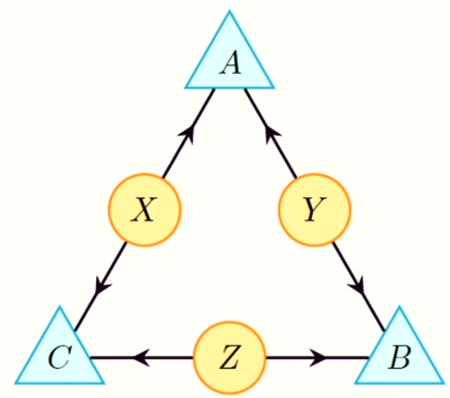
M’ = G → G’ Inflation of M



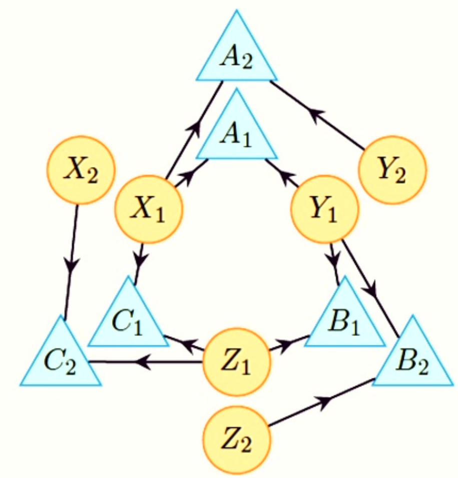
$\{A_1, A_2, B_1\}$ and $\{A_1, A_2, B_2\}$ are not injectable sets

But they satisfy additional equalities due to symmetries of the inflation

Causal model M



M' = G → G' Inflation of M

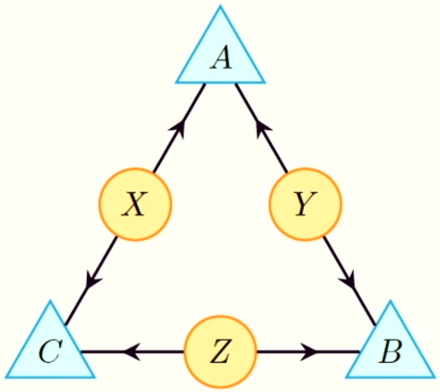


$\{A_1, A_2, B_1\}$ and $\{A_1, A_2, B_2\}$ are not injectable sets

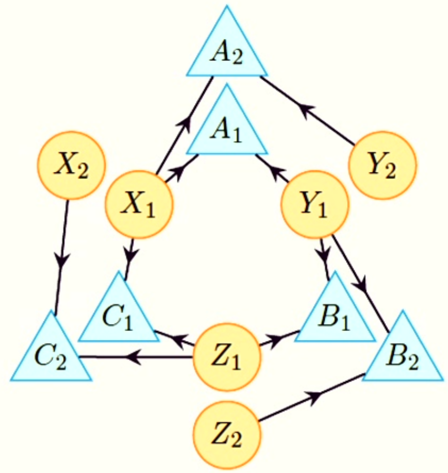
But they satisfy additional equalities due to symmetries of the inflation

$$\mathrm{P}_{A_1 A_2 B_1}(aa'b) = \mathrm{P}_{A_1 A_2 B_2}(aa'b)$$

Causal model M



M' = G → G' Inflation of M



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$$\mathrm{P}_{A_1 A_2 B_2}(aa'b) = \mathrm{P}_{A_2 A_1 B_2}(aa'b)$$

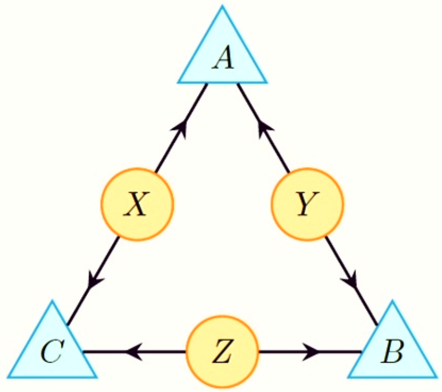
Recall effect of symmetries in deriving marginal compatibility constraints

$\exists Q_{XYZ}$ with $P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ}$ as marginals

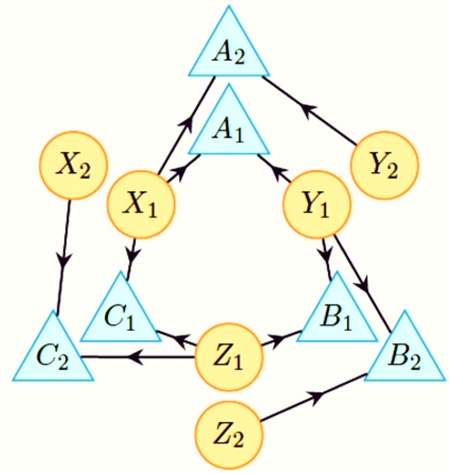
$$\forall x, y : P_{XY}(xy) = \sum_z Q_{XYZ}(xyz)$$
$$\forall y, z : P_{YZ}(yz) = \sum_x Q_{XYZ}(xyz)$$
$$\forall x, z : P_{XZ}(xz) = \sum_y Q_{XYZ}(xyz)$$
$$\forall x : P_X(x) = \sum_{y,z} Q_{XYZ}(xyz)$$
$$\forall y : P_Y(y) = \sum_{x,z} Q_{XYZ}(xyz)$$
$$\forall z : P_Z(z) = \sum_{x,y} Q_{XYZ}(xyz)$$
$$\forall x, y, z : 0 \leq Q_{XYZ}(xyz) \leq 1$$
$$\forall a, a' : Q_{XYZ}(xaa') = Q_{XYZ}(xa'a)$$

Still linear quantifier elimination!

Causal model M



M' = G $\rightarrow$ G' Inflation of M



$\{A_1, A_2, B_1\}$ and $\{A_1, A_2, B_2\}$ are not injectable sets

But they satisfy additional equalities due to symmetries of the inflation

$$\mathrm{P}_{A_1 A_2 B_1}(aa'b) = \mathrm{P}_{A_1 A_2 B_2}(aa'b)$$

$$\mathrm{P}_{A_1 A_2 B_1}(aa'b) = \mathrm{P}_{A_1 A_2 B_2}(a'ab)$$

$$\mathrm{P}_{A_1 A_2 B_2}(aa'b) = \mathrm{P}_{A_2 A_1 B_2}(aa'b)$$

Polynomial inequality constraints for causal compatibility with the original DAG

⇐

Polynomial equality constraints for causal compatibility with the inflated DAG (from d-separation relations)

+

Linear inequality constraints for marginal compatibility, potentially including symmetry constraints on the joint distribution (from linear quantifier elimination)

Next lecture: More on inflation