Title: Strong Gravity Lecture - 230320

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Collection: Strong Gravity (2022/2023)

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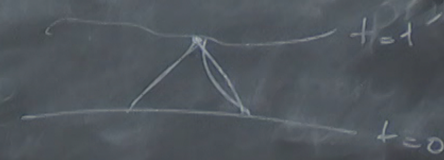
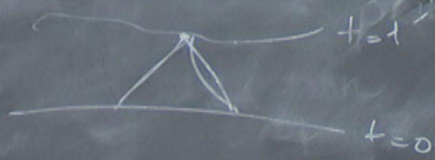
Types of Partial Differential Equations



Wave Eqn $\qquad \left(-\partial_t^2 + \partial_x^2\right) u = 0 \qquad$ Hyperbolic PDE

Laplace Eqn $\qquad \left(\partial_x^2 + \partial_y^2\right) u = 0 \qquad$ Elliptic PDE

Heat Eqn $\qquad \left(\partial_t - \partial_x^2\right) u = 0 \qquad$ Parabolic PDE



Hyperbolic PDE

Elliptic PDE

Parabolic PDE

General Second, linear PDE

$$a\partial_x^2 u + b\,\partial_x\partial_y u + c\,\partial_y^2 u + d\,\partial_x u + e\,\partial_y u + fu = 0$$

$b^2 - 4ac > 0$ hyperbolic $\quad (-\partial_x^2 + \partial_y^2)\tilde{u} = \text{l.o.t.}$

$= 0$ parabolic $\quad \partial_x^2 \tilde{u} = \text{l.o.t.}$

< 0 elliptic $\quad (\partial_x^2 + \partial_y^2)\tilde{u} = \text{l.o.t.}$

$\vec{u}$

Consider $\partial_t \vec{u} + \underline{A}\,\partial_x \vec{u} + \underline{B}\,\vec{u} = 0$ (*)

$\underline{A}$ has all real eigenvectors $\Rightarrow$ (*) weakly hyperbolic

Also have complete set of eigenvectors $\Rightarrow$ (*) strongly hyperbolic

Well posedness

$$\|u(t)\| \leq K e^{\alpha t} \|u(t=0)\|$$

independent of $u(t=0)$

Well posedness

$$\|u(t)\| \leq K e^{\alpha t} \|u(t=0)\|$$

independent of $u(t=0)$

... eigenvectors $\Rightarrow$ $(*)$ strongly hyperbolic

$$\vec{u} = \begin{pmatrix} u \\ \partial_t u \\ \partial_x u \end{pmatrix} \qquad \partial_t \vec{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \partial_x \vec{u} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{u}$$

Eigenvalues: $0, \pm 1$, Eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$(\partial_t - \partial_x)u$, $(\partial_t + \partial_x)u$

$\underline{A}$ has all real eigenvectors $\Rightarrow$ (*) weakly hyperbolic

Also have complete set of eigenvectors $\Rightarrow$ (*) strongly hyperbolic

Example Wave Eqn $\quad \vec{u} = \begin{pmatrix} u \\ \partial_t u \\ \partial_x u \end{pmatrix}$

$$\partial_t \vec{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \partial_x \vec{u} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{u}$$

Eigenvalues: $0, \pm 1$, Eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$(\partial_t - \partial_x)u$, $(\partial_t + \partial_x)u$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{u}$$

, Eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$(\partial_t - \partial_x)u$, $(\partial_t + \partial_x)u$

Laplace Eqn

$$\underline{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$(\partial_t^2 + \partial_x^2)$

Eigenvalues: $0, \pm i$

Have $\frac{1}{\omega}$ wavelength $\sim e^{\omega t}$

→ learn conditional dist. $(y|x)$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{u}$$

Eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$(\partial_t - \partial_x)u \, , \, (\partial_t + \partial_x)u$

Laplace Eqn

$$\underline{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$(\partial_t^2 + \partial_x^2)$

Eigenvalues: $0, \pm i$

Have $\frac{1}{\omega}$ wavelength $\sim e^{\omega t + i\omega x}$

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix}$$

$$(A - \lambda \mathbb{I})^n \vec{v} \neq 0$$

$$(A - \lambda \mathbb{I})^{n+1} \vec{v} = 0$$

Eigenvalue 1 (multiplicity 2)

Eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} u \\ v \end{pmatrix} \sim \begin{pmatrix} \frac{1}{2} i\omega t \\ 1 \end{pmatrix} e^{i\omega(t+x)}$$

→ learn conditional dist. $P(y|x)$

$L_1 = 2$

$\vec{u}$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$(\partial_t - \partial_x)u, \quad (\partial_t + \partial_x)u$$

Laplace Eqn

$$\underline{A} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$$

Eigenvalues: $0, \pm i$

Have $\frac{1}{\omega}$ wavelength $\sim e^{\omega t + i\omega x}$

$$\partial_t \vec{u} + A\,\partial_x \vec{u} = 0$$

$$\sim e^{i(\omega t + kx)}$$

$$i\omega + \lambda i k = 0$$

$$(\partial_t^2 + \partial_x^2)$$

$\rightarrow$ learn conditional dist. $p(y|x)$

$\vec{u}$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$(\partial_t - \partial_x)u$, $(\partial_t + \partial_x)u$

Laplace Eqn

$$\underline{A} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$$

Eigenvalues: $0, \pm i$

Have $\frac{1}{\omega}$ wavelength $\sim e^{\omega t + i\omega x}$

$\partial_t \vec{u} + \underline{A}\,\partial_x \vec{u} = 0$

$\sim e^{i(\omega t + kx)}$

$i\omega + \lambda i k = 0$

$(\partial_t^2 + \partial_x^2)$

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix}$$

$$(A - \lambda \mathbb{I})^n \vec{v} \neq 0$$

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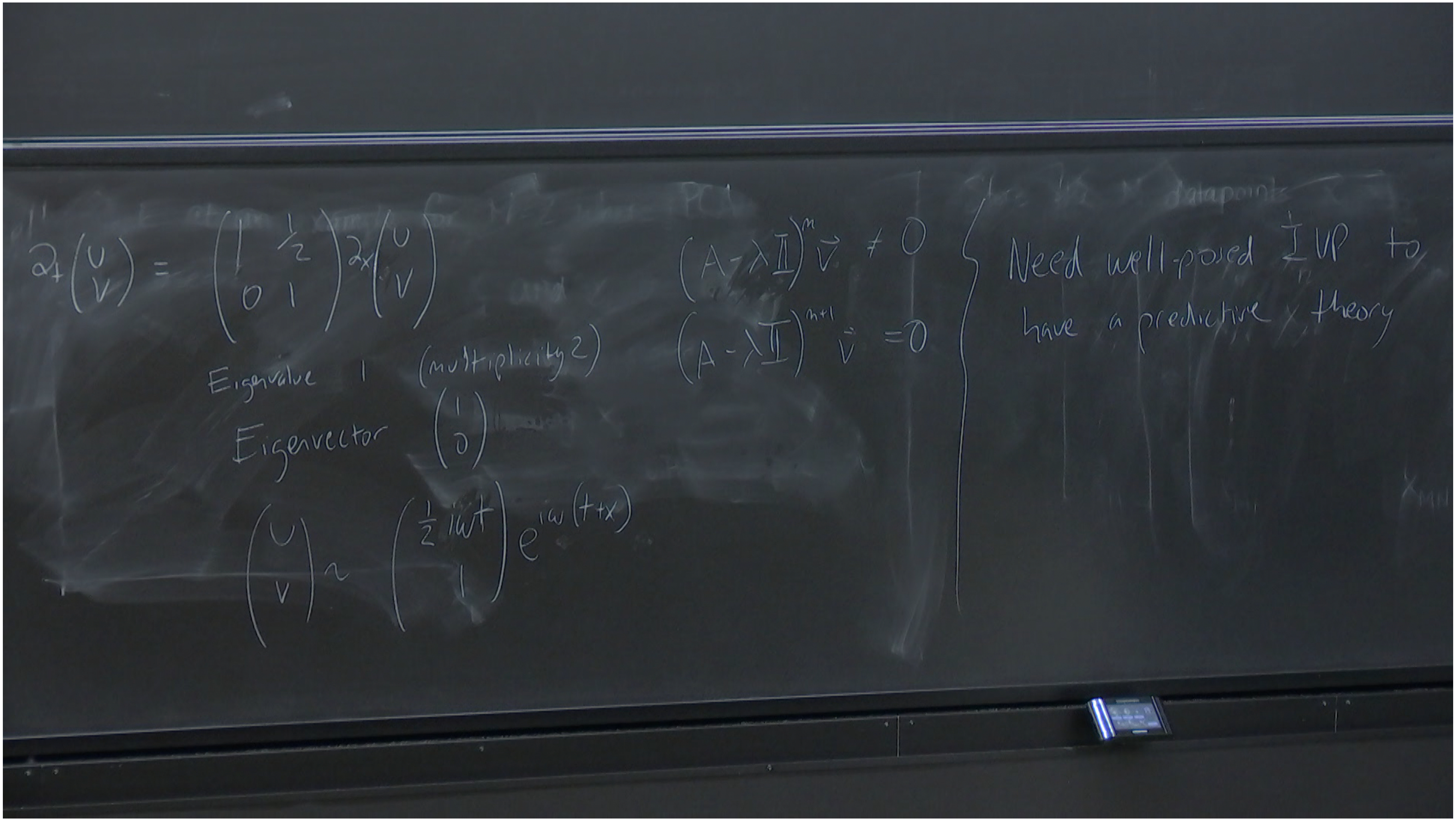
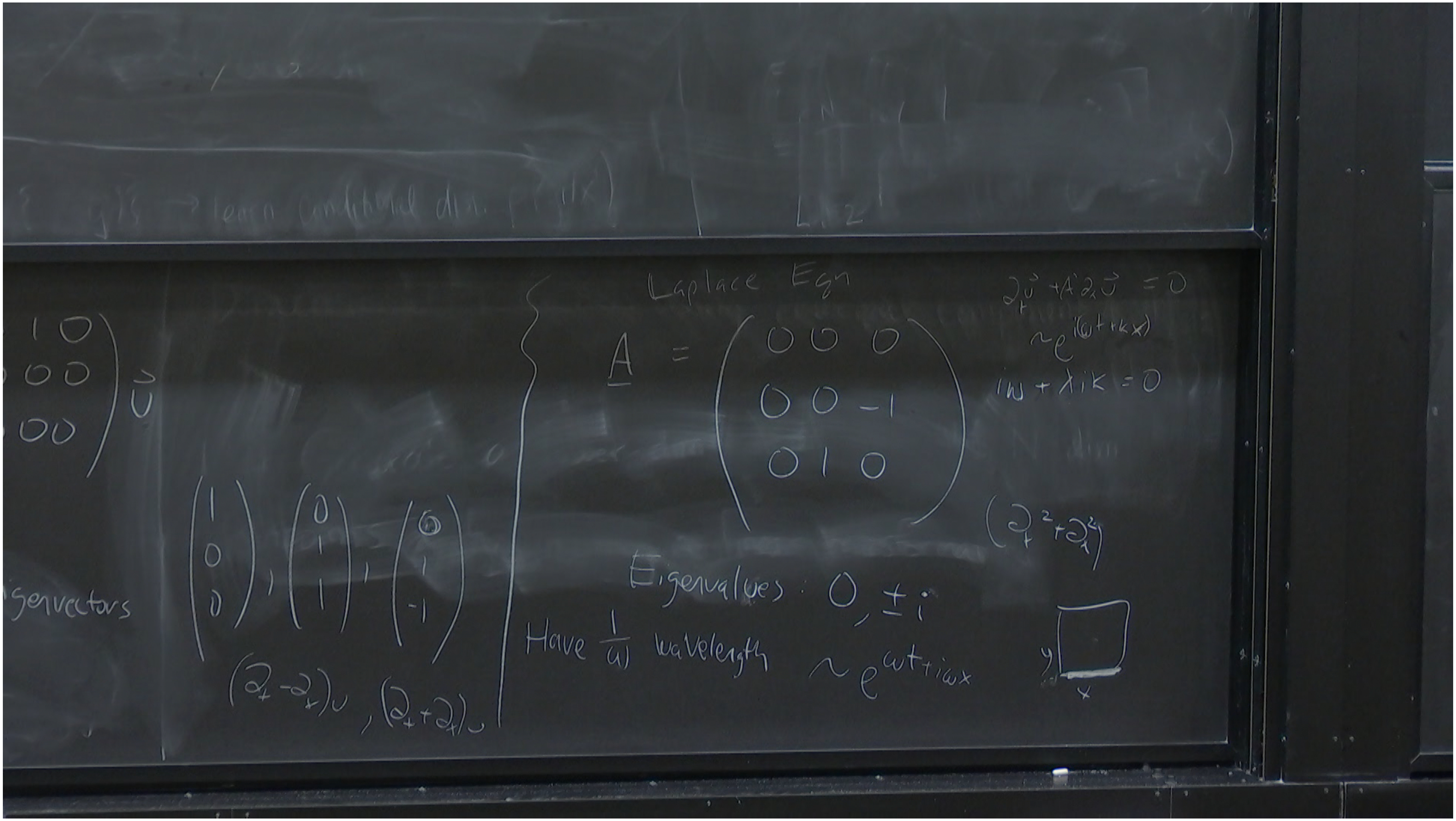
Eigenvalue 1 (multiplicity 2)

Eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} u \\ v \end{pmatrix} \sim \begin{pmatrix} \frac{1}{2} i\omega t \\ 1 \end{pmatrix} e^{i\omega(t+x)}$$


$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix}$
Eigenvalue 1 (multiplicity 2)
Eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\begin{pmatrix} u \\ v \end{pmatrix} \sim \begin{pmatrix} \frac{1}{2} i\omega t \\ 1 \end{pmatrix} e^{i\omega(t+x)}$
$(A - \lambda \mathbb{I})^n \vec{v} \neq 0$
$(A - \lambda \mathbb{I})^{n+1} \vec{v} = 0$
Need well-posed IVP to have a predictive theory

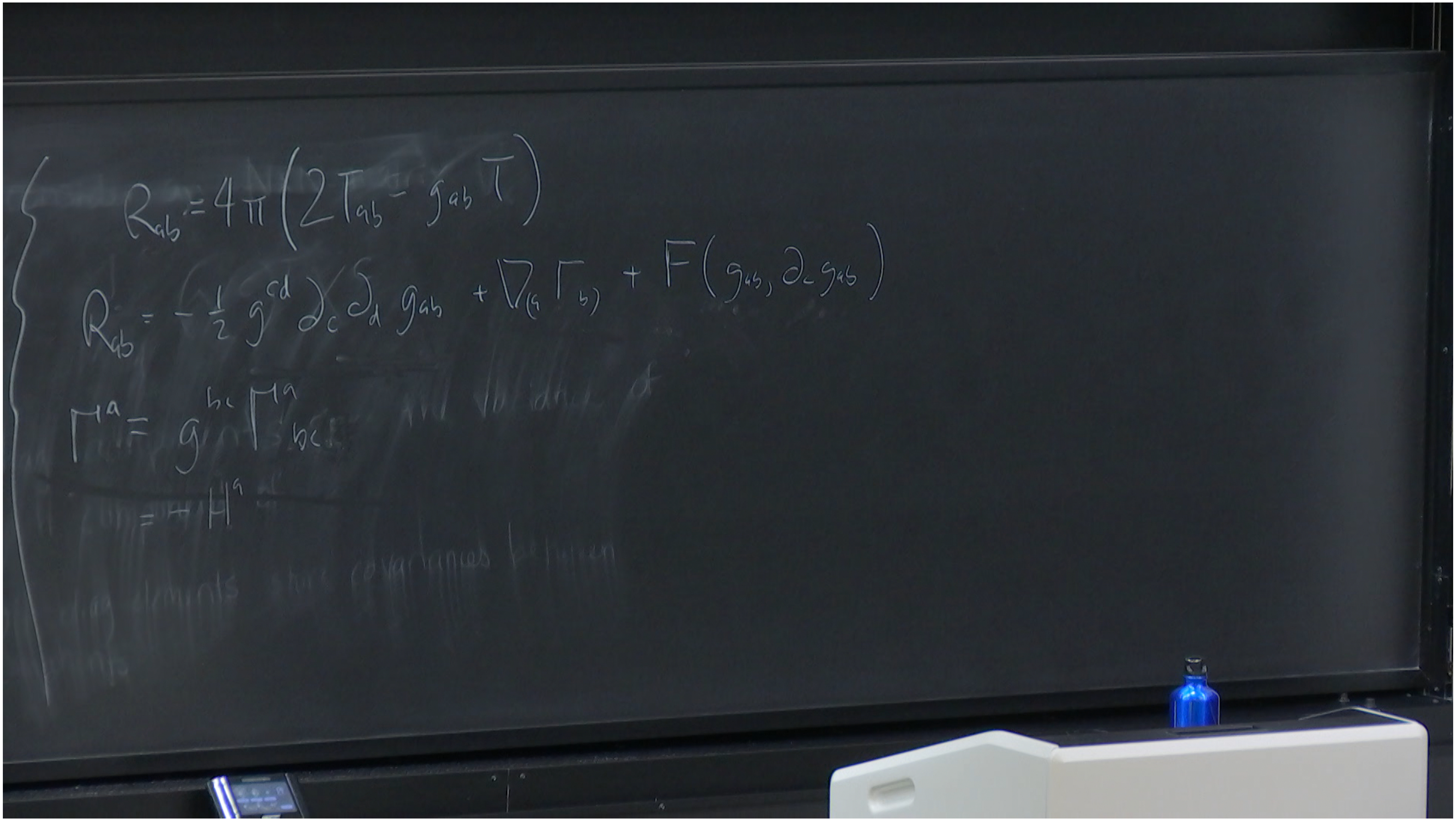

Laplace Eqn
Eigenvalues: 0, ±i
Have 1/ω wavelength

Generalized Harmonic Formulation of EFEs

Fix gauge

$$\Box x^a = \nabla_b \nabla^b x^a = H^a$$

Promote H^a to independent variables


$R_{ab} = 4\pi \left(2T_{ab} - g_{ab} T\right)$
$R_{ab} = -\frac{1}{2} g^{cd} \partial_c \partial_d g_{ab} + \nabla_{(a} \Gamma_{b)} + F(g_{ab}, \partial_c g_{ab})$
$\Gamma^a = g^{bc} \Gamma^a{}_{bc}$
$= -H^a$

GH Eqns: $8\pi\left(2T_{ab}-g_{ab}T\right) = \underbrace{g^{cd}\partial_c\partial_d g_{ab}}_{\text{principal part}} + \partial_b g^{cd}\partial_c g_{ad}$ (*)

$$+\partial_a g^{cd}\partial_c g_{bd} + \partial_a H_b + \partial_b H_a$$

$$-2H_d\Gamma^d{}_{ab} + 2\Gamma^c{}_{db}\Gamma^d{}_{ca}$$

$$L(H^a) = 0 \quad (**)$$

(*) Simple choice

$$H_a = \tilde{F}_a(g_{ab})$$

Simplest choice

$$H_a = 0$$

Evolve $\{ g_{ab}, \partial_t g_{ab}, H_a, \partial_t H_a \}$

$$C^a = H^a - \Box x^a$$

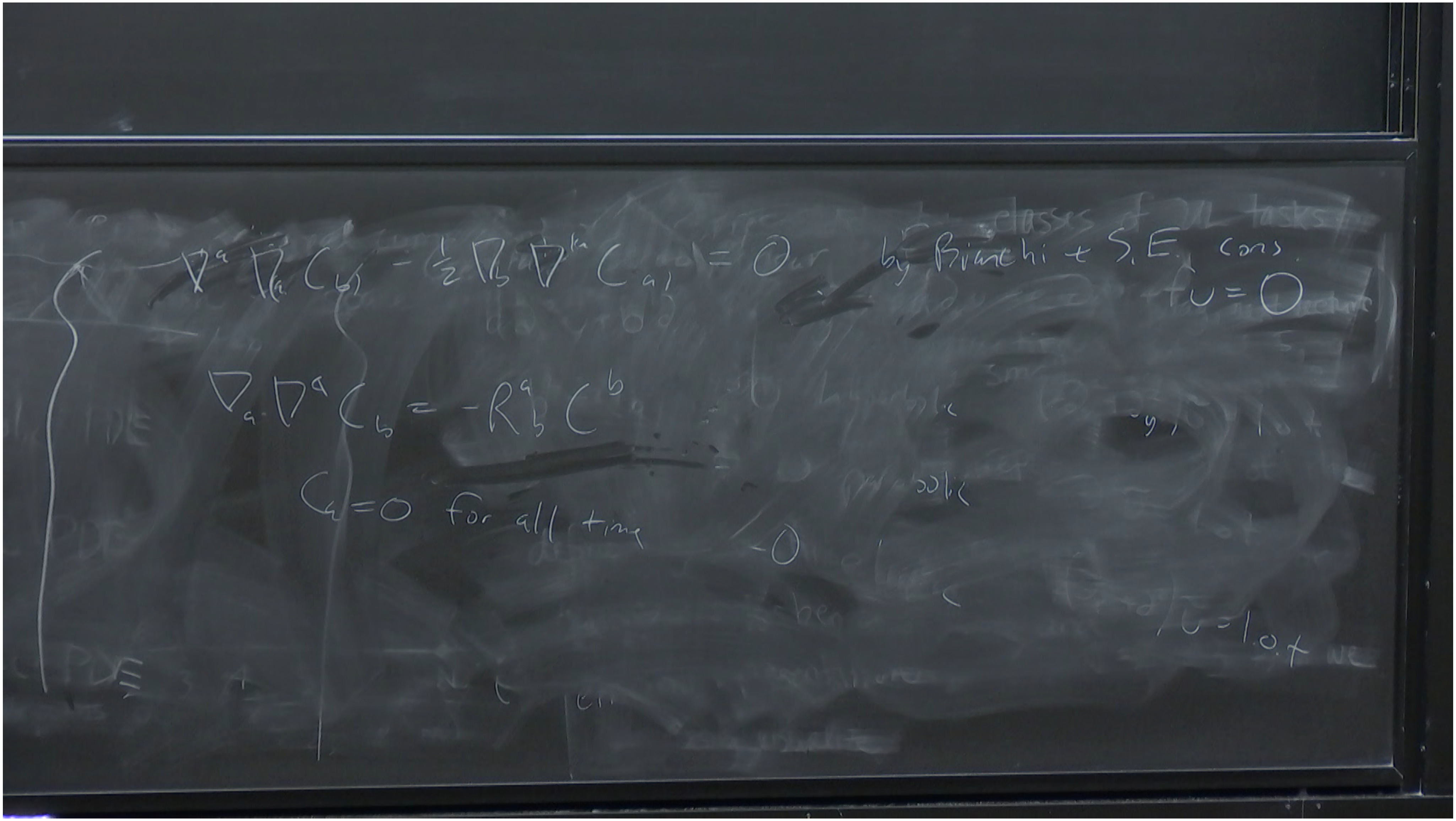
From (*) $\quad R_{ab} - 4\pi\left(2T_{ab} - g_{ab}T\right) = \nabla_{(a}C_{b)}$

$n^a n^b$

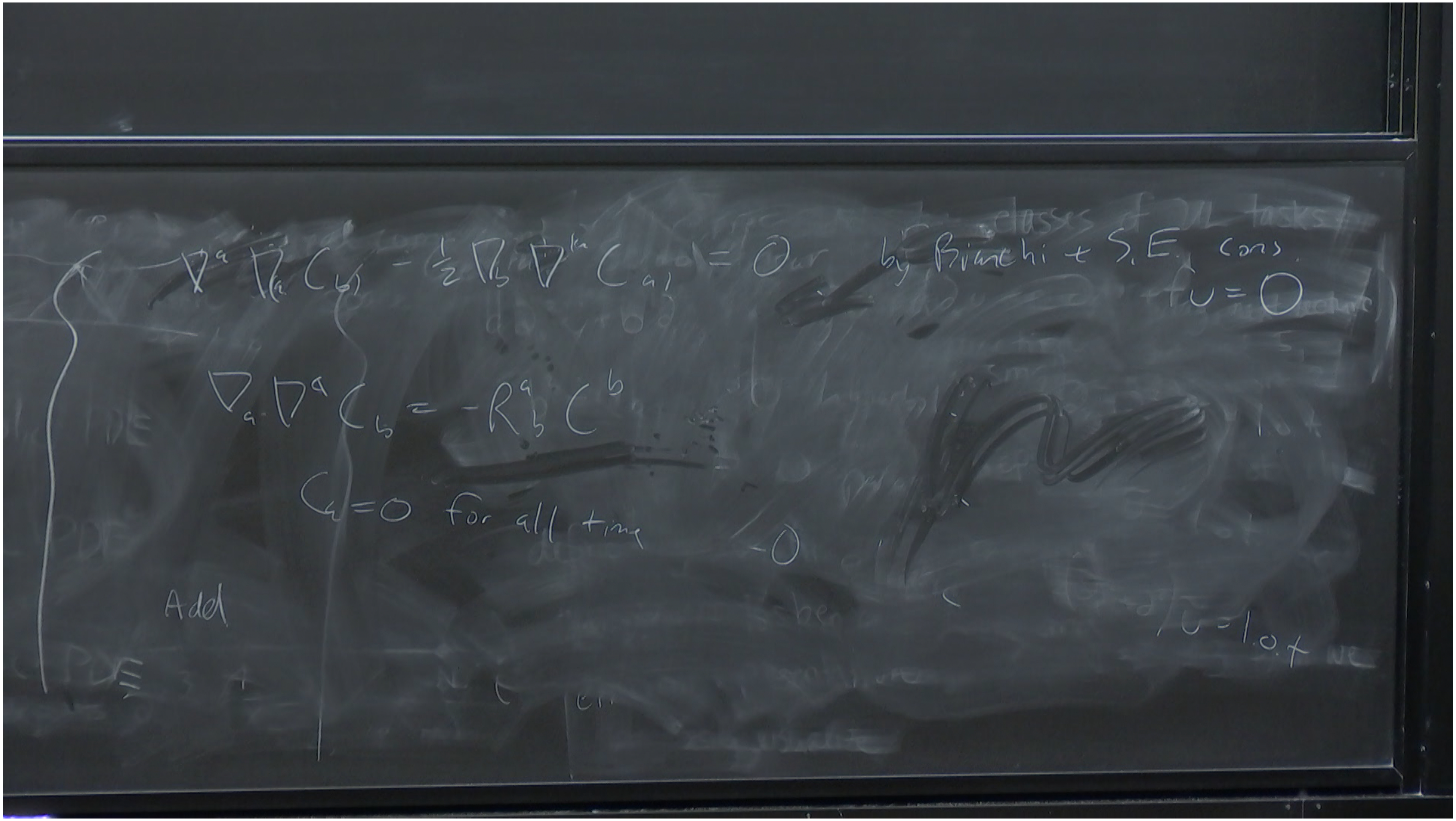
$\gamma^{ab} n^c$

$C_a = \partial_t C_a = 0 \iff$ Hamiltonian/Mom Constraint at $t=0$

$$\left(R_{ab} - \frac{1}{2}R g_{ab}\right) - 8\pi T_{ab} = \nabla_{(a}C_{b)} - \frac{1}{2} g_{ab} g^{cd} \nabla_{(c}C_{d)}$$



$\nabla^a \nabla_{(a} C_{b)} - \frac{1}{2} \nabla_b \nabla^{a} C_{a} = 0$
by Bianchi + S.E. cons.
$u = 0$
$\nabla_a \nabla^a C_b = -R^a_b C^b$
$C_a = 0$ for all time


$\nabla^a \nabla_{[a} C_{b]} - \frac{1}{2} \nabla_b \nabla^{ka} C_{a]} = 0$
by Bianchi + S.E. cons
$\nabla_a \nabla^a C_b = -R^a_b C^b$
$C_b = 0$ for all time
Add