Title: Strong Gravity Lecture - 230321

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Collection: Strong Gravity (2022/2023)

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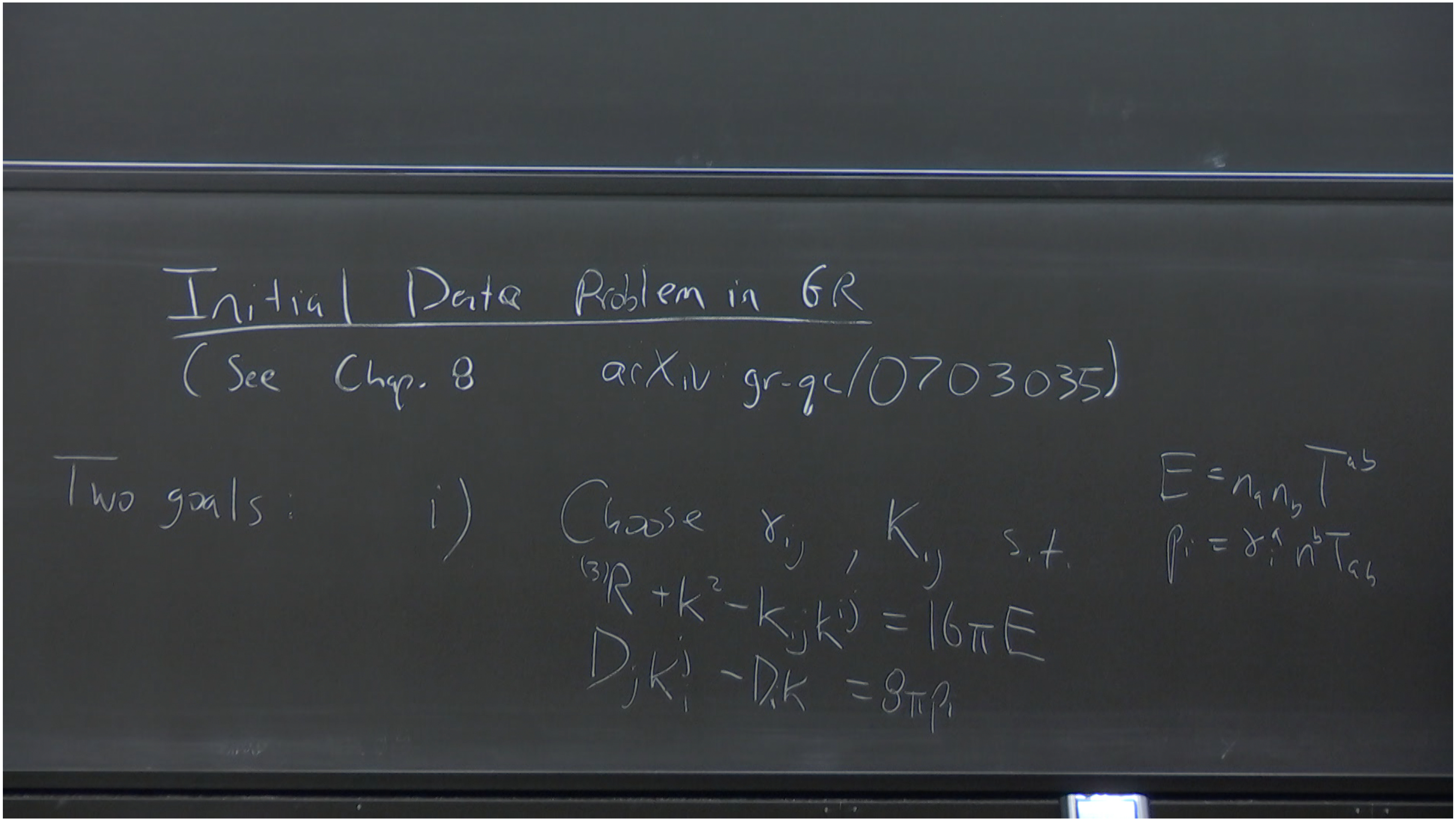
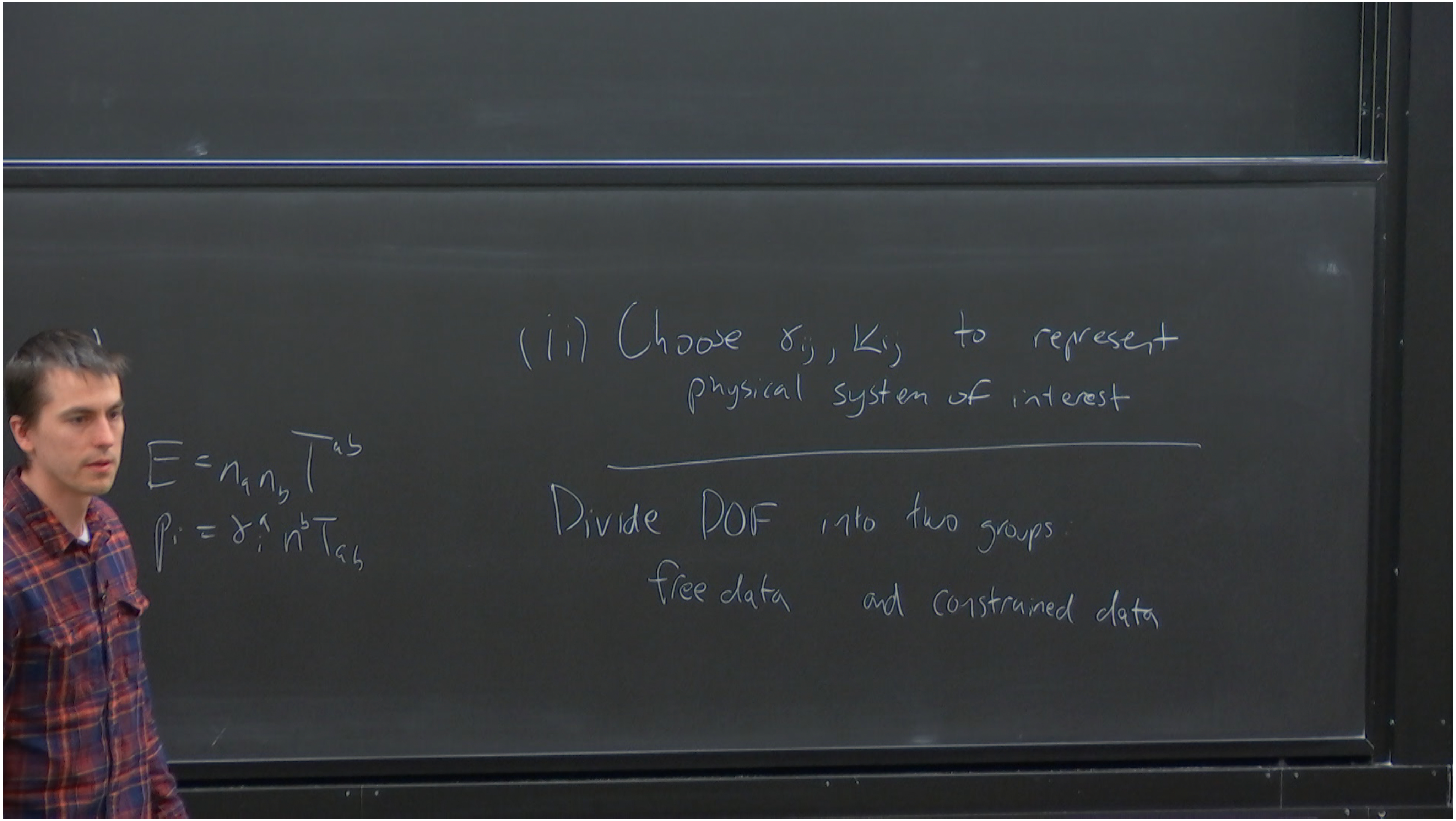
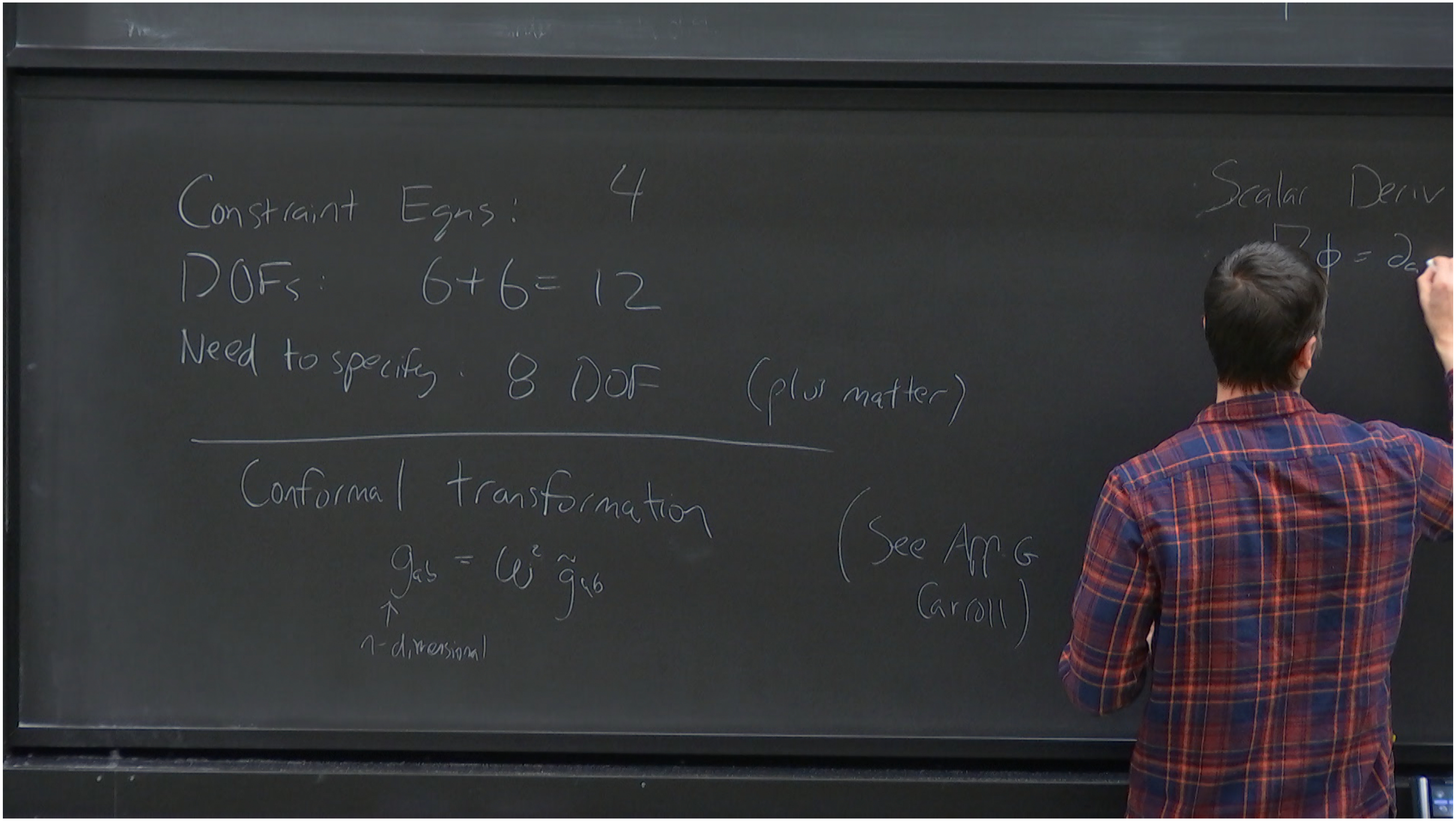
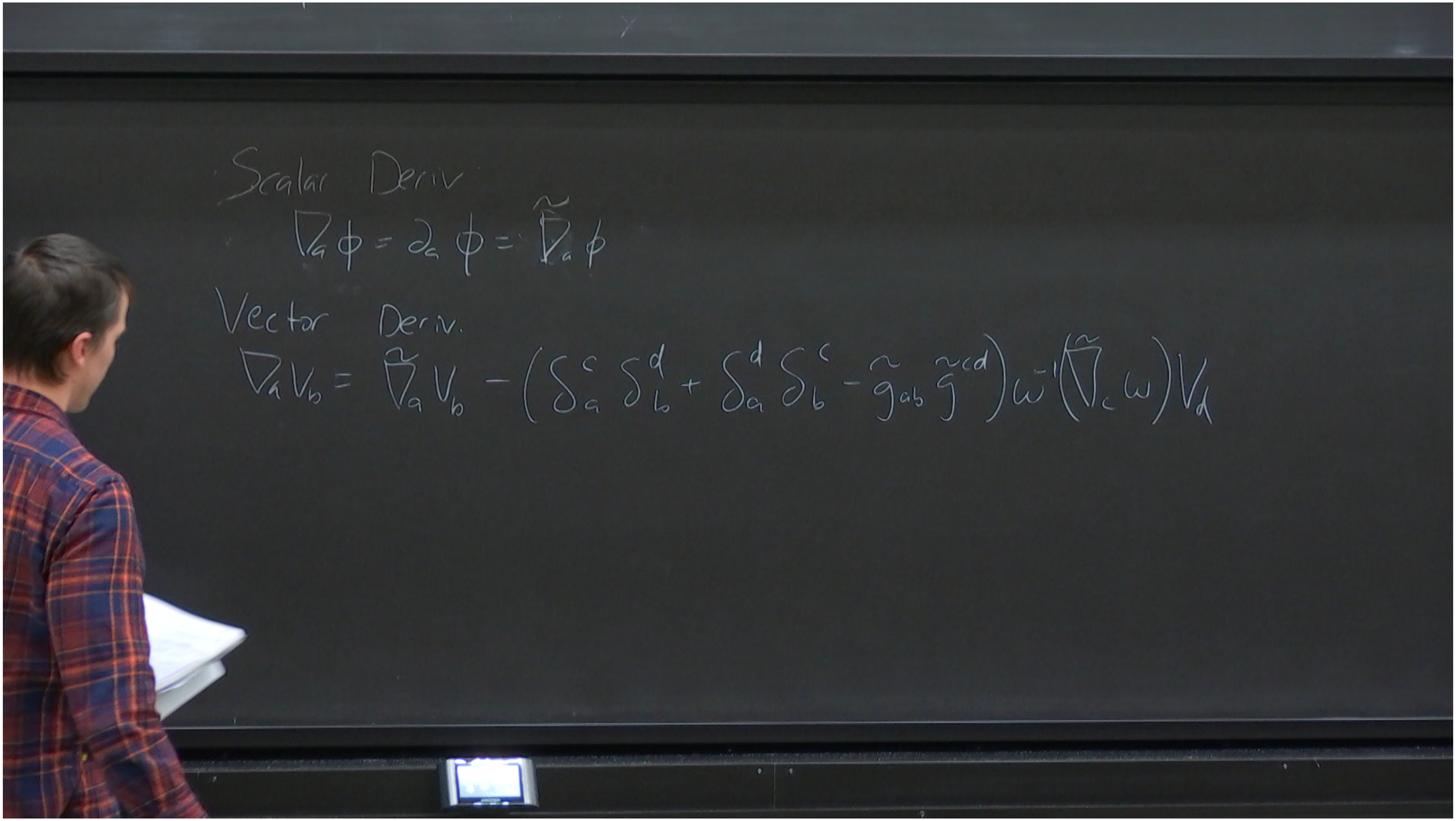

Initial Data Problem in GR
(See Chap. 8 arXiv: gr-qc/0703035)
Two goals:
i) Choose γ_{ij}, K_{ij} s.t.
${}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi E$
$D_j K^j_i - D_i K = 8\pi p_i$
$E = n_a n_b T^{ab}$
$p_i = \gamma_i^a n^b T_{ab}$


$E = n_a n_b T^{ab}$
$p_i = \gamma_i^a n^b T_{ab}$
(ii) Choose γ_{ij}, K_{ij} to represent
physical system of interest
Divide DOF into two groups:
free data and constrained data


Constraint Eqns: 4
DOFs: 6+6=12
Need to specify: 8 DOF (plus matter)
Conformal transformation
$g_{ab} = \omega^2 \tilde{g}_{ab}$
n-dimensional
(See App. G Carroll)
Scalar Deriv
$\nabla \phi = \partial_a$



Scalar Deriv
$\nabla_a \phi = \partial_a \phi = \tilde{\nabla}_a \phi$
Vector Deriv.
$\nabla_a V_b = \tilde{\nabla}_a V_b - \left(\delta_a^c \delta_b^d + \delta_a^d \delta_b^c - \tilde{g}_{ab} \tilde{g}^{cd}\right) \omega^{-1} \left(\tilde{\nabla}_c \omega\right) V_d$

Scalar Deriv

$$\nabla_a \phi = \partial_a \phi = \tilde{\nabla}_a \phi$$

Vector Deriv.

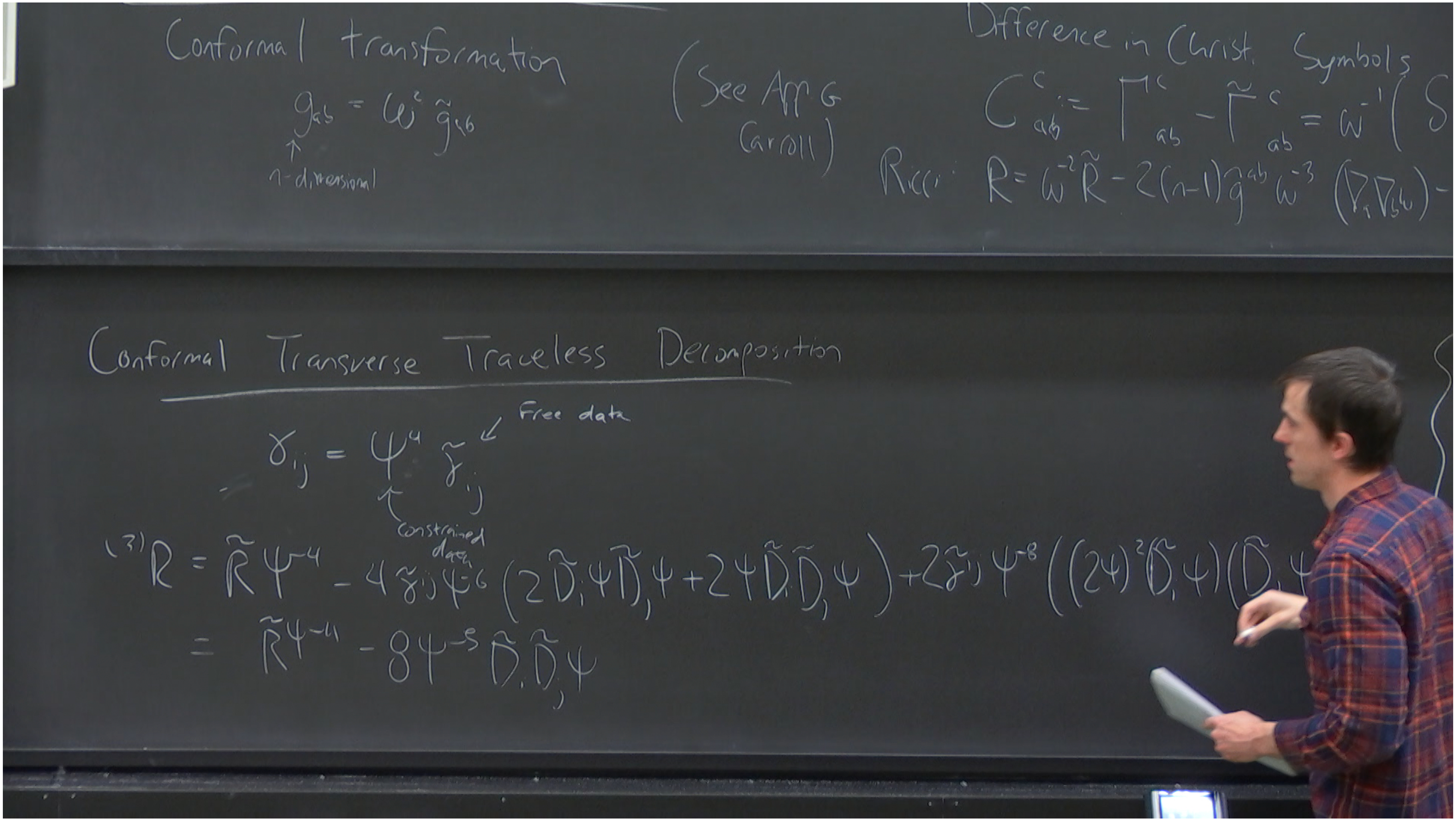
$$\nabla_a V_b = \tilde{\nabla}_a V_b - \left(\delta_a^c \delta_b^d + \delta_a^d \delta_b^c - \tilde{g}_{ab} \tilde{g}^{cd}\right) \omega^{-1} \left(\tilde{\nabla}_c \omega\right) V_d$$

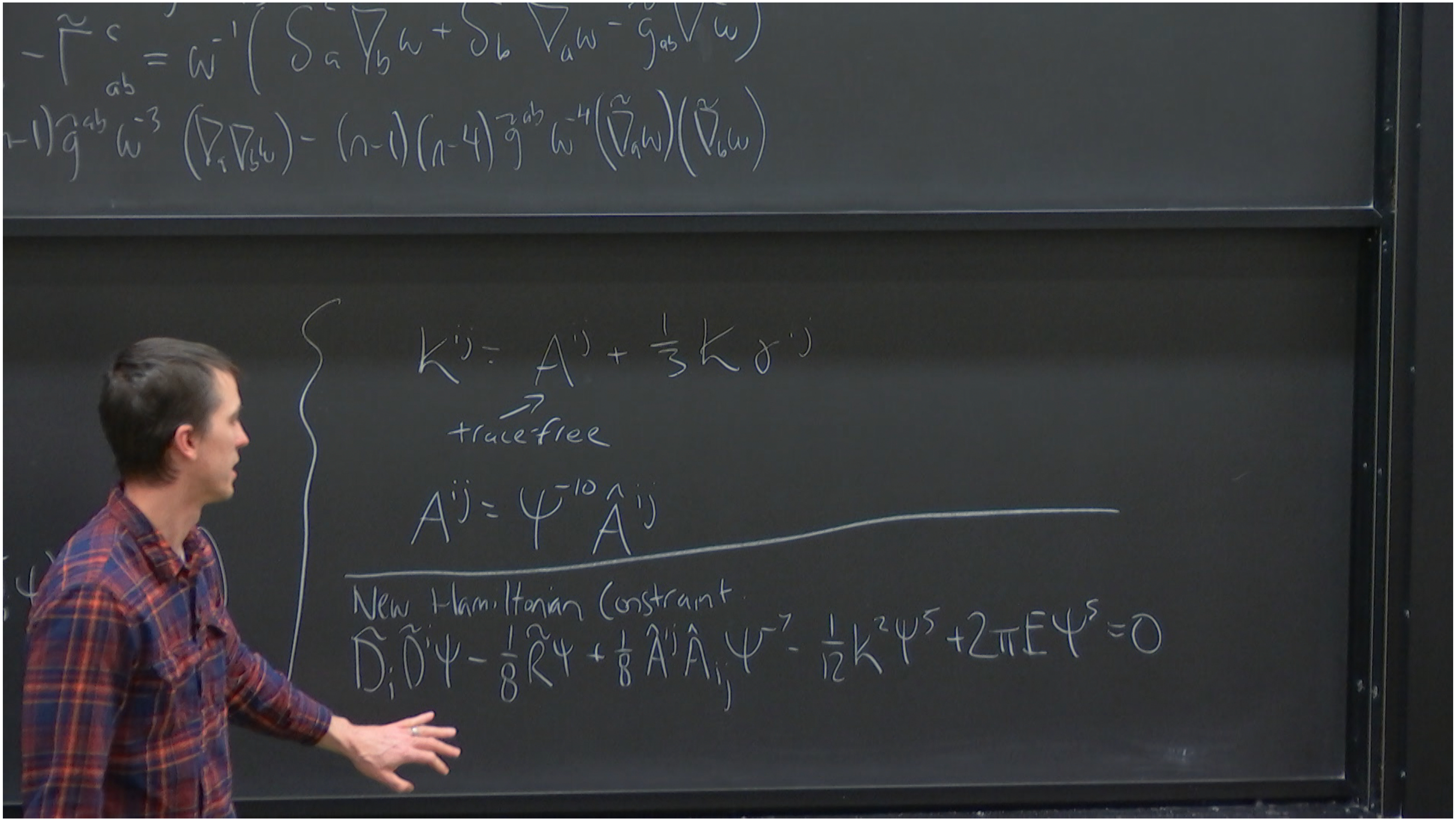
Difference in Christ. Symbols

$$C^c_{ab} := \Gamma^c_{ab} - \tilde{\Gamma}^c_{ab} = \omega^{-1}\left(\delta^c_a \tilde{\nabla}_b \omega + \delta^c_b \tilde{\nabla}_a \omega - \tilde{g}_{ab} \tilde{\nabla}^c \omega\right)$$

Ricci

$$R = \omega^{-2} \tilde{R} - 2(n-1) \tilde{g}^{ab} \omega^{-3} \left(\tilde{\nabla}_a \tilde{\nabla}_b \omega\right) - (n-1)(n-4) \tilde{g}^{ab} \omega^{-4} \left(\tilde{\nabla}_a \omega\right)\left(\tilde{\nabla}_b \omega\right)$$


Conformal transformation
$g_{ab} = \omega^2 \tilde{g}_{ab}$
n-dimensional
(See App G Carroll)
Difference in Christ. Symbols
$C^c_{ab} := \Gamma^c_{ab} - \tilde{\Gamma}^c_{ab} = \omega^{-1}(S$
Ricci: $R = \omega^{-2}\tilde{R} - 2(n-1)\tilde{g}^{ab}\omega^{-3}(\tilde{\nabla}_a\tilde{\nabla}_b\omega) -$
Conformal Transverse Traceless Decomposition
$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$
Free data
constrained data
${}^{(3)}R = \tilde{R}\psi^{-4} - 4\tilde{\gamma}^{ij}\psi^{-6}(2\tilde{D}_i\psi\tilde{D}_j\psi + 2\psi\tilde{D}_i\tilde{D}_j\psi) + 2\tilde{\gamma}^{ij}\psi^{-8}((2\psi)^2(\tilde{D}_i\psi)(\tilde{D}_j\psi$
$= \tilde{R}\psi^{-4} - 8\psi^{-5}\tilde{D}_i\tilde{D}_j\psi$



$-\tilde{\Gamma}^c_{ab} = \omega^{-1}(\delta^c_a \nabla_b \omega + \delta^c_b \nabla_a \omega - g_{ab}\nabla^c \omega)$
$(n-1)\tilde{g}^{ab}\omega^{-3}(\nabla_a\nabla_b\omega) - (n-1)(n-4)\tilde{g}^{ab}\omega^{-4}(\tilde{\nabla}_a\omega)(\tilde{\nabla}_b\omega)$
$K^{ij} = A^{ij} + \frac{1}{3}K\gamma^{ij}$
trace-free
$A^{ij} = \psi^{-10}\hat{A}^{ij}$
New Hamiltonian Constraint
$\tilde{D}_i\tilde{D}^i\psi - \frac{1}{8}\tilde{R}\psi + \frac{1}{8}\hat{A}^{ij}\hat{A}_{ij}\psi^{-7} - \frac{1}{12}K^2\psi^5 + 2\pi E\psi^5 = 0$

$$= \omega^{-1}\left(\delta_a \nabla_b \omega + \delta_b \nabla_a \omega - g_{ab} \nabla \omega \right)$$

$$(\nabla_a \nabla_b \omega) - (n-1)(n-4)\tilde{g}^{ab}\omega^{-4}(\tilde{\nabla}_a \omega)(\tilde{\nabla}_b \omega)$$

$$K^{ij} = A^{ij} + \frac{1}{3} K \gamma^{ij}$$

trace-free

free data

$$A^{ij} = \psi^{-10} \hat{A}^{ij}$$

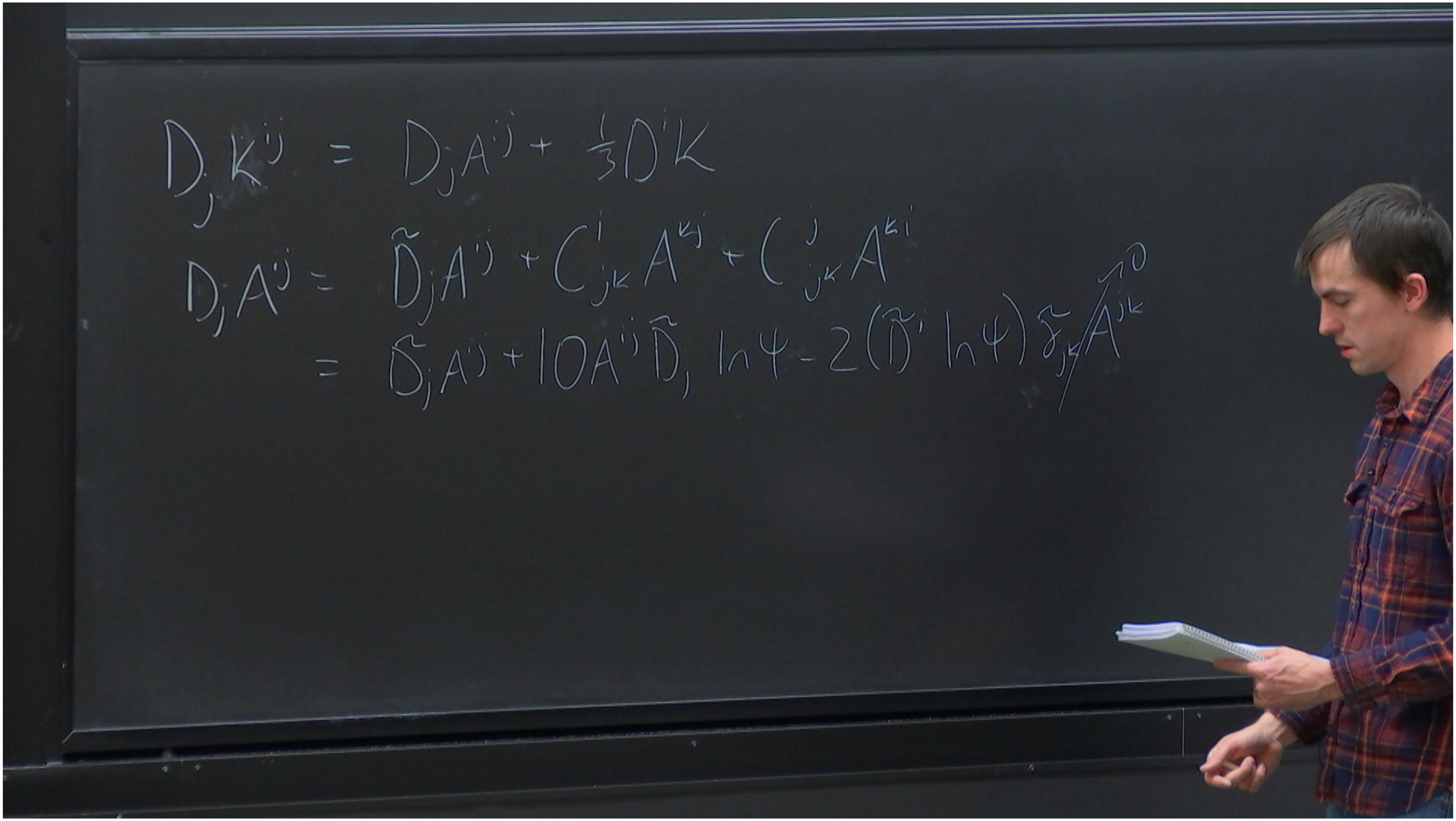
New Hamiltonian Constraint

$$\hat{D}_i \hat{D}^i \psi - \frac{1}{8}\tilde{R}\psi + \frac{1}{8}\hat{A}^{ij}\hat{A}_{ij}\psi^{-7} - \frac{1}{12}K^2\psi^5 + 2\pi E \psi^5 = 0$$

$$D_j K^{ij} = D_j A^{ij} + \frac{1}{3} D^i K$$

$$D_j A^{ij} = \tilde{D}_j A^{ij} + C^i_{jk} A^{kj} + C^j_{jk} A^{ik}$$

$$= \tilde{D}_j A^{ij} + 10 A^{ij} \tilde{D}_j \ln\psi - 2 (\tilde{D}^i \ln\psi) \tilde{\gamma}_{jk} A^{jk}$$


$D_j K^{ij} = D_j A^{ij} + \frac{1}{3} D^i K$
$D_j A^{ij} = \tilde{D}_j A^{ij} + C^i_{jk} A^{kj} + C^j_{jk} A^{ki}$
$= \tilde{D}_j A^{ij} + 10 A^{ij} \tilde{D}_j \ln\psi - 2 (\tilde{D}^i \ln\psi) \cancelto{0}{\tilde{\gamma}_{jk} A^{jk}}$

$$D_j K^{ij} = D_j A^{ij} + \frac{1}{3} D^i K$$

$$\tilde{\gamma}_{ij} \hat{A}^{ij} =$$

$$D_j A^{ij} = \tilde{D}_j A^{ij} + C^i_{jk} A^{kj} + C^j_{jk} A^{ki}$$

$$= \tilde{D}_j A^{ij} + 10 A^{ij} \tilde{D}_j \ln\psi - 2(\tilde{D}^i \ln\psi) \cancelto{0}{\tilde{\gamma}_{jk} A^{jk}}$$

$$= \psi^{-10} \tilde{D}_j (\psi^{10} A^{ij}) = \psi^{-10} \tilde{D}_j \hat{A}^{ij}$$

Further Decompose: $\hat{A}^{ij} = (\tilde{\mathbb{L}} X)^{ij} + \hat{A}^{ij}_{TT}$

long. part

transverse

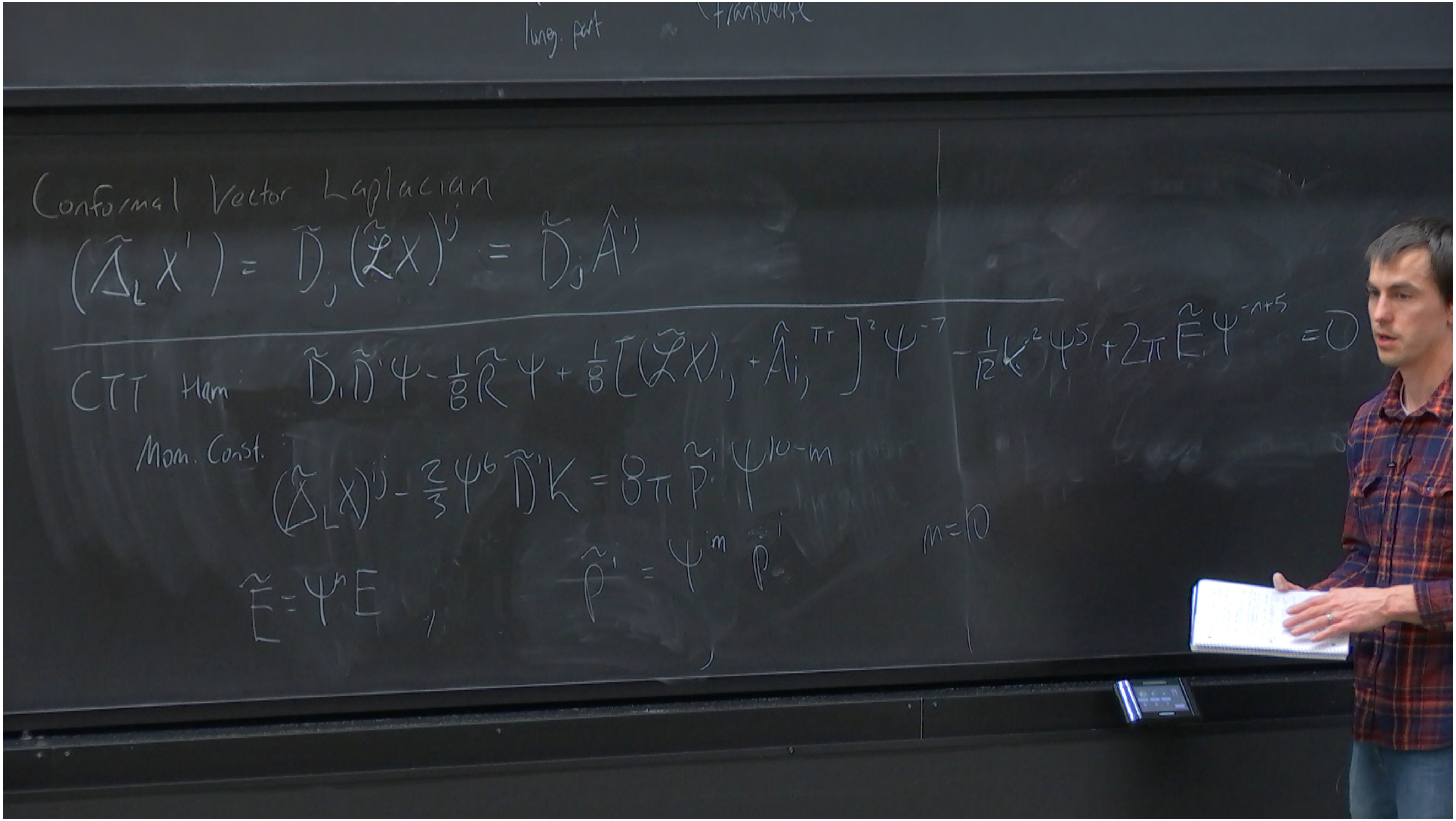
Transverse & Traceless

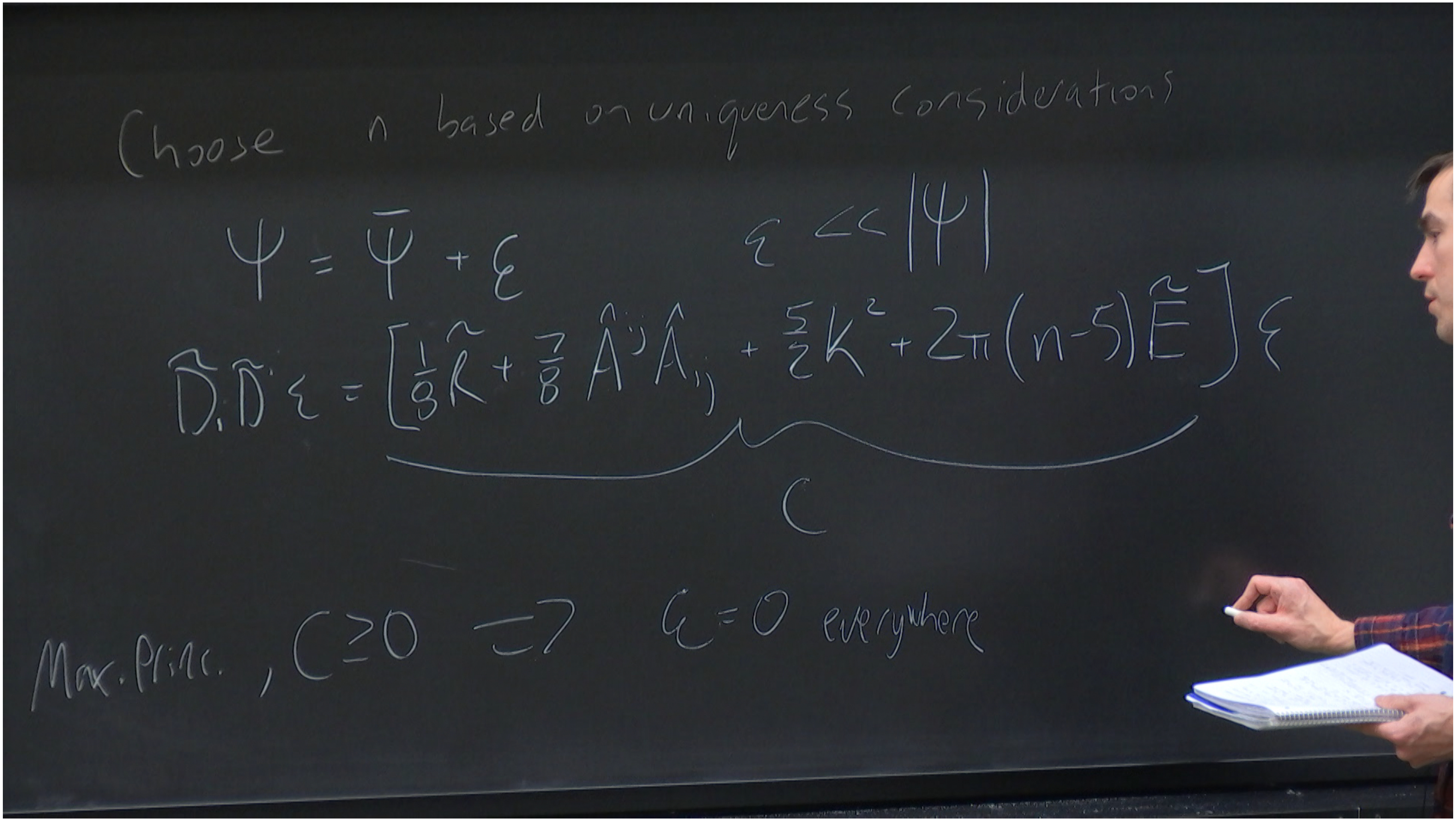
$$\tilde{\gamma}_{ij}\hat{A}^{ij}_{TT} = D_j \hat{A}^{ij}_{TT} = 0$$

X^i is vector "potential" which we will solve for

$\mathcal{L}$ is conformal Killing Operator

$$(\mathcal{L}X)^{ij} = \tilde{D}^i X^j + \tilde{D}^j X^i - \frac{2}{3}\left(\tilde{D}_k X^k\right)\tilde{\gamma}^{ij}$$


long. part
transverse
Conformal Vector Laplacian
$(\tilde{\Delta}_L X^i) = \tilde{D}_j (\tilde{\mathcal{L}} X)^{ij} = \tilde{D}_j \hat{A}^{ij}$
CTT Ham
$\tilde{D}_i \tilde{D}^i \psi - \frac{1}{8}\tilde{R}\psi + \frac{1}{8}[(\tilde{\mathcal{L}}X)_{ij} + \hat{A}_{ij}^{TT}]^2 \psi^{-7} - \frac{1}{12}K^2\psi^5 + 2\pi \tilde{E}\psi^{-n+5} = 0$
Mom. Const.
$(\tilde{\Delta}_L X)^{ij} - \frac{2}{3}\psi^6 \tilde{D}^i K = 8\pi \tilde{p}^i \psi^{10-m}$
$m=10$
$\tilde{E} = \psi^n E$
$\tilde{p}^i = \psi^m p^i$


Choose n based on uniqueness considerations
$\Psi = \bar{\Psi} + \varepsilon \qquad \varepsilon \ll |\Psi|$
$\tilde{D}_i \tilde{D}^i \varepsilon = \underbrace{\left[\frac{1}{8}\tilde{R} + \frac{7}{8}\hat{A}^{ij}\hat{A}_{ij} + \frac{5}{12}K^2 + 2\pi(n-5)\tilde{E}\right]}_{C}\varepsilon$
Max. Princ., $C \geq 0 \Rightarrow \varepsilon = 0$ everywhere

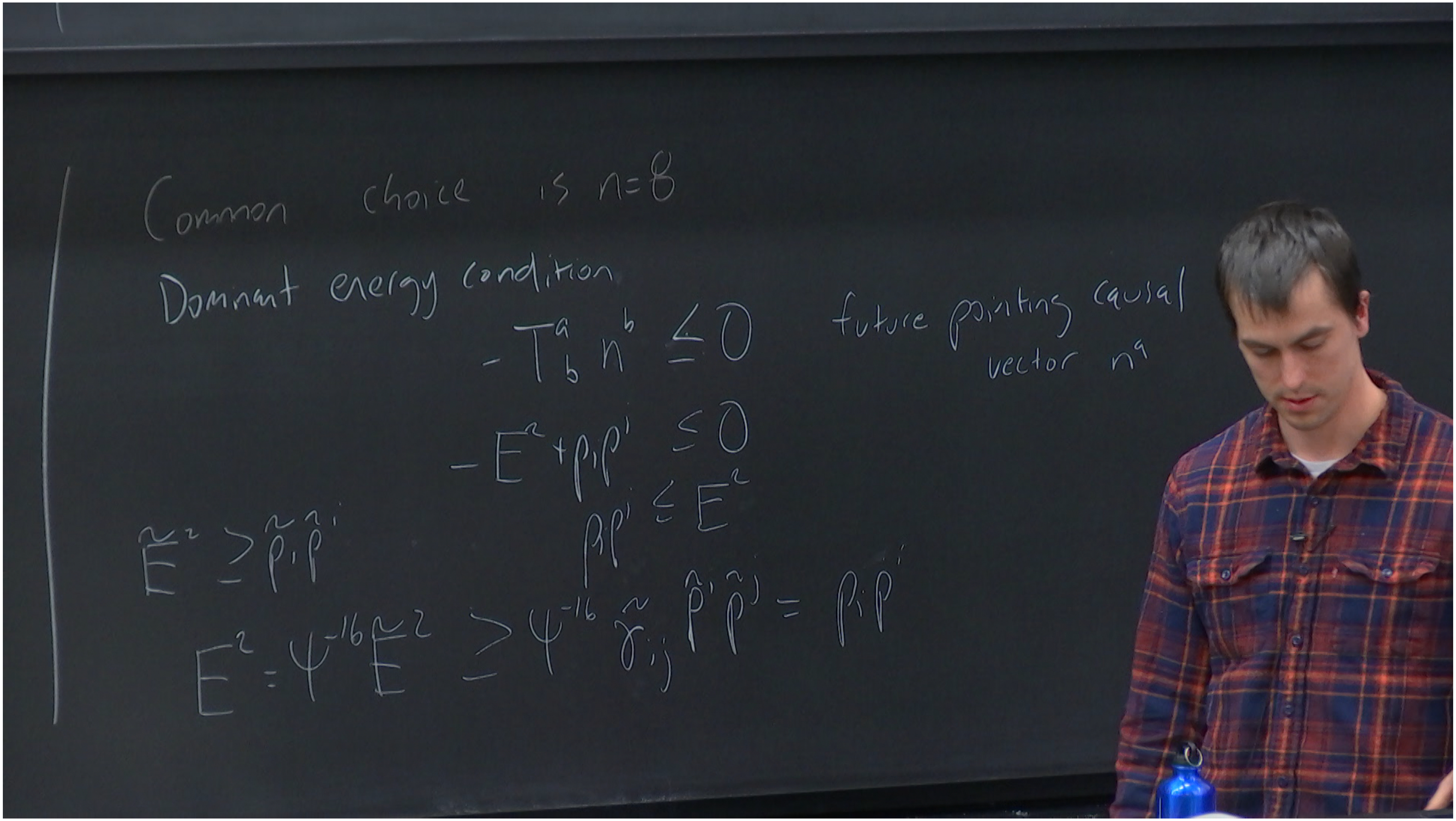
Choose n based on uniqueness considerations

$$\psi = \bar{\psi} + \varepsilon \qquad \varepsilon \ll |\psi|$$

$$\tilde{D}_i \tilde{D}^i \varepsilon = \underbrace{\left[\frac{1}{8}\tilde{R} + \frac{7}{8}\hat{A}^{ij}\hat{A}_{ij} + \frac{5}{2}K^2 + 2\pi(n-5)\tilde{E}\right]}_{C}\varepsilon$$

$$n \geq 5$$

Max. Princ., $C \geq 0 \implies \varepsilon = 0$ everywhere


Common choice is n=8
Dominant energy condition
$-T^a_b n^b \leq 0$
future pointing causal vector n^a
$-E^2 + p_i p^i \leq 0$
$p_i p^i \leq E^2$
$\tilde{E}^2 \geq \tilde{p}_i \tilde{p}^i$
$E^2 = \psi^{-16} \tilde{E}^2 \geq \psi^{-16} \tilde{\gamma}_{ij} \hat{p}^i \hat{p}^j = p_i p^i$