Title: Strong Gravity Lecture - 230314

Speakers: William East

Collection: Strong Gravity (2022/2023)

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$$M, \quad g_{ab}, \quad T_{ab}$$

$$G_{ab} = 8(\text{today})\, T_{ab}$$

3+1 Decomposition of Spacetime

References: GR by Wald Chap. 10.2

3+1 Formalism Gourgoulhon arXiv: gr-qc/0703035

Hypersurfaces of Spacetime

4-dim manifold $\mathcal{M}$

3-dim manifold $\hat{\Sigma}$

Embedding of $\hat{\Sigma}$ in $\mathcal{M}$

$\Phi : \hat{\Sigma} \to \mathcal{M}$, 1:1, Φ, Φ^{-1} continuous

Then

Level se

Then $\Phi(\hat{\Sigma}) = \Sigma$ is called a hypersurface

Level set $\Sigma : \{ x^a \in M \mid F(x^a) = 0 \}$

$\partial_a F$ is normal Σ

$\iff \partial_a F \, v^a = 0$ v^a tangent to Σ

Φ, Φ^{-1} continuous

$\partial_a F$ timel

called a hypersurface

$| \; F(x^a) = 0 \}$

Σ

$\partial_a F$ timelike, call Σ spacelike

$\partial_a F$ spacelike call Σ timelike

$\partial_a F$ null call Σ null

$\Sigma \to \mathcal{M}$, 1:1, Φ, Φ^{-1} continuous

Choose coördinates $x^a = (t, x^i)$ x^i coördinates on $\hat{\Sigma}$

$\uparrow_{\text{3-index}}$

with $\Sigma = \{ x^a \in \mathcal{M} \mid t = 0 \}$

embedding

$$\Phi : \hat{\Sigma} \to \mathcal{M}$$
$$x^i \to (0, x^i)$$

...nal

$$n_a = \frac{-\nabla_a t}{\sqrt{-\nabla_a t \nabla^a t}} \qquad n_a n^a = -1$$

$n^a = (1, \vec{0})$ (in Minkowski)

$\Phi : \hat{\Sigma} \to M$

$v^i \to (0, v^i)$

push-forward mapping

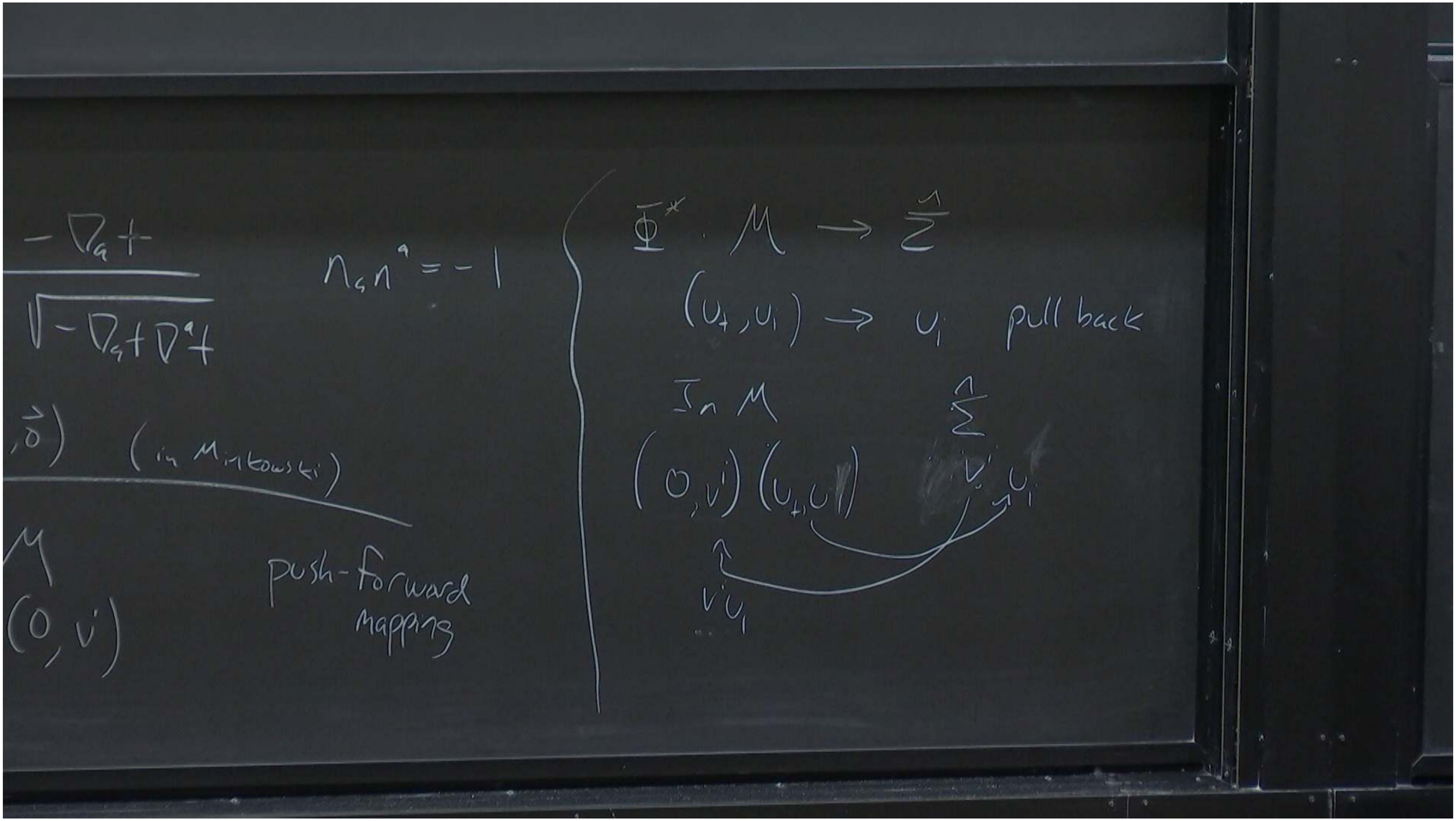
$\Phi^* : M \to \hat{\Sigma}$

$(v_t, v_i) \to v_i$ pull back

In M $\qquad \hat{\Sigma}$

$(0, v^i)\ (v_t, v^i) \qquad v^i\ v^i$

$v^i v_i$


$-\nabla_a t$
$\sqrt{-\nabla_a t \nabla^a t}$
$n_a n^a = -1$
(in Minkowski)
push-forward mapping
$(0, v^i)$
$\Phi^* : M \to \hat{\Sigma}$
$(u_t, u_i) \to u_i$ pull back
In M
$\hat{\Sigma}$
$(0, v^i)$ (u_t, u_i)
$v^i u_i$

Caution $\quad M \qquad \hat{\Sigma}$

$$(v^t, v^i) \not\to V^i$$

Vectors $\quad V^a \to \gamma^i_b V^b$

For metric M, induced metric γ_{ij} on Σ

is $\gamma_{ij} = g_{ij}$ in our adopted coördinates

$$V^a = \underbrace{-V^b n_b n^a}_{\text{orthogonal}} + \Big(\underbrace{V^a + V^b n_b n^a}_{\text{tangent}} \Big)$$

$$=: \left(\gamma^a_b V^b \right)$$

$\gamma^a_b =$

$n_a \gamma^a_b V^b$

$$\gamma^a_b = \delta^a_b + n_b n^a$$

$$n_a \gamma^a_b v^b = n^a v_b - n_a v^a = 0$$

Σ $\qquad$ M

$$v_i \rightarrow \gamma^i_a v_i$$

$$\gamma_{ab} = \gamma^i_a \gamma^j_b \gamma_{ij} = g_{ab} + n_a n_b$$

if $u^a n_a = v^a n_a = 0$

$$\gamma_{ab} u^a v^b = g_{ab} u^a v^b$$

if $\tilde{u}^a = \lambda n^a$

$$\gamma_{ab} \tilde{u}^a \tilde{v}^b = 0$$

$- (\gamma_b V)$

Intrinsic Curvature   Same as $(g_{ab}, M) \to (\gamma_{ij}, \Sigma)$

Introduce covariant derivative compatible γ_{ij}

(also assume torsion free)

$$D_i \gamma_{jk} = 0$$

$\Gamma^{(3)i}_{jk}$ Christoffel symbols defined w/ γ_{ij}

3D Riemann tensor $(D_i D_j - D_j D_i) V^k = {}^{(3)}R^k{}_{lij} V^l$

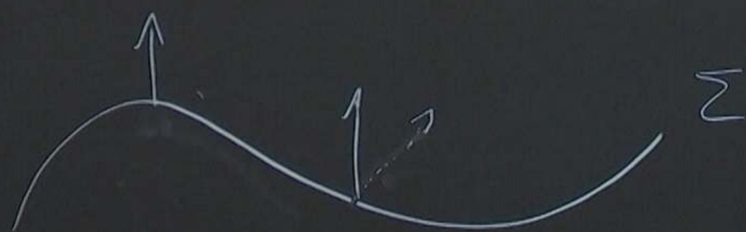
$$^{(3)}R_{ij} = {}^{(3)}R^{k}{}_{ikj}$$

$$^{(3)}R = \gamma^{ij}\,{}^{(3)}R_{ij}$$

$$D_a T^{b_1 b_2 \dots}{}_{c_1 c_2 \dots} = \gamma^{b_1}{}_{d_1}\gamma^{b_2}{}_{d_2}\cdots\gamma^{e_1}{}_{c_1}\gamma^{e_2}{}_{c_2}\cdots\gamma^{f}{}_{a}\nabla_f T^{d_1\dots}{}_{e_1\dots}$$

$$D_a\gamma_{bc} = \gamma_b{}^{d}\gamma_c{}^{e}\gamma^{f}{}_{a}\nabla_f\left(g_{de} + n_d n_e\right) = 0$$

Extrinsic Curvature



$$K_{ij} = -\nabla_j n_i$$

Example: Cylinder as hypersurface in $\mathbb{R}^3$

$$\Sigma : \{ (x,y,z) \in \mathbb{R}^3 \mid \rho^2 = x^2 + y^2 = 1 \}$$

$\mathbb{R}^3$

$^2 = 1 \}$

Unit normal $n_a = \left(\frac{x}{\rho}, \frac{y}{\rho}, 0 \right)$

$$\nabla_b n_a = \begin{pmatrix} \frac{y^2}{\rho^3} & \frac{-xy}{\rho^3} & 0 \\ \frac{-xy}{\rho^3} & \frac{x^2}{\rho^3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J^a_i = \frac{dx^a}{dx^i}$$

$i = \{\phi, z\}$

$\tan\phi = \frac{y}{x}$

$$K_{ij} = -\nabla_a n_b J^a_i J^b_j$$

$$K_{\phi\phi} = \frac{-(x^2+y)^2}{\rho^3}\Bigg|_{\rho=1} = -1$$

$$\frac{dx^a}{dx^i}$$

$$\left.\frac{(x^2+y)^2}{\rho^3}\right|_{\rho=1} = -1$$

$$K_{ab} = \gamma_a^i \gamma_b^j K_{ij}$$

$$= -\nabla_b n_a - \underbrace{n^c(\nabla_c n_a)}_{a_a \text{ "acceleration"}} n_b$$