Title: Strong Gravity Lecture - 230302

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Collection: Strong Gravity (2022/2023)

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$$\Delta = 0 \qquad r_\pm = M \pm \sqrt{M^2 - a^2}$$

Kerr Schild Cartesian coordinates

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \frac{2Mr^3}{r^4 + a^2 z^2}\left[dt + \frac{r(x\,dx + y\,dy)}{a^2 + r^2} + \frac{a(y\,dx - x\,dy)}{a^2 + r^2} + \frac{z}{r} \ldots\right.$$

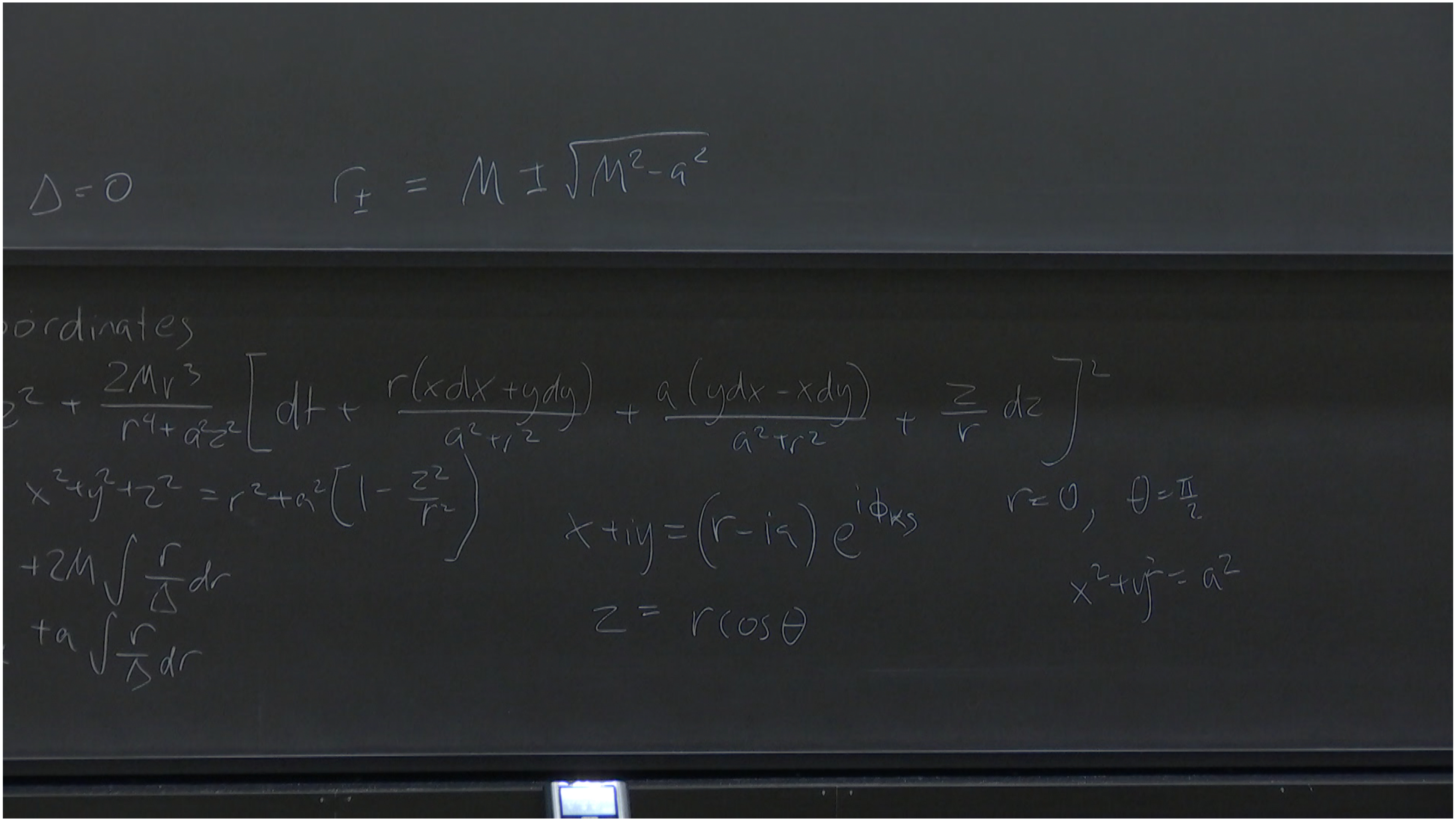
$$x^2 + y^2 + z^2 = r^2 + a^2\left(1 - \frac{z^2}{r^2}\right)$$

$$t_{KS} = t_{BL} + 2M \int \frac{r}{\Delta}\,dr$$

$$\phi_{KS} = \phi_{BL} + a \int \frac{r}{\Delta}\,dr$$

$$x + iy = (r - ia)\,e^{i\phi_{KS}}$$

$$z = r\cos\theta$$


$\Delta = 0$
$r_\pm = M \pm \sqrt{M^2 - a^2}$
oordinates
$^2 + \frac{2Mr^3}{r^4 + a^2 z^2}\left[dt + \frac{r(xdx + ydy)}{a^2 + r^2} + \frac{a(ydx - xdy)}{a^2 + r^2} + \frac{z}{r}dz\right]^2$
$x^2 + y^2 + z^2 = r^2 + a^2\left(1 - \frac{z^2}{r^2}\right)$
$x + iy = (r - ia)\, e^{i\phi_{KS}}$
$r = 0, \quad \theta = \frac{\pi}{2}$
$+ 2M \int \frac{r}{\Delta} dr$
$x^2 + y^2 = a^2$
$+ a \int \frac{r}{\Delta} dr$
$z = r\cos\theta$

Metric in Boyer Lindquist coördinates

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta\,(r^2+a^2-\Delta)}{\Sigma}\,dt\,d\phi + \left(\frac{(r^2+a^2)^2 - \Delta \ldots}{\Sigma}\right.$$

$$\Sigma = r^2 + a^2\cos^2\theta \quad , \quad \Delta = r^2 + a^2 - 2Mr$$

$$\Delta = 0 \qquad r_\pm = M \pm \sqrt{M^2 - a^2}$$

Kerr Schild Cartesian coördinates

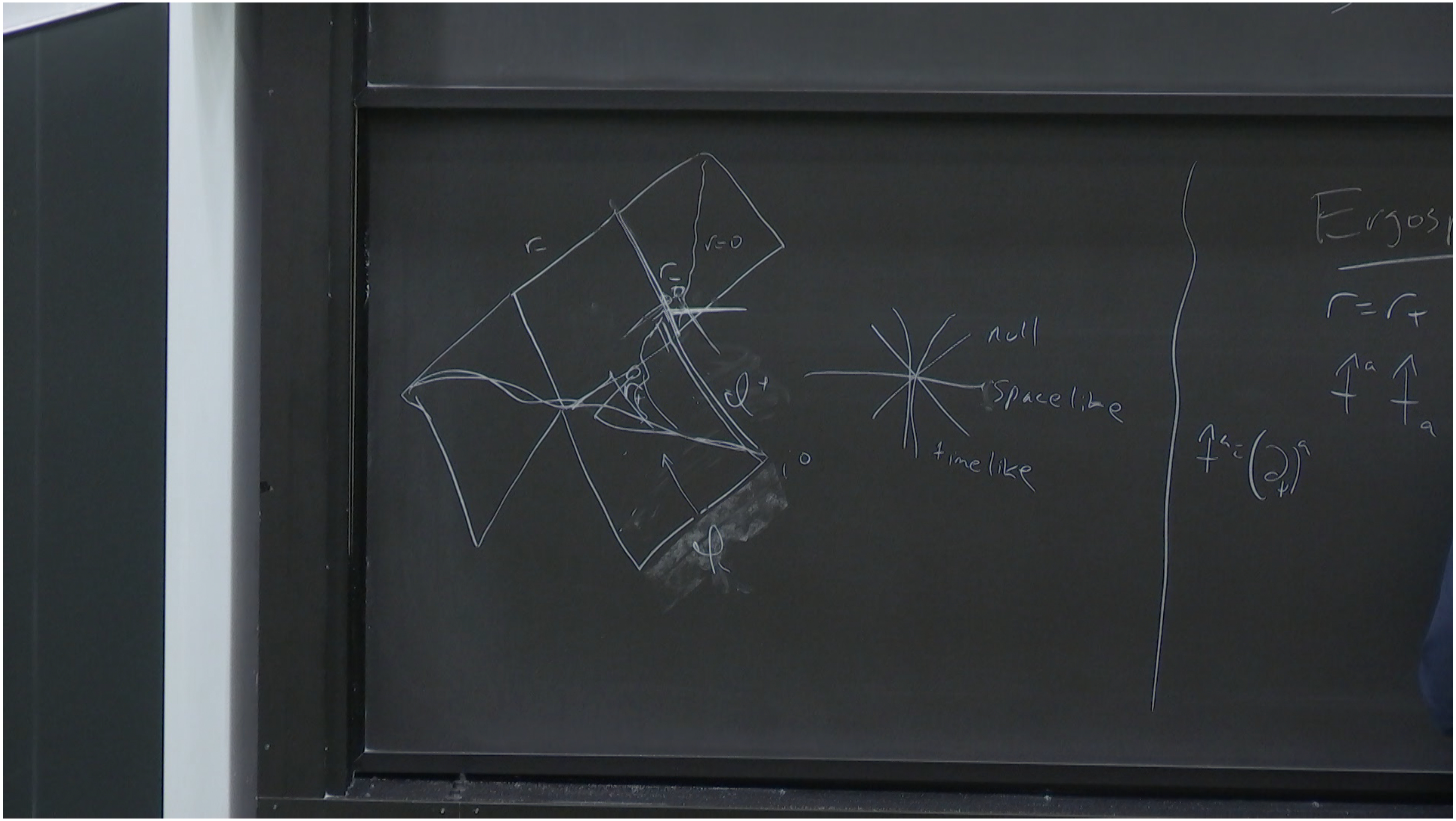
ates

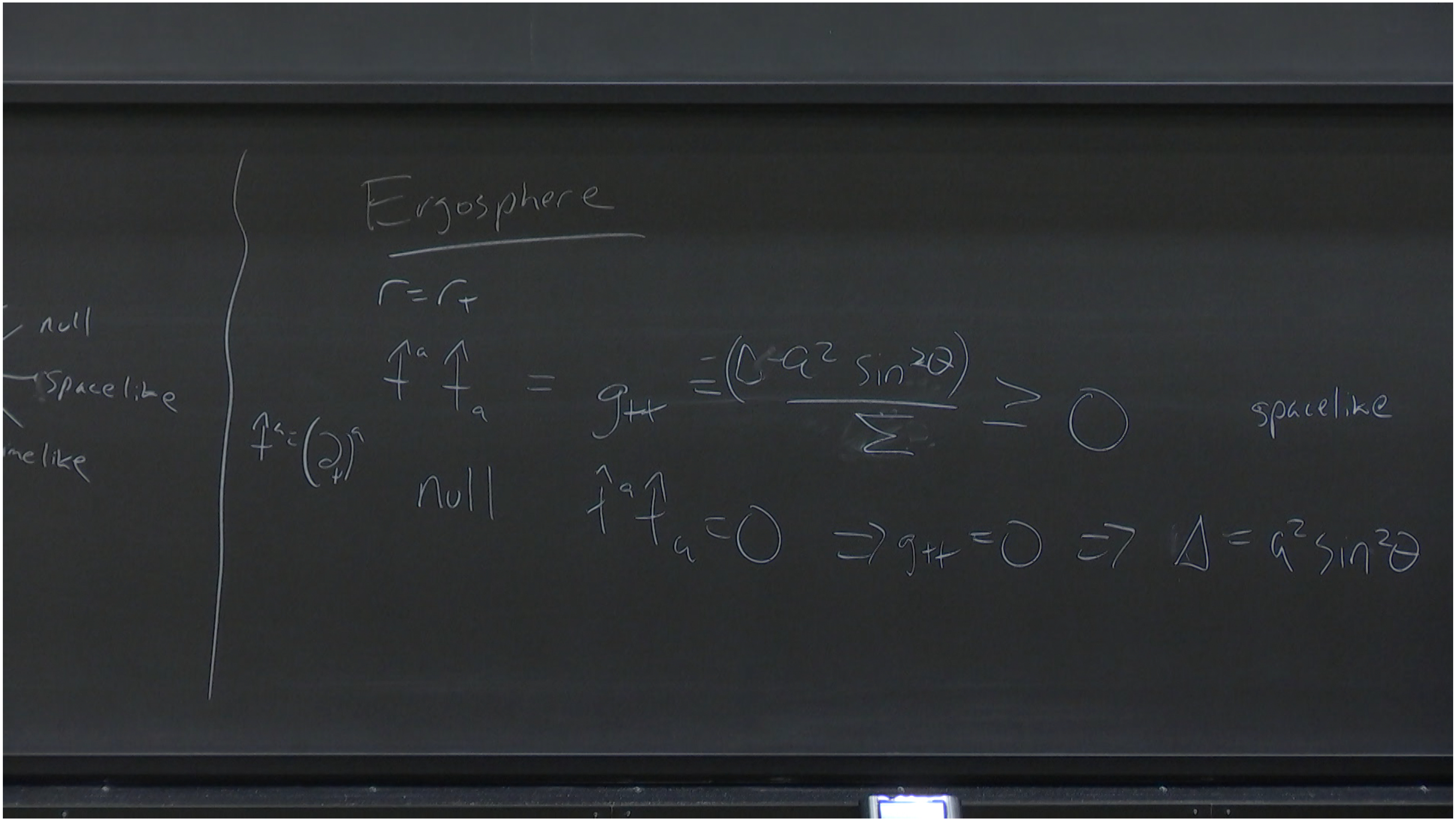
$$\ldots + a^2 - \Delta) \, dt \, d\phi + \left(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \right] \sin^2\theta \, d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

$$r^2 + a^2 - 2Mr$$

$$r_\pm = M \pm \sqrt{M^2 - a^2}$$

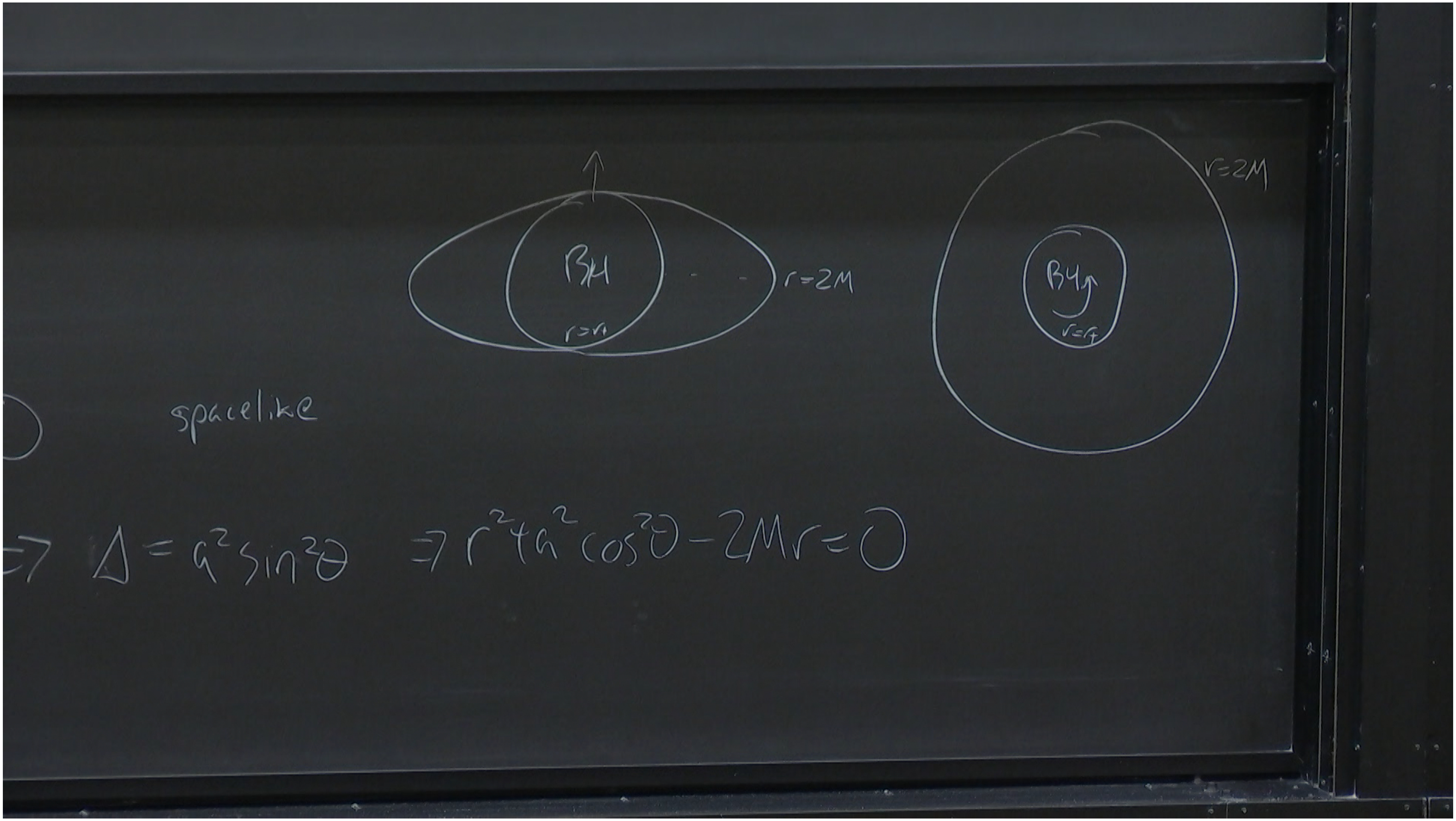
$$\ldots \left[dt + \frac{r(x\,dx + y\,dy)}{a^2 + r^2} + \frac{a(y\,dx - x\,dy)}{a^2 + r^2} + \frac{z}{r} dz \right]^2$$


r=0
null
Spacelike
timelike
Ergos
r=r+


Ergosphere
$r = r_+$
null
spacelike
$\hat{t}^a = (\partial_t)^a$
$\hat{t}^a \hat{t}_a = g_{tt} = \frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} \geq 0$
spacelike
null
$\hat{t}^a \hat{t}_a = 0 \Rightarrow g_{tt} = 0 \Rightarrow \Delta = a^2 \sin^2\theta$

$$\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} \geq 0 \qquad \text{spacelike}$$

$$0 \Rightarrow g_{tt} = 0 \Rightarrow \Delta = a^2 \sin^2\theta \Rightarrow r^2 + a^2\cos^2\theta - 2Mr = 0$$


BH
r=r+
r=2M
BH
r=r+
r=2M
spacelike
⇒ Δ = a² sin²θ ⇒ r² + a² cos²θ − 2Mr = 0

$$-1 = \underline{g_{ab}u^a u^b} = \underbrace{g_{tt}(u^t)^2 + g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 + g_{\phi\phi}(u^\phi)^2}_{\geq 0}$$

$$+ 2 g_{t\phi} u^t u^\phi \leq -1$$

$$g_{t\phi} < 0 \qquad u^t = \frac{dt}{d\tau} > 0, \quad u^\phi = \frac{d\phi}{d\tau} > 0$$

$$\underbrace{g_{tt}(u^t)^2 + g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 + g_{\phi\phi}(u^\phi)^2}_{\geq 0}$$

$$+ 2 g_{t\phi} u^t u^\phi \leq -1$$

$$g_{t\phi} < 0 \qquad u^t = \frac{dt}{d\tau} > 0, \quad u^\phi = \frac{d\phi}{d\tau} > 0$$

$$\alpha\left(\hat{t}^a + \Omega \hat{\phi}^a\right) \qquad \hat{\phi}^a \chi_a = 0, \qquad g_{\phi t} + g_{\phi\phi}\Omega = 0$$

$$\Omega = \frac{d\phi}{dt} = -\frac{g_{\phi t}}{g_{\phi\phi}}$$

$$r \to r_+$$

$$\Omega = \bar{a}/2r_+ = \Omega_H$$

$$\chi^a = \hat{t}^a + \Omega_H \hat{\phi}^a$$

For $\bar{a} \ll 1$

$$r_+ = \left(2 - \tfrac{1}{2}\bar{a}^2\right) M$$

$$\Omega_H = \bar{a}/4M$$

$\bar{a} = 1 - \varepsilon$, $\varepsilon \ll 1$,

$$r_+ = \left(1 + \sqrt{2\varepsilon}\right) M$$

$$\Omega_H = \frac{1 - \sqrt{2\varepsilon}}{2M}$$

Geodesics on Kerr

Geodesic Eqn: $u^a \nabla_a u^b = 0$

$u^a = \frac{dx^a}{d\tau}$ ← proper time

$u^a u_a = -1$

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a{}_{bc} u^b u^c = 0$$

$\nabla_{(c} K_{ab)} = 0$

$\nabla_{(a} K_{b)} = 0$

$u^a \nabla_a (K^b u_b) = 0$

Constants: $\tilde{E} = -\hat{t}^a u_a = -u_t = -(g_{tt} u^t + g_{t\phi} u^\phi)$

$\tilde{J} = \hat{\phi}^a u_a = u_\phi = g_{\phi\phi} u^\phi + g_{t\phi} u^t$

$= -1$

$$C \doteq K_{ab} u^a u^b$$

$g_{t\phi} u^\phi)$

$g_{t\phi} u^t$

$$= \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta \frac{d\phi}{dt} - \frac{2Mar\sin^2\theta}{\Sigma} \frac{dt}{dt}$$

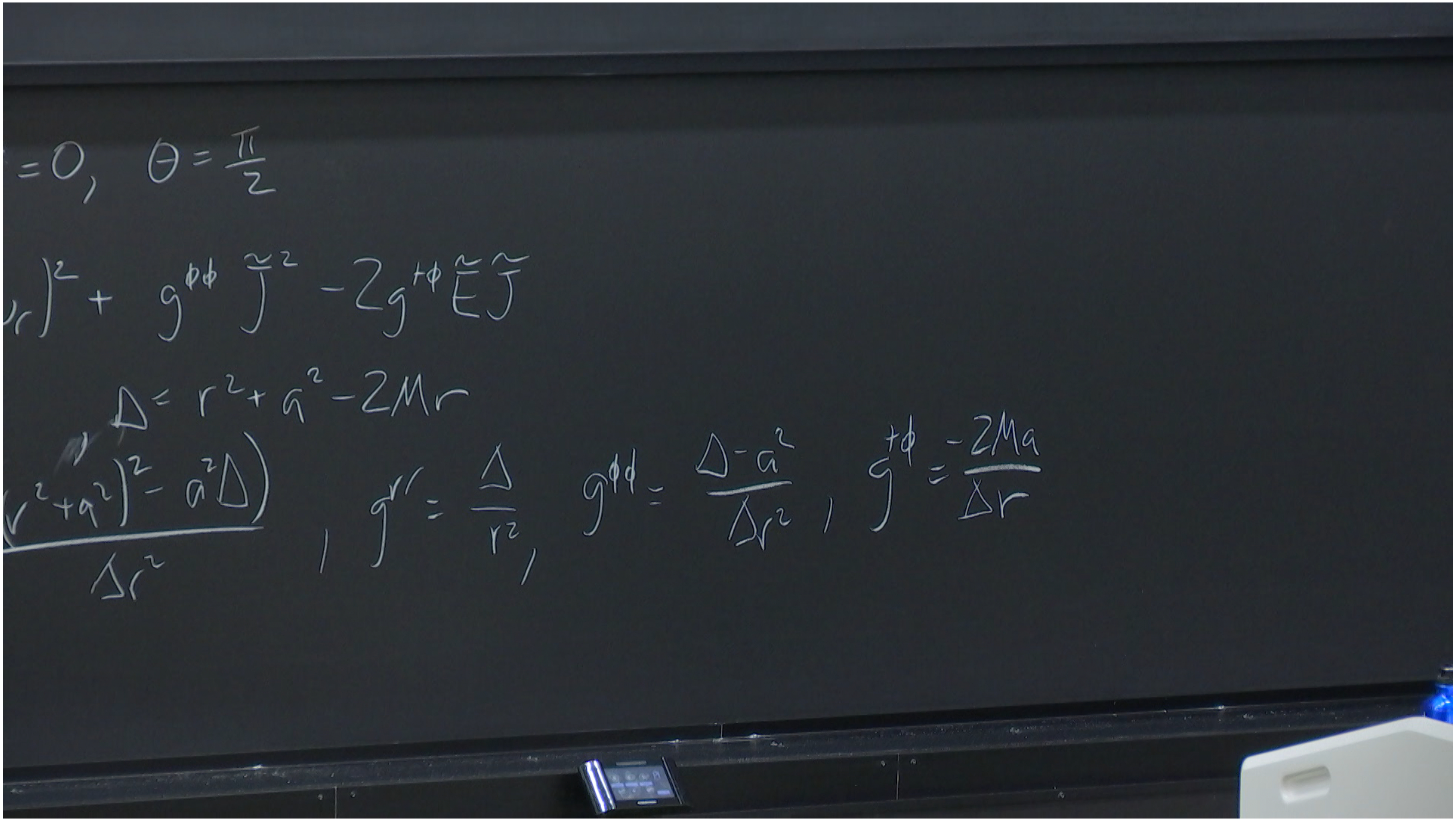
$$u^a \nabla_a (K^{ab} u_b) = 0$$

② Restrict Equatorial orbits $u^\theta = 0$, $\theta = \frac{\pi}{2}$

$$-1 = g^{ab} u_a u_b = g^{tt}\tilde{E}^2 + g^{rr}(u_r)^2 + g^{\phi\phi}\tilde{J}^2 - 2g^{t\phi}\tilde{E}\tilde{J}$$

$$u_r = g_{rr} u^r = \frac{r^2}{\Delta}\frac{dr}{d\tau}, \quad \Delta = r^2 + a^2 - 2Mr$$

$$g^{tt} = \frac{-\left((r^2+a^2)^2 - a^2\Delta\right)}{\Delta r^2}, \quad g^{rr} = \frac{\Delta}{r^2}, \quad g^{\phi}$$


$= 0, \quad \theta = \frac{\pi}{2}$
$r)^2 + g^{\phi\phi}\tilde{J}^2 - 2g^{t\phi}\tilde{E}\tilde{J}$
$\Delta = r^2 + a^2 - 2Mr$
$\frac{(r^2+a^2)^2 - a^2\Delta)}{\Delta r^2}, \quad g^{rr} = \frac{\Delta}{r^2}, \quad g^{\phi\phi} = \frac{\Delta - a^2}{\Delta r^2}, \quad g^{t\phi} = \frac{-2Ma}{\Delta r}$

$$(*) \Rightarrow \quad -1 = \frac{\Delta}{r^2} u_r^2 + \frac{a^2\Delta - (r^2+a^2)^2}{\Delta r^2}\tilde{E}^2 + \left(\frac{\Delta - a^2}{\Delta r^2}\right)\tilde{J}^2 + \frac{4Ma}{r}\tilde{E}\tilde{J}$$

$$-\Delta^2 u_r^2 = (1-\tilde{E}^2) r^4 - 2Mr^3 + \left[a^2(1-\tilde{E}^2) + \tilde{J}^2\right] r^2 - 2M(a$$

$$:= V(\tilde{E}, \tilde{J}, r)$$

$$\leq 0$$

Bound $\tilde{E} < 1$

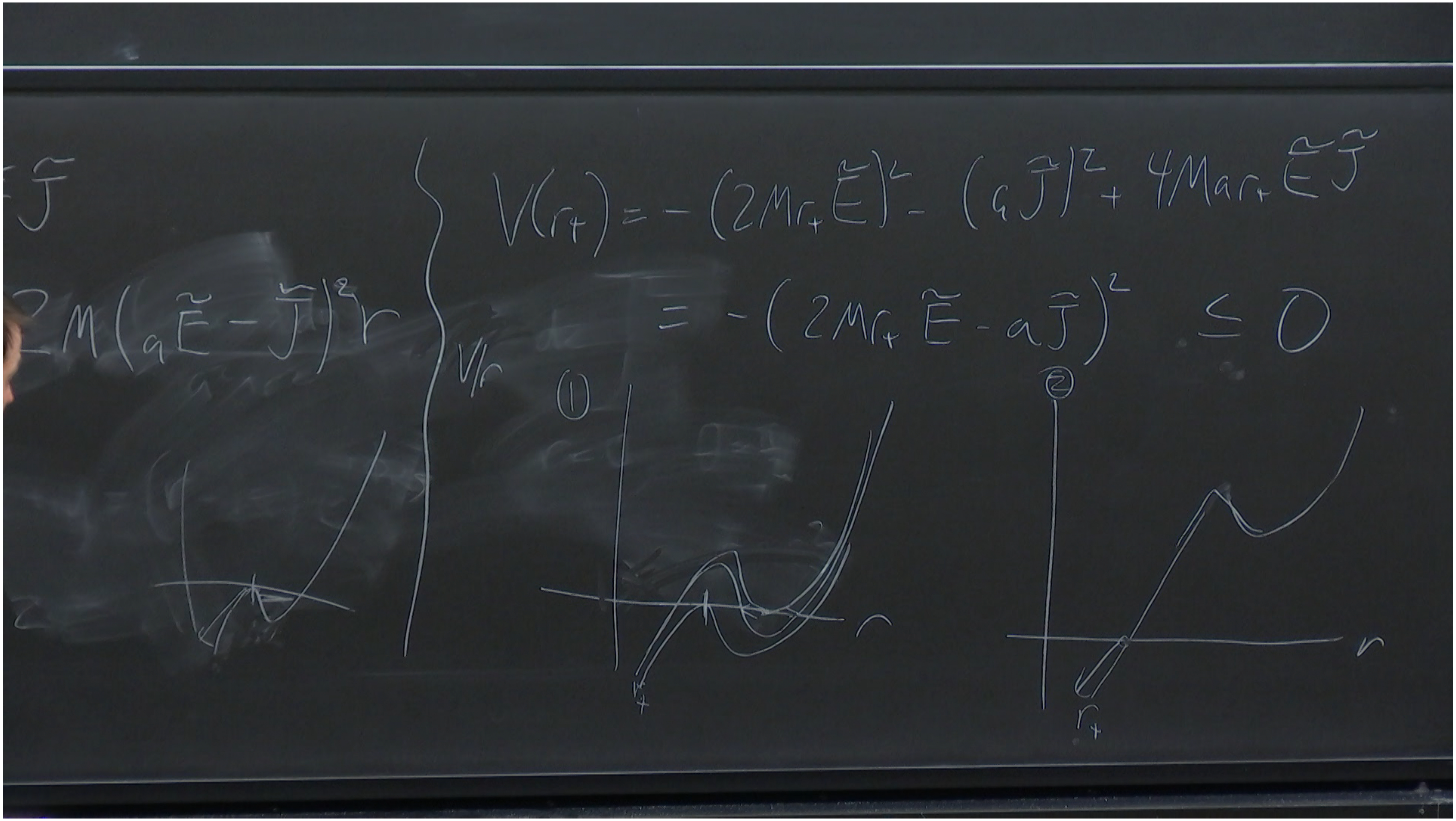
$$+\frac{a^2\Delta-(r^2+a^2)^2}{\Delta r^2}\tilde{E}^2+\left(\frac{\Delta-a^2}{\Delta r^2}\right)\tilde{J}^2+\frac{4Ma}{r}\tilde{E}\tilde{J}$$

$$-\tilde{E}^2)r^4-2Mr^3+\left[a^2(1-\tilde{E}^2)+\tilde{J}^2\right]r^2-2M(a\tilde{E}-\tilde{J})^2r$$

$$(\tilde{E},\tilde{J},r)$$

Bound $\tilde{E}<1$

0


$V(r_+) = -(2Mr_+\tilde{E})^2 - (a\tilde{J})^2 + 4Mar_+\tilde{E}\tilde{J}$
$= -(2Mr_+\tilde{E} - a\tilde{J})^2 \leq 0$
$2M(a\tilde{E} - \tilde{J})^2 r$
V/r
①
②
r
r_+