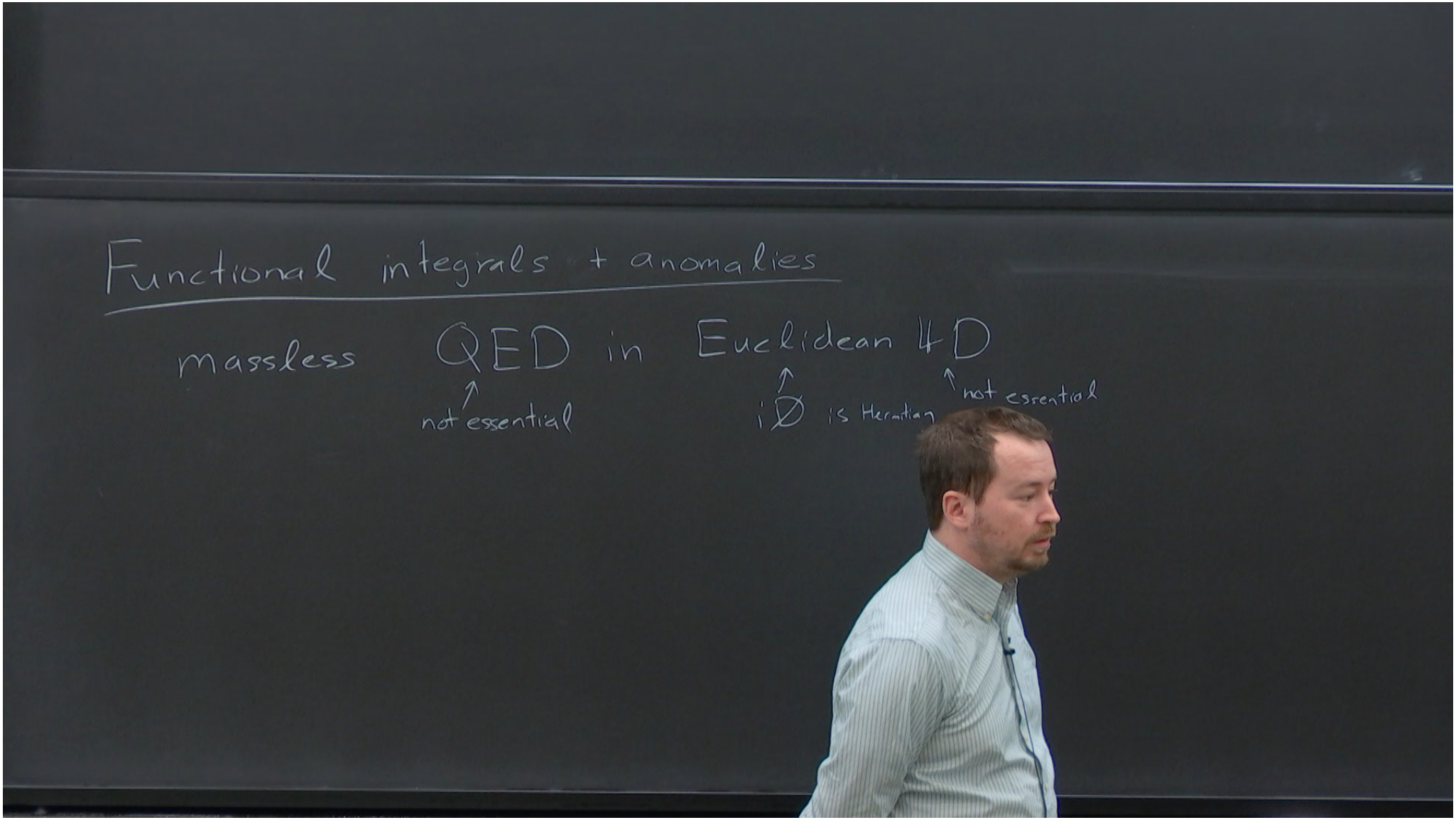
Title: Quantum Fields and Strings Lecture - 230302

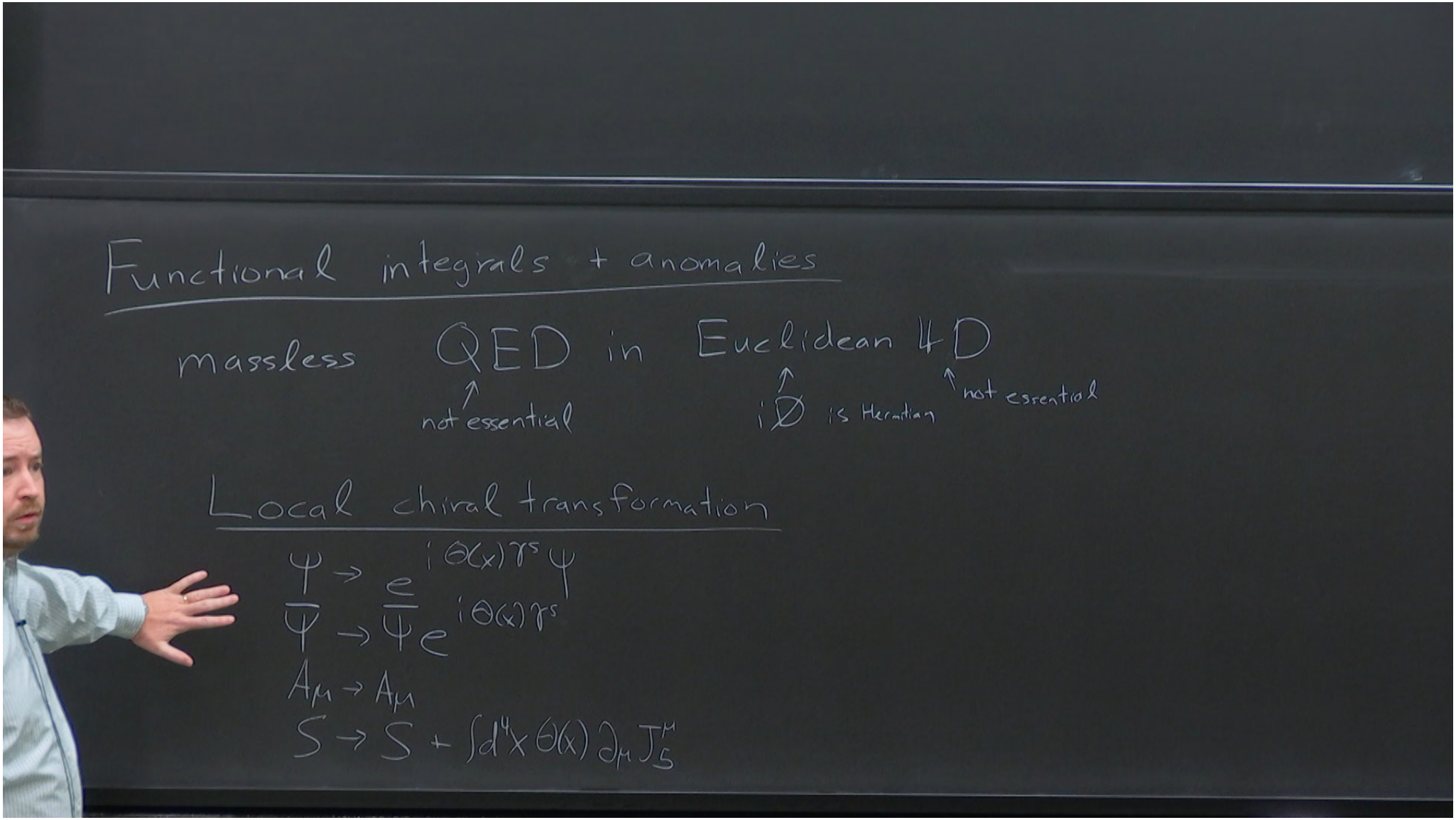
Speakers: Dan Wohns

Collection: Quantum Fields and Strings (2022/2023)

Date: March 02, 2023 - 11:30 AM

URL: https://pirsa.org/23030016


Functional integrals + anomalies
massless QED in Euclidean 4D
not essential
not essential


Functional integrals + anomalies
massless QED in Euclidean 4D
not essential
$i\not{D}$ is Hermitian
not essential
Local chiral transformation
$\psi \to e^{i\theta(x)\gamma^5}\psi$
$\bar{\psi} \to \bar{\psi} e^{i\theta(x)\gamma^5}$
$A_\mu \to A_\mu$
$S \to S + \int d^4x\, \theta(x)\, \partial_\mu J_5^\mu$

$$Z[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S[A]}$$

classical background

quantum

$$0 = \left.\frac{\delta Z'}{\delta \theta}\right|_{\theta=0} = \frac{\delta}{\delta\theta}\left[\int \mathcal{D}\psi' \mathcal{D}\bar{\psi}' e^{-S[A] - \theta \cdot \partial_\mu J_5^\mu}\right]\Big|_{\theta=0}$$

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functional + matrix trace

$$\mathcal{D}\psi' = \mathcal{D}\psi\, e^{\mathrm{Tr}\, i\theta\gamma^5}$$

$$Z[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S[A]}$$

$A$: classical background; $\psi, \bar{\psi}$: quantum

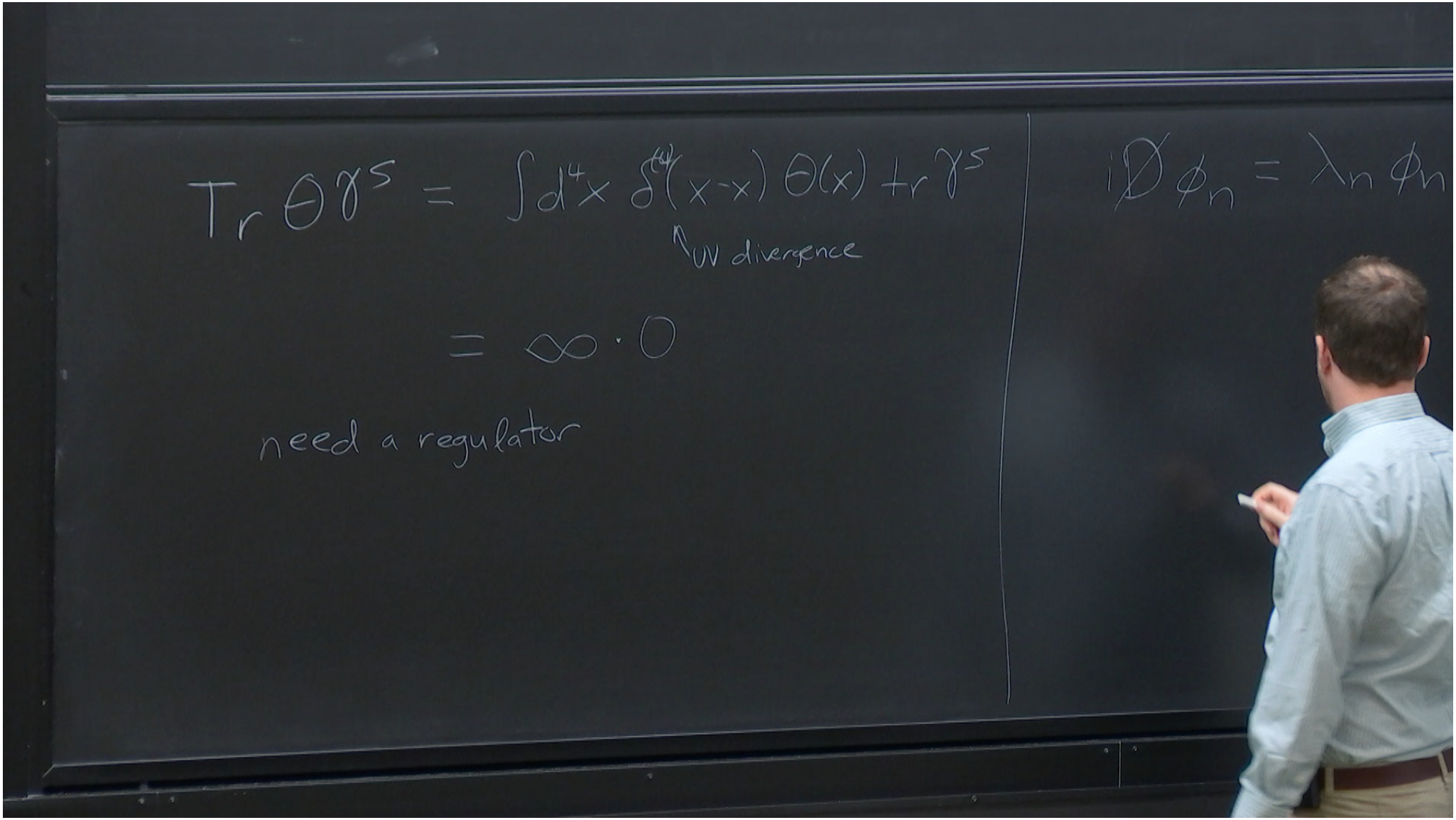
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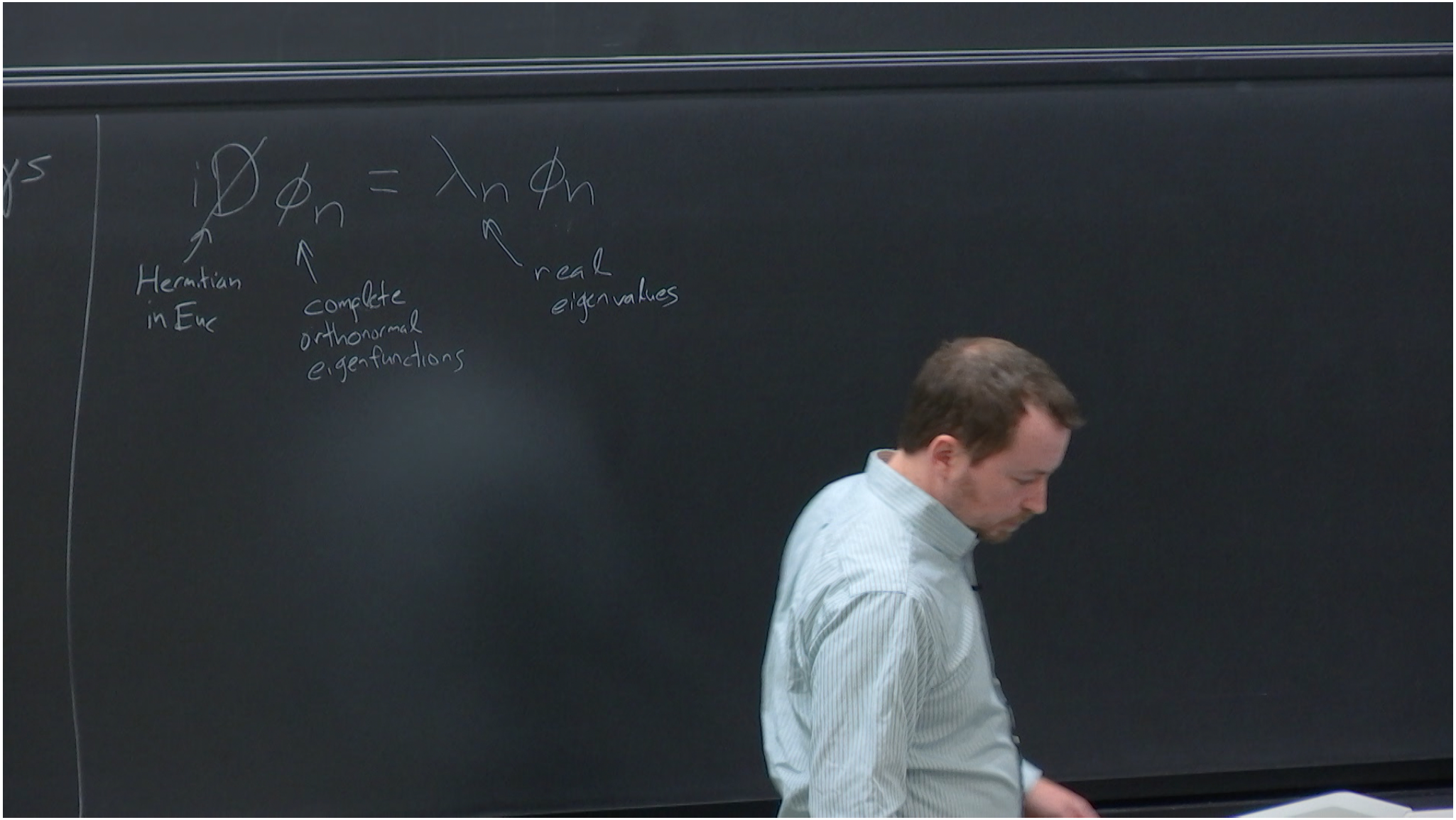
functional + matrix trace

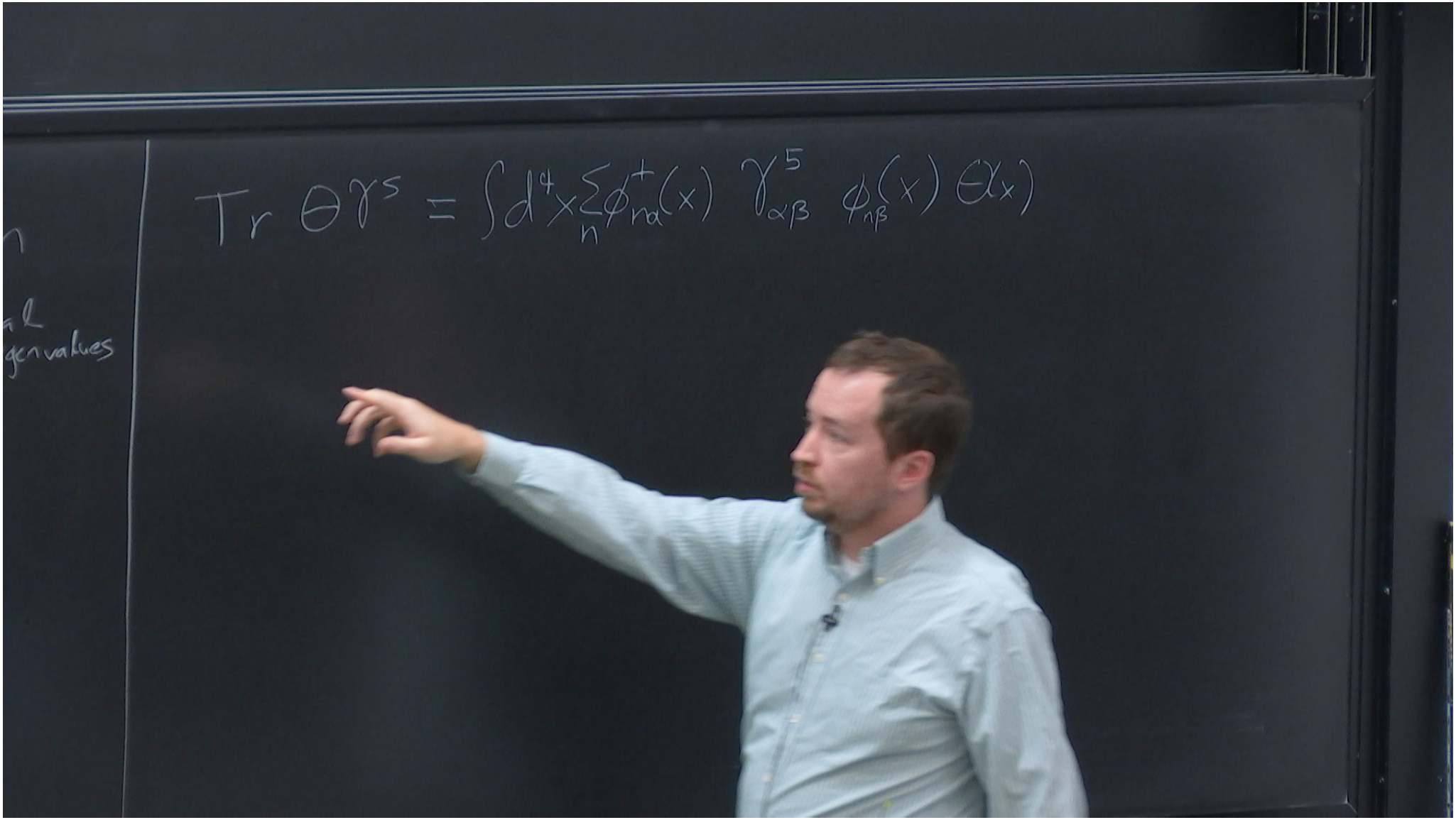
$$\mathcal{D}\psi' = \mathcal{D}\psi\, e^{i\mathrm{Tr}\theta\gamma^5}$$

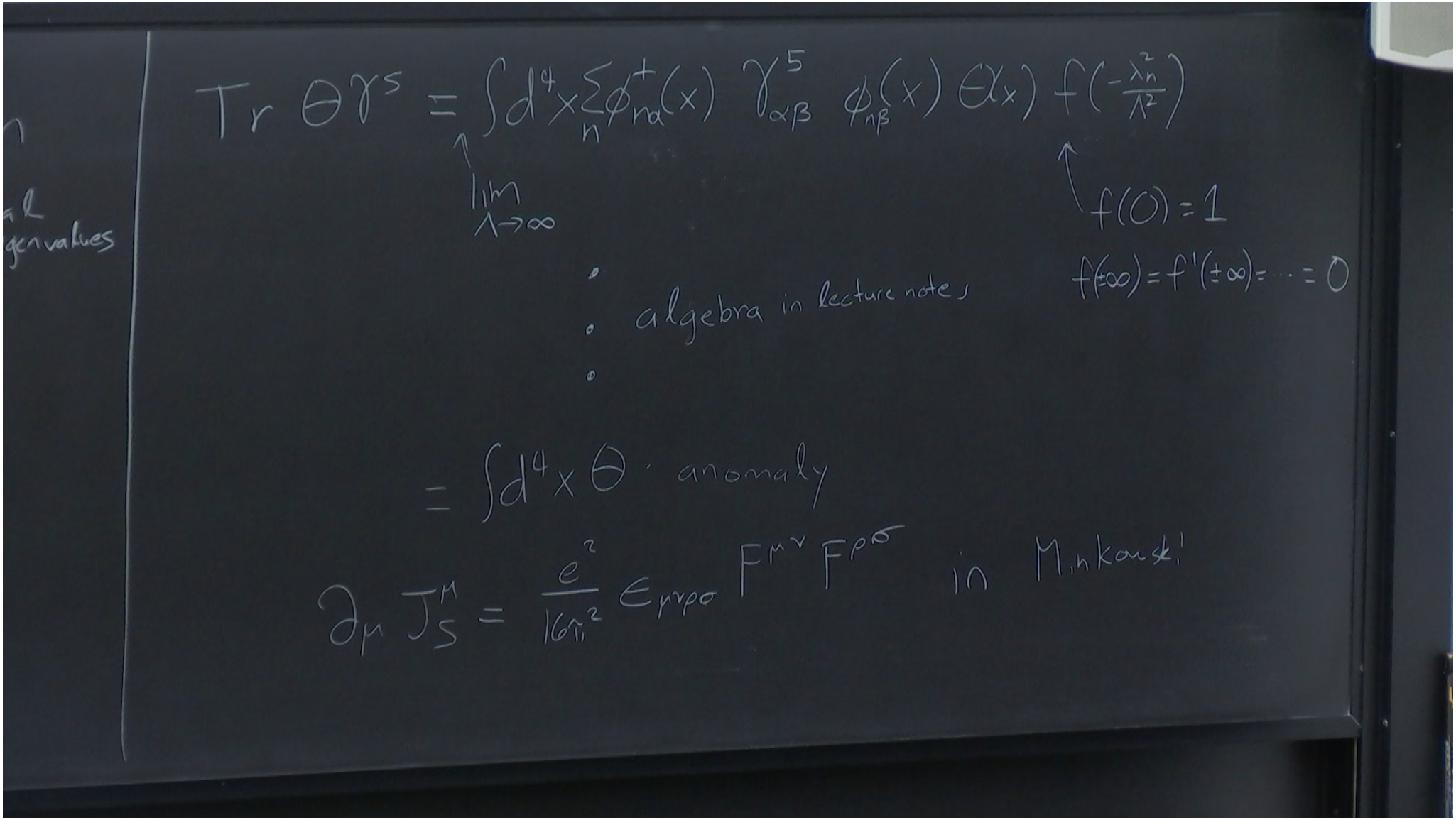
$$\mathcal{D}\bar{\psi}' = \mathcal{D}\bar{\psi}\, e^{i\mathrm{Tr}\theta\gamma^5}$$

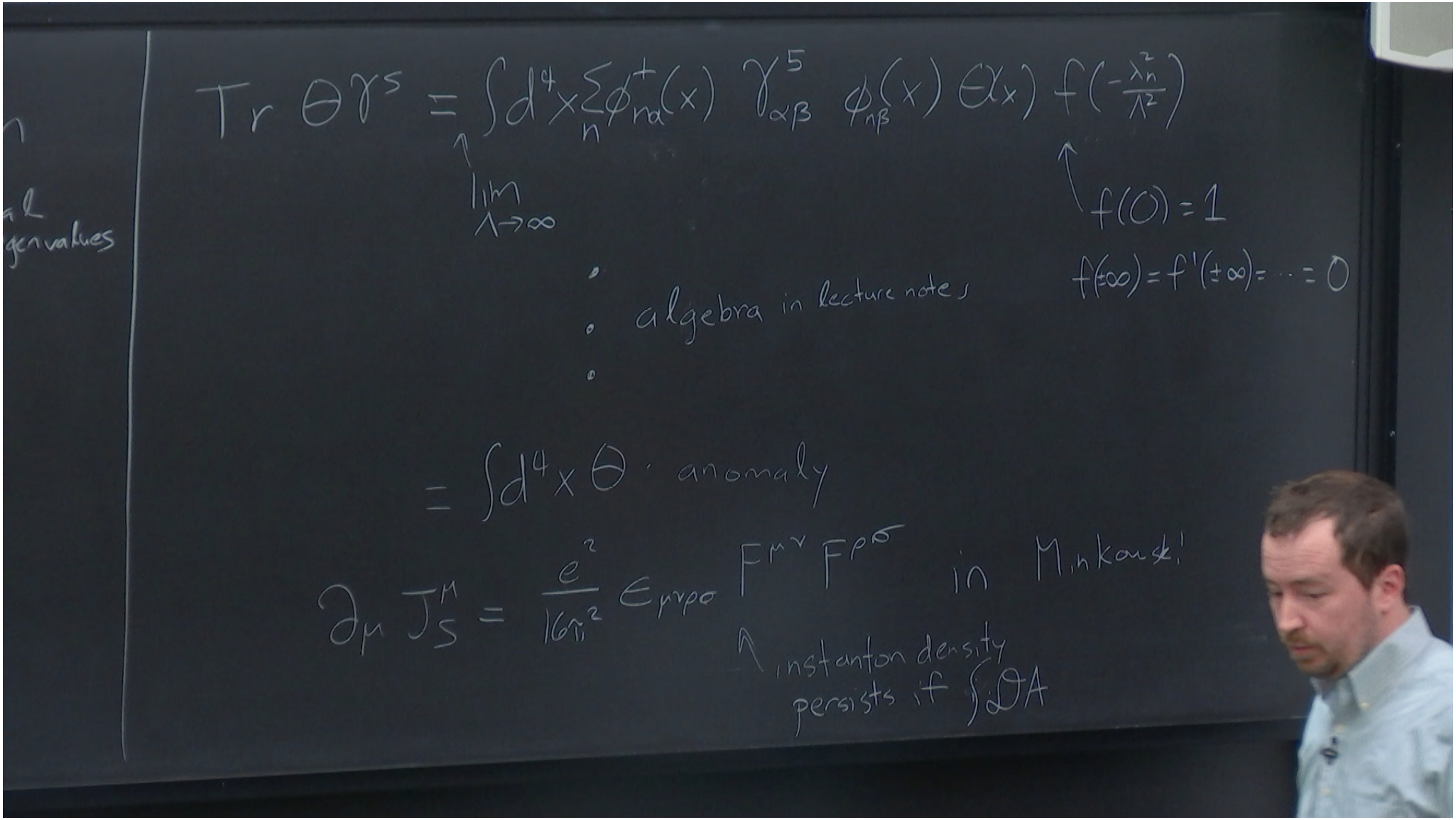
need to compute $\mathrm{Tr}\,\theta\gamma^5$


$Tr\,\theta\gamma^5 = \int d^4x\, \delta^{(4)}(x-x)\,\theta(x)\,tr\,\gamma^5$
UV divergence
$= \infty \cdot 0$
need a regulator
$i\not{D}\phi_n = \lambda_n \phi_n$


$i\not{D}\phi_n = \lambda_n \phi_n$
Hermitian in Euc
complete orthonormal eigenfunctions
real eigenvalues


$\mathrm{Tr}\,\Theta\gamma^5 = \int d^4x \sum_n \phi^\dagger_{n\alpha}(x)\ \gamma^5_{\alpha\beta}\ \phi_{n\beta}(x)\ \Theta(x)$


$\mathrm{Tr}\,\theta\gamma^5 = \int d^4x \sum_n \phi^+_{n\alpha}(x)\,\gamma^5_{\alpha\beta}\,\phi_{n\beta}(x)\,\theta(x)\,f\left(-\frac{\lambda_n^2}{\Lambda^2}\right)$
$\lim_{\Lambda\to\infty}$
$f(0)=1$
$f(\pm\infty)=f'(\pm\infty)=\cdots=0$
algebra in lecture notes
$= \int d^4x\,\theta\cdot$ anomaly
$\partial_\mu J^\mu_5 = \frac{e^2}{16\pi^2}\,\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ in Minkowski


$\mathrm{Tr}\,\theta\gamma^5 = \int d^4x \sum_n \phi^+_{n\alpha}(x)\,\gamma^5_{\alpha\beta}\,\phi_{n\beta}(x)\,\theta(x)\,f\left(-\frac{\lambda_n^2}{\Lambda^2}\right)$
$\lim_{\Lambda\to\infty}$
$f(0) = 1$
$f(\pm\infty) = f'(\pm\infty) = \cdots = 0$
algebra in lecture notes
$= \int d^4x\,\theta \cdot \text{anomaly}$
$\partial_\mu J^\mu_5 = \frac{e^2}{16\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,F^{\mu\nu}F^{\rho\sigma}$ in Minkowski
instanton density
persists if $\int \mathcal{D}A$

$\bar{\psi} \to \bar{\psi} e^{i\theta(x)\gamma^5}$

$A_\mu \to A_\mu$

$S \to S + \int d^4x\, \theta(x)\, \partial_\mu J_5^\mu$

## Zero modes

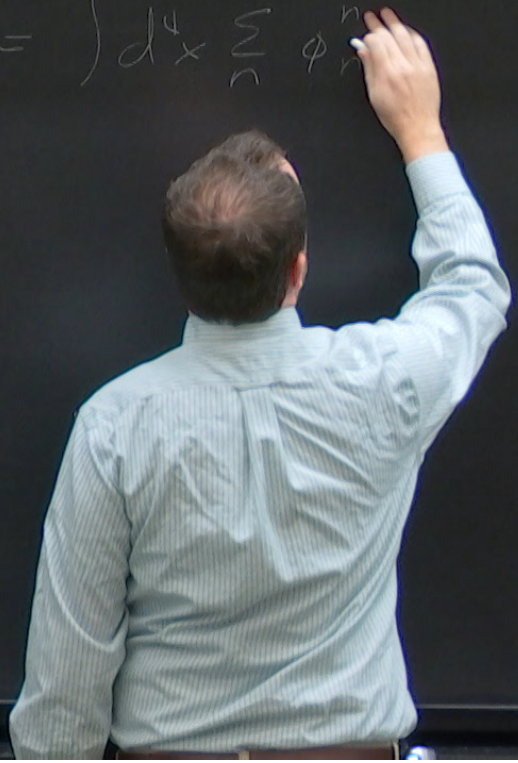
$\gamma^5 \phi_n$

$iD\!\!\!/\, \gamma^5 \phi_n = -\gamma^5 (iD\!\!\!/)\, \phi_n$

$= -\lambda_n \gamma^5 \phi_n$

if $\lambda_n \neq 0$, $\phi_n$ and $\gamma^5 \phi_n$ are orthogonal

$\mathrm{Tr}\, \theta \gamma^5 = \int d^4x \sum_n \phi_n^{\dagger}$



$$\bar{\psi} \to \bar{\psi} e^{i\theta(x)\gamma^5}$$

$$A_\mu \to A_\mu$$

$$S \to S + \int d^4x\, \theta(x)\, \partial_\mu J_5^\mu$$

## Zero modes

$\gamma^5 \phi_n$

$$i\not{D}\gamma^5\phi_n = -\gamma^5(i\not{D})\phi_n$$

$$= -\lambda_n \gamma^5 \phi_n$$

if $\lambda_n \neq 0$, $\phi_n$ and $\gamma^5\phi_n$ are orthogonal

$$\mathrm{Tr}\,\theta\gamma^5 = \int d^4x \sum_n \phi_n^\dagger \gamma^5 \phi_n \theta$$

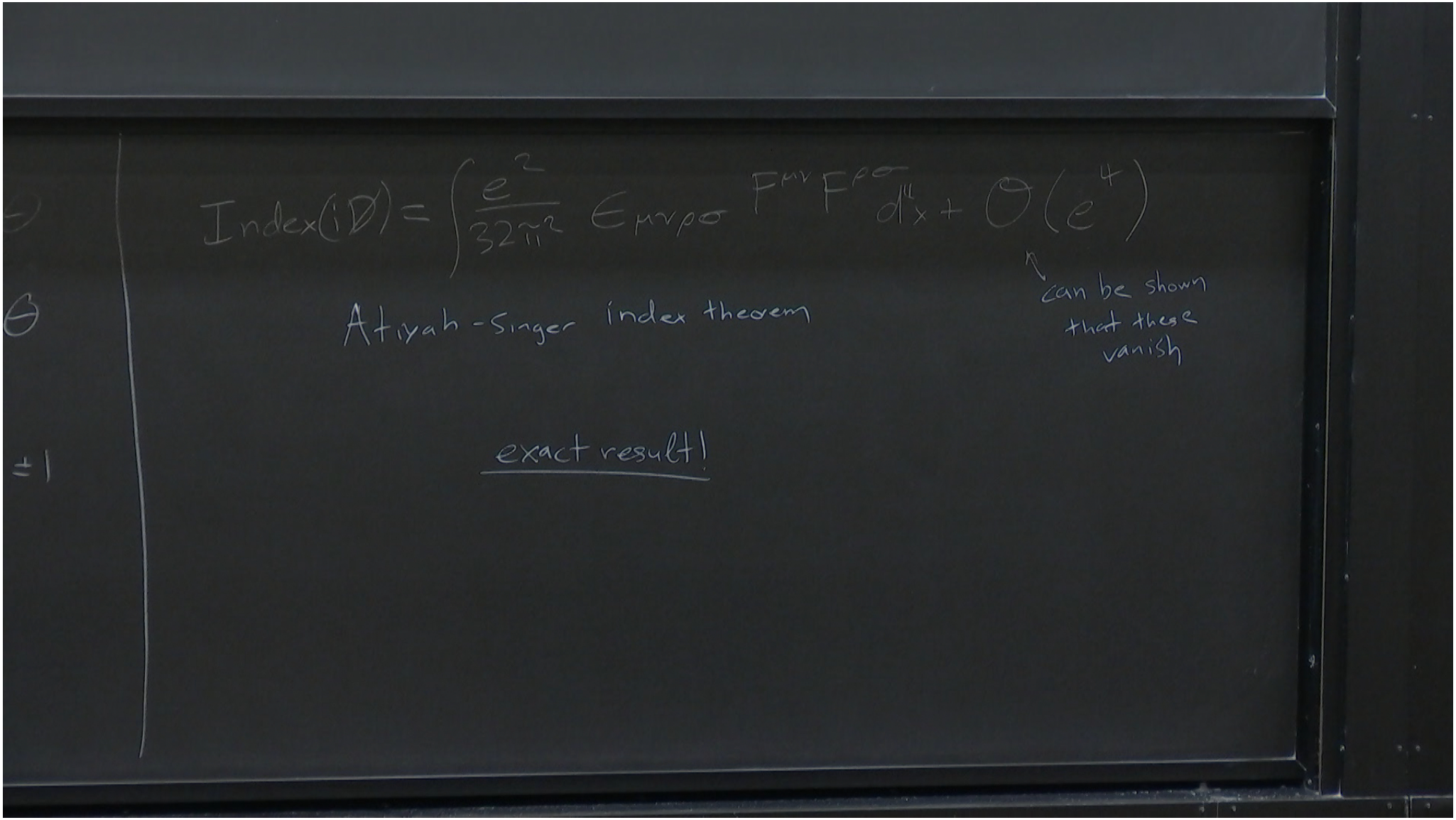
$$= \int d^4x \sum_{\text{zero modes}} \phi_n^\dagger \gamma^5 \phi_n \theta$$

$\gamma^5$ has eigenvalues $\pm 1$

$$= n_+ - n_-$$

= integer

$$= \mathrm{Index}(i\not{D})$$


$\text{Index}(i\not{D}) = \int \frac{e^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} d^4x + \mathcal{O}(e^4)$
Atiyah-Singer index theorem
can be shown that these vanish
exact result!

$\partial_\mu J_5^\mu$

$$\mathrm{Tr}\,\theta\gamma^5 = \int d^4x \sum_n \phi_n^\dagger \gamma^5 \phi_n \theta$$

$$= \int d^4x \sum_{\substack{\text{zero} \\ \text{modes}}} \phi_n^\dagger \gamma^5 \phi_n \theta$$

$\gamma^5$ has eigenvalues $\pm 1$

$$= n_+ - n_-$$

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$$\mathrm{Index}(i\not{D}) = \int \frac{e^2}{32\pi^2}$$

Atiyah-Singe

exa



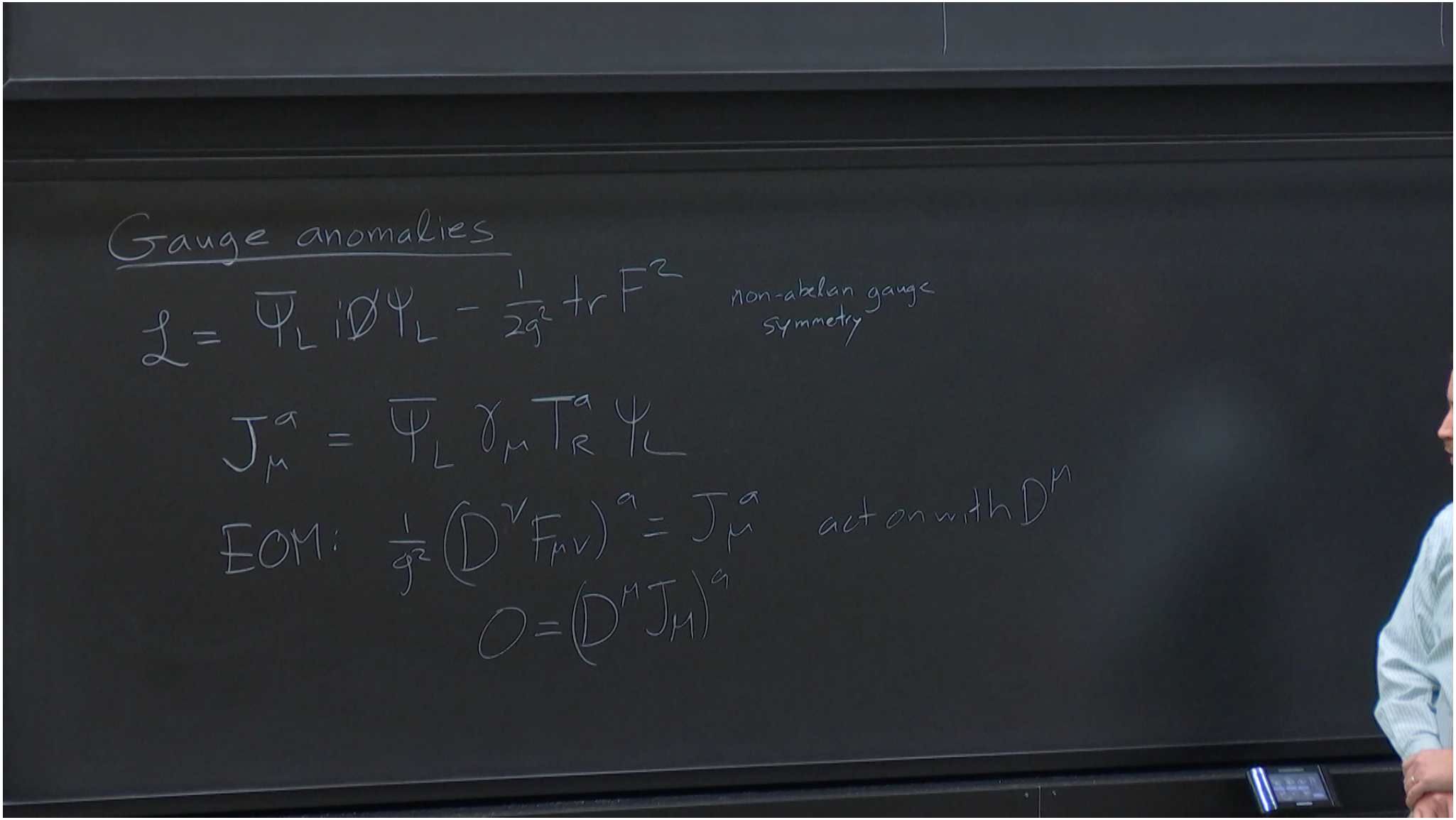
## Gauge anomalies

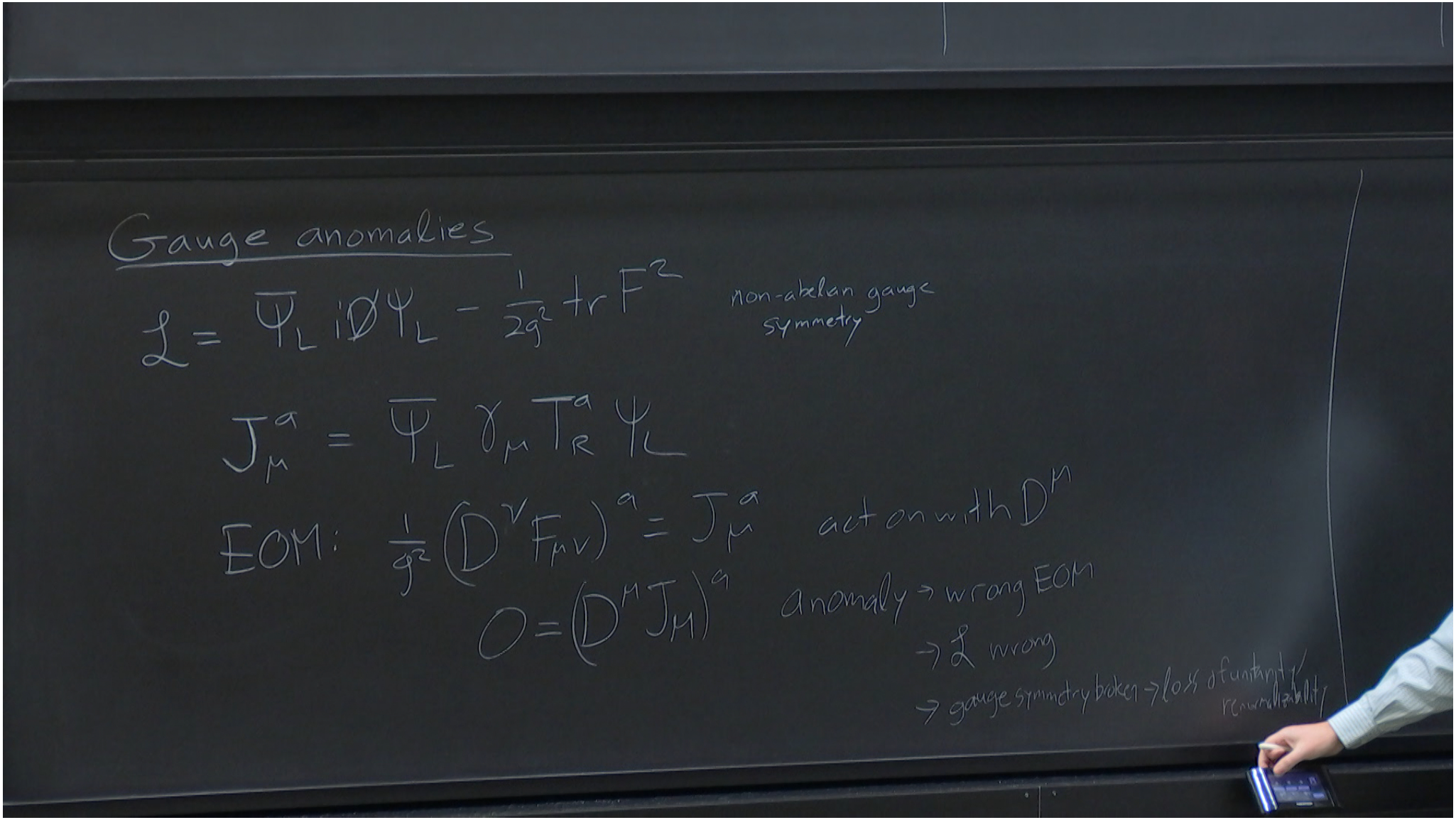
$$\mathcal{L} = \bar{\psi}_L i \not{D} \psi_L - \frac{1}{2g^2} \mathrm{tr}\, F^2$$

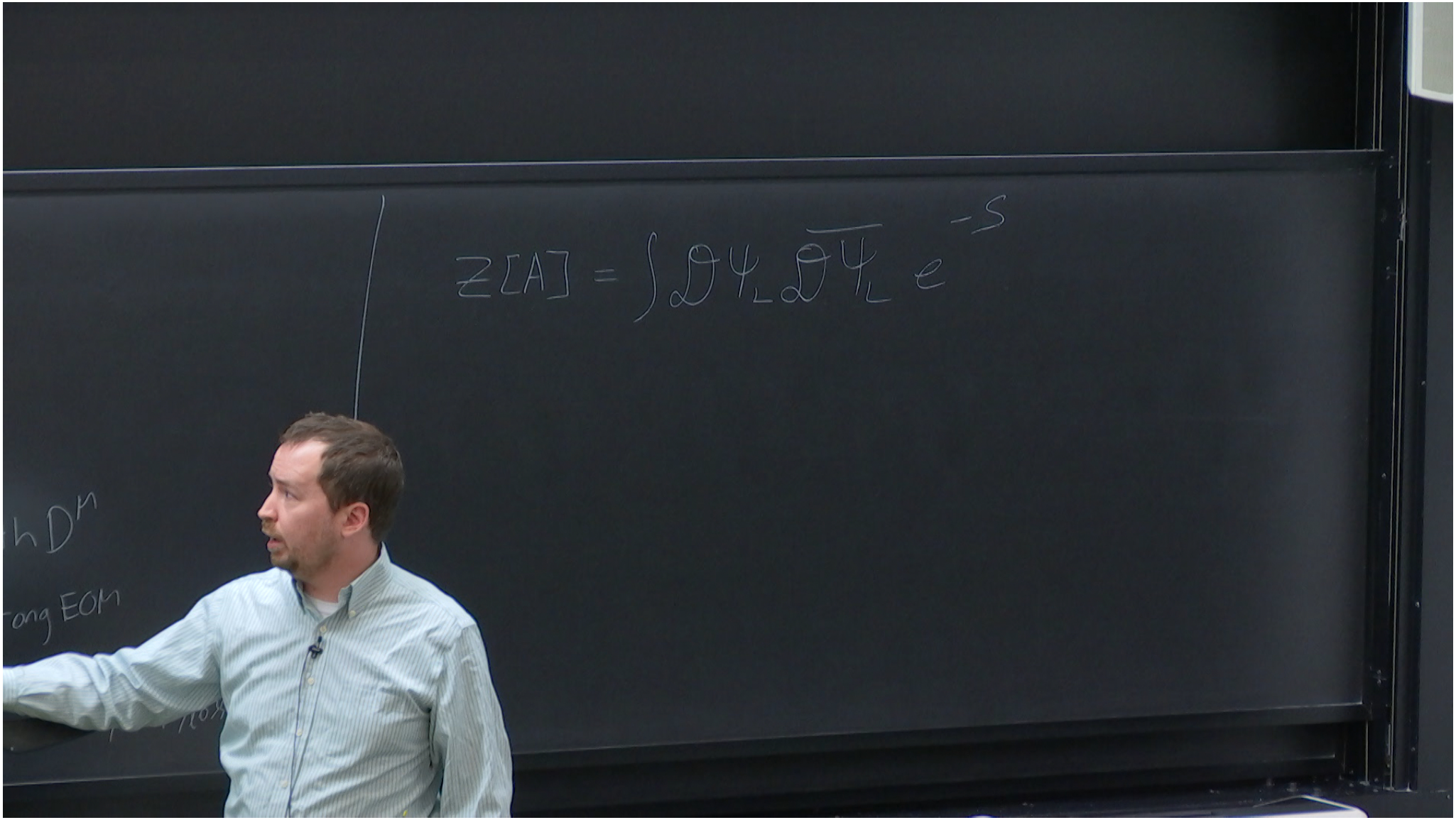
non-abelian gauge symmetry

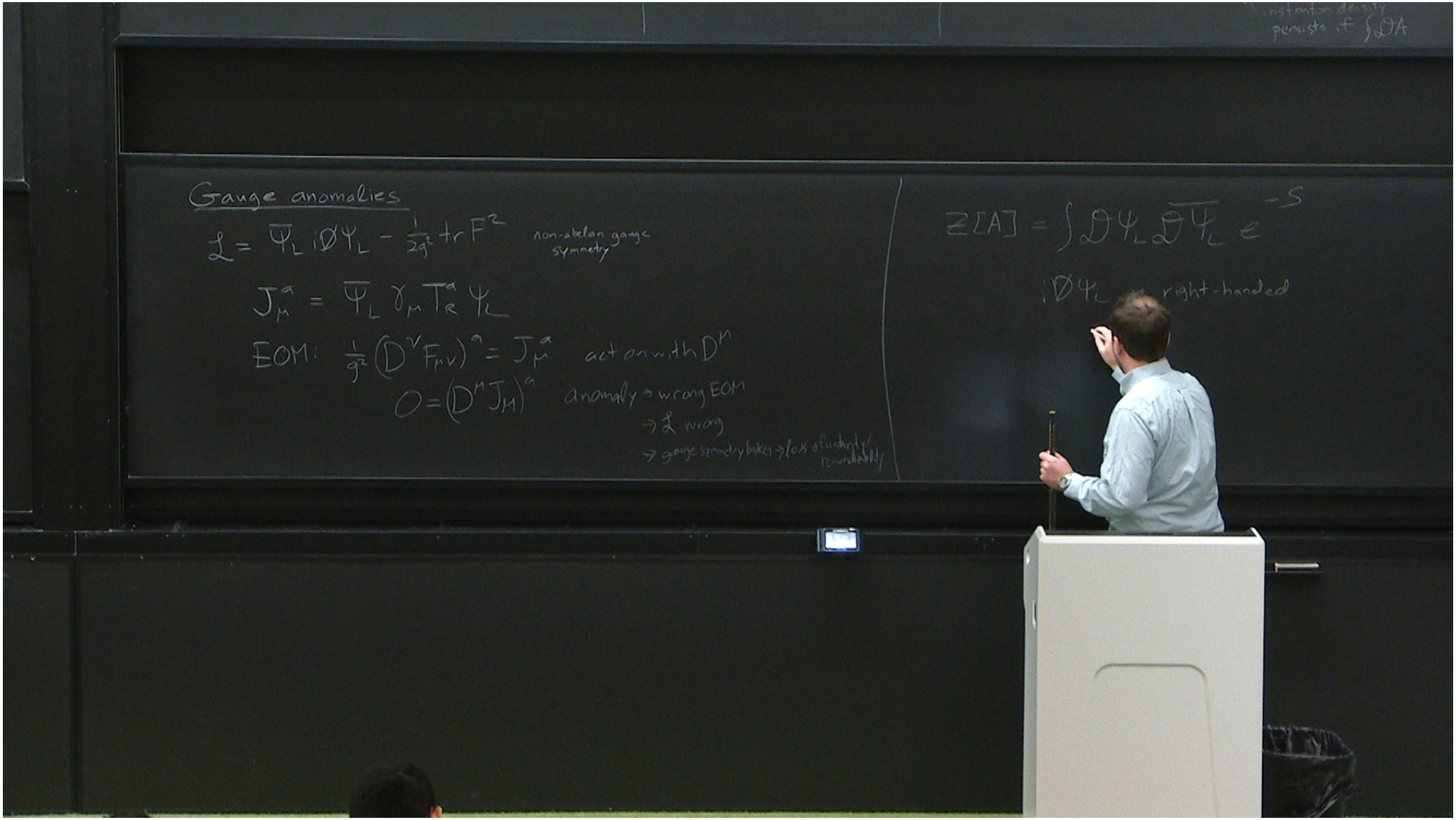
$$J^a_\mu = \bar{\psi}_L \gamma_\mu T^a_R \psi_L$$

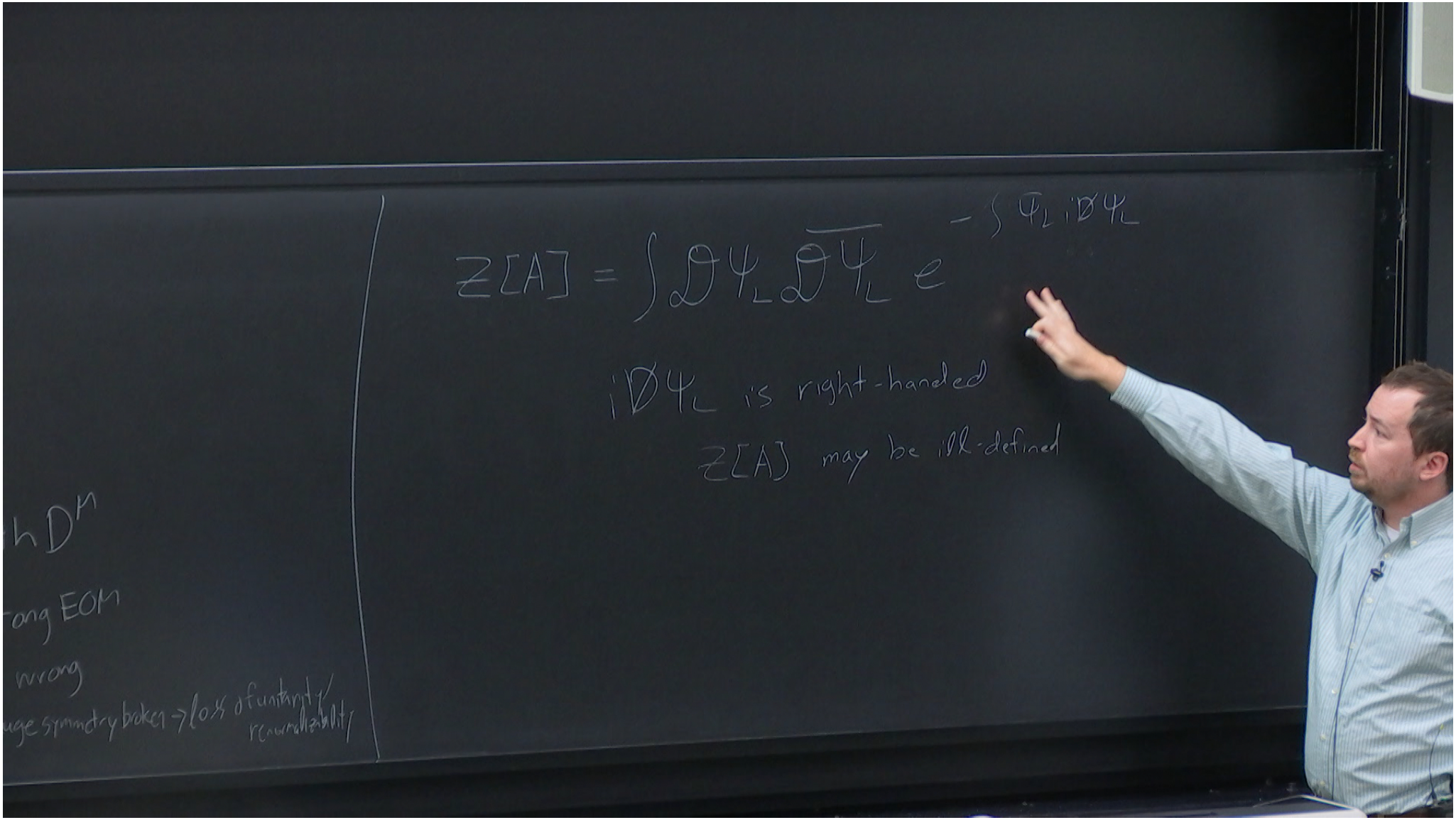


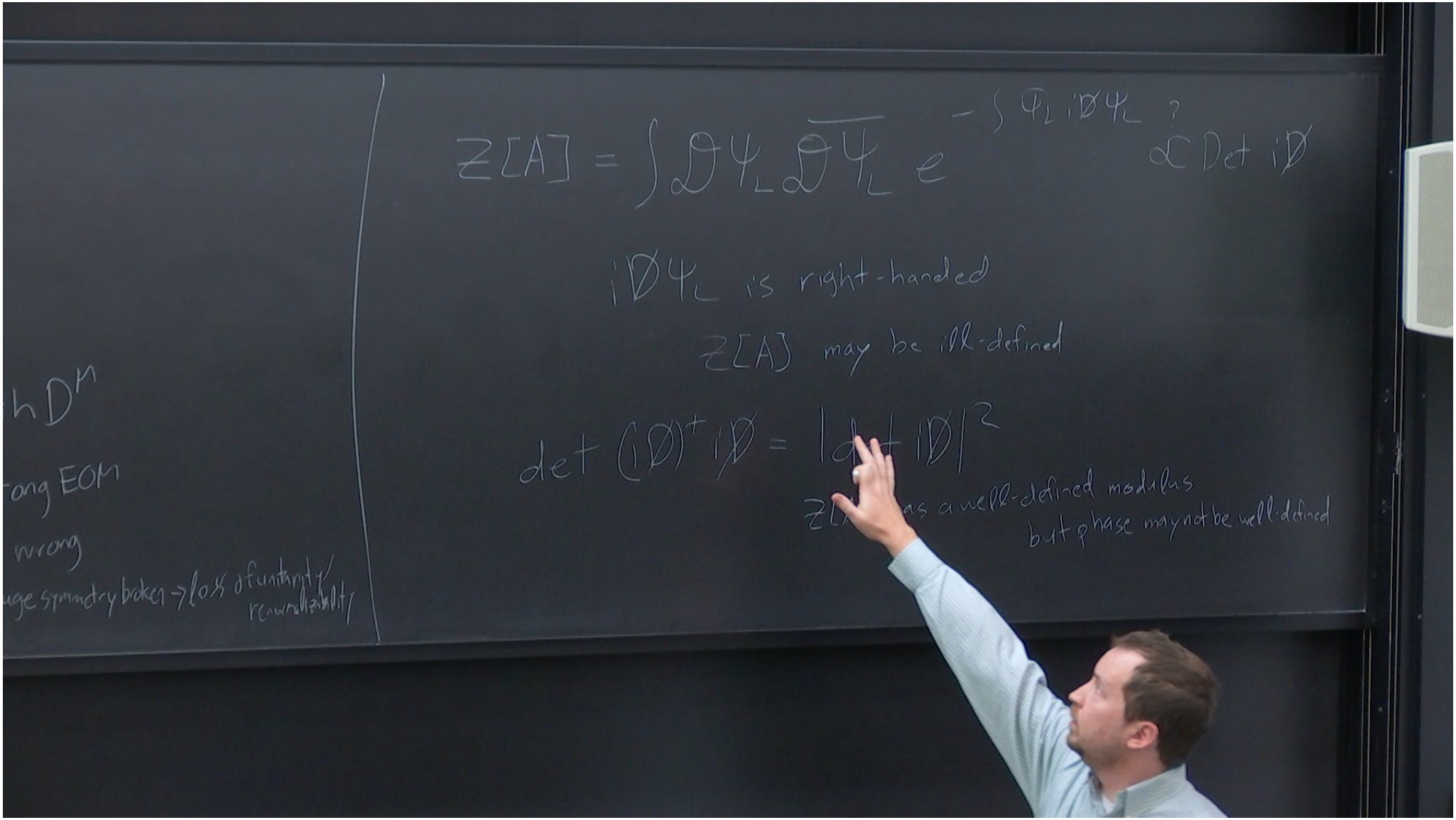

Gauge anomalies
$\mathcal{L} = \bar{\psi}_L i\not{D} \psi_L - \frac{1}{2g^2} \mathrm{tr} F^2$
non-abelian gauge symmetry
$J^a_\mu = \bar{\psi}_L \gamma_\mu T^a_R \psi_L$
EOM: $\frac{1}{g^2}(D^\nu F_{\mu\nu})^a = J^a_\mu$ act on with $D^\mu$
$0 = (D^\mu J_\mu)^a$

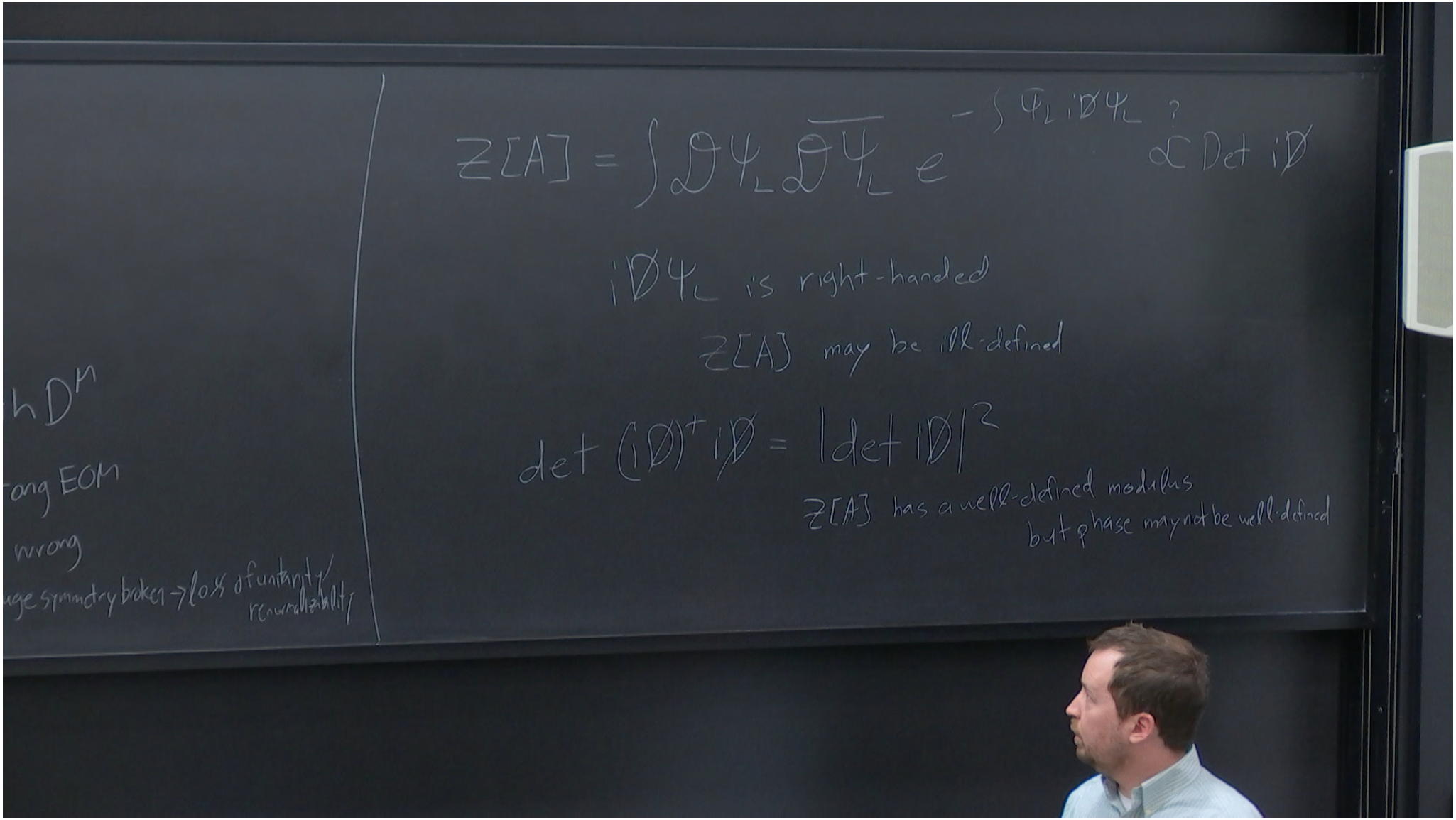

Gauge anomalies
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$0 = (D^\mu J_\mu)^a$ anomaly → wrong EOM
→ $\mathcal{L}$ wrong
→ gauge symmetry broken → loss of unitarity / renormalizability


$Z[A] = \int \mathcal{D}\psi_L \mathcal{D}\bar{\psi}_L e^{-S}$
$hD^{\mu}$
$\gamma^{\mu}$ EOM


Gauge anomalies
$\mathcal{L} = \bar{\psi}_L i\not{D}\psi_L - \frac{1}{2g^2}\mathrm{tr}F^2$
non-abelian gauge symmetry
$J^a_\mu = \bar{\psi}_L \gamma_\mu T^a_R \psi_L$
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$Z[A] = \int \mathcal{D}\psi_L \mathcal{D}\bar{\psi}_L e^{-S}$
$i\not{D}\psi_L$ right-handed
instanton density persists if $\int \mathcal{D}A$


$Z[A] = \int \mathcal{D}\psi_L \mathcal{D}\bar{\psi}_L \, e^{-\int \bar{\psi}_L i\not{D} \psi_L}$
$i\not{D}\psi_L$ is right-handed
$Z[A]$ may be ill-defined
$hD^\mu$
long EOM
wrong
uge symmetry broken → loss of unitarity/renormalizability


$Z[A] = \int \mathcal{D}\psi_L \mathcal{D}\bar{\psi}_L \, e^{-\int \bar{\psi}_L i\not{D} \psi_L}$ ? $\propto$ Det $i\not{D}$
$i\not{D}\psi_L$ is right-handed
$Z[A]$ may be ill-defined
$\det (i\not{D})^\dagger i\not{D} = |\det i\not{D}|^2$
as a well-defined modulus
but phase may not be well-defined
ong EOM
wrong
ge symmetry broken → loss of unitarity/


$Z[A] = \int \mathcal{D}\psi_L \mathcal{D}\bar{\psi}_L \, e^{-\int \bar{\psi}_L i\not{D} \psi_L}$ ? $\propto$ Det $i\not{D}$
$i\not{D}\psi_L$ is right-handed
$Z[A]$ may be ill-defined
$\det (i\not{D})^\dagger i\not{D} = |\det i\not{D}|^2$
$Z[A]$ has a well-defined modulus
but phase may not be well-defined
$hD^\mu$
ong EOM
wrong
uge symmetry broken → loss of unitarity/renormalizability

→ gauge symmetry broken → loss of unitarity / renormalizability

gauge anomaly $\propto \mathrm{tr}\, T_R^a T_R^{(b} T_R^{c)}$ - may vanish for
- can cancel

Summary: anomaly $\Leftrightarrow \mathcal{D}\psi' = \mathcal{D}\psi$

global anomaly :(

gauge anomaly :)

