Title: Symplectic groupoid and log-canonical coordinates on the Teichmueller space of closed genus two surfaces.

Speakers: Michael Shapiro

Series: Mathematical Physics

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Abstract: The coordinate functions on a Poisson variety are log-canonical if the Poisson bracket of two coordinate functions equals a constant times the product of these functions. We consider the symplectic groupoid of unipotent upper-triangular matrices equipped with canonical Poisson bracket. We described a system of log-canonical coordinates and the corresponding cluster structure. As a bonus, we discovered a system of log-canonical coordinates on Teichmueller space of closed genus 2 surfaces. This is joint work with L. Chekhov.

Zoom link: https://pitp.zoom.us/j/94716952708?pwd=R2RiQWRpcHFMYlJLMlB0UjlPVGZkQT09

Symplectic groupoid and log-canonical coordinates for Teichmüller space of closed genus 2 surfaces

joint w/ L. Chekhov

Bondal Poisson structure on unipotent real upper-triangular $n \times n$ matrices.

$\xleftrightarrow[g,\, s = 1,2]{}$

Fock-Chekhov Teichmüller space $T_{g,s}$ of genus g surfaces with s holes
Goldmann Poisson bracket

Q: Find "Darboux type coordinates"

It is done.

Bonus:

When $s \geq 1$ "shear coordinates" are "Darboux type" coordinates for Poisson bracket

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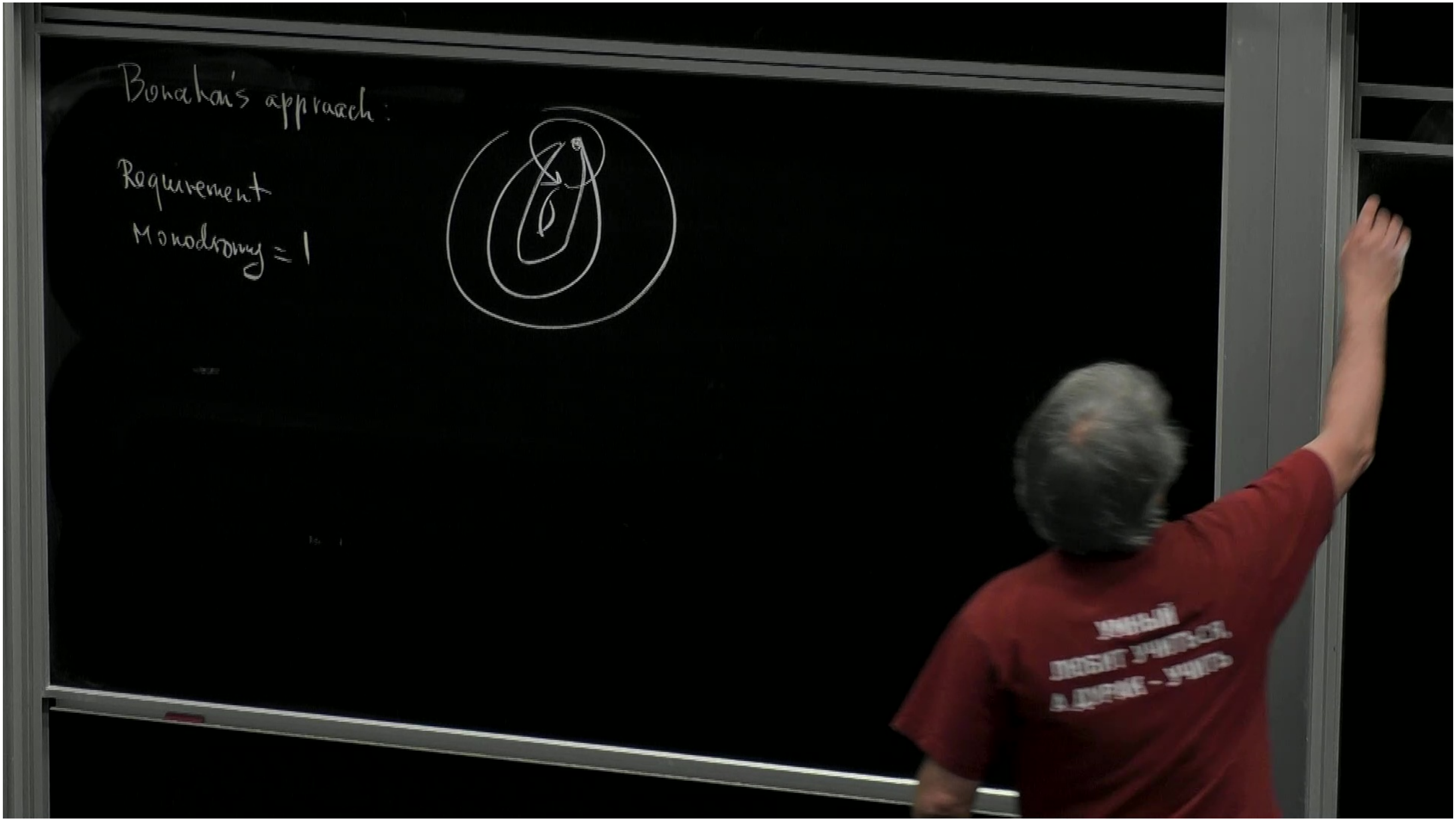
Bonus: obtained "shear coordinates for $s=0$".


Bonahon's approach:
Requirement
Monodromy = 1

Symplectic groupoid of unipotent upper-triangular matrices.

$$\mathcal{A}_n = \left\{ \begin{pmatrix} 1 & & a_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix} \right\} \quad a_{ij} \in \mathbb{R}$$

$$\mathcal{M} = \{ (B, A) \mid B \in GL_n,\ A \in \mathcal{A}_n,\ BAB^t \in \mathcal{A}_n \}.$$

$$\begin{array}{ccc} & \mathcal{M} & \\ {}^{s}\swarrow & & \searrow^{t} \\ \mathcal{A}_n & & \mathcal{A}_n \end{array} \qquad \begin{array}{ccc} & (B, A) & \\ {}^{s}\swarrow & & \searrow^{t} \\ A & & \tilde{A} = BAB^t. \end{array}$$

$\mathcal{A}_n \hookrightarrow \mathcal{M} \qquad A \to (E, A)$

$i: \mathcal{M} \to \mathcal{M} \qquad (B, A) \longmapsto (B^{-1}, BAB^t)$

$$m: \underbrace{\big((C, BAB^t), (B, A)\big)}_{\mathcal{M}^{(2)}} \longmapsto (CB, A)$$

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$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathcal{A}_n & & \mathcal{A}_n \end{tikzcd} \qquad \begin{tikzcd} & (B, A) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & \tilde{A} = BAB^t \end{tikzcd}$$

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$$\begin{tikzcd} & \mathcal{M}^{(2)} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{M} \arrow[dr, "t"'] & & \mathcal{M} \arrow[dl, "s"] \\ & \mathcal{A}_n & \end{tikzcd}$$

Statement (A. Weinstein '87) $\mathcal{M}^{(2)}$ $\exists$ natural sympl. structure ω on $\mathcal{M}$ such that $\boxed{m^* \omega = p_1^* \omega + p_2^* \omega}$

$$\mathcal{A}_n \xleftarrow{s} \mathcal{M} \xrightarrow{t} \mathcal{A}_n$$

$s_* \tilde{\omega}$ is a Poisson bracket on $\mathcal{A}_n$.

$t_* \omega^* = - s_* \omega^*$

Bondal '00, computed expression of Poisson bracket on $\mathcal{A}_n$ ....

The Poisson bracket on $\mathcal{A}_n$ is so called reflection Poisson bracket.

$$\{\overset{1}{A} \overset{\otimes}{,} \overset{2}{A}\} = r\overset{1}{A}\overset{2}{A} + \overset{1}{A} r^{t_1} \overset{2}{A} - \overset{2}{A} r^{t_1} \overset{1}{A} - \overset{2}{A}\overset{1}{A} r$$

r = trigonometric r matrix

(*) Braid group action on $\mathcal{A}_n$

generators $A \to B_{i,i+1} A B_{i,i+1}^t$

$$B_{i,i+1} = \begin{array}{cc} & \begin{array}{cc} i & i+1 \end{array} \\ \begin{array}{c} i \\ i+1 \end{array} & \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \boxed{\begin{matrix} x_{i,i+1} & -1 \\ 1 & 0 \end{matrix}} & \\ & & & \ddots \; 1 \end{pmatrix} \end{array}$$

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathcal{A}_n & & \mathcal{A}_n \end{tikzcd}$$

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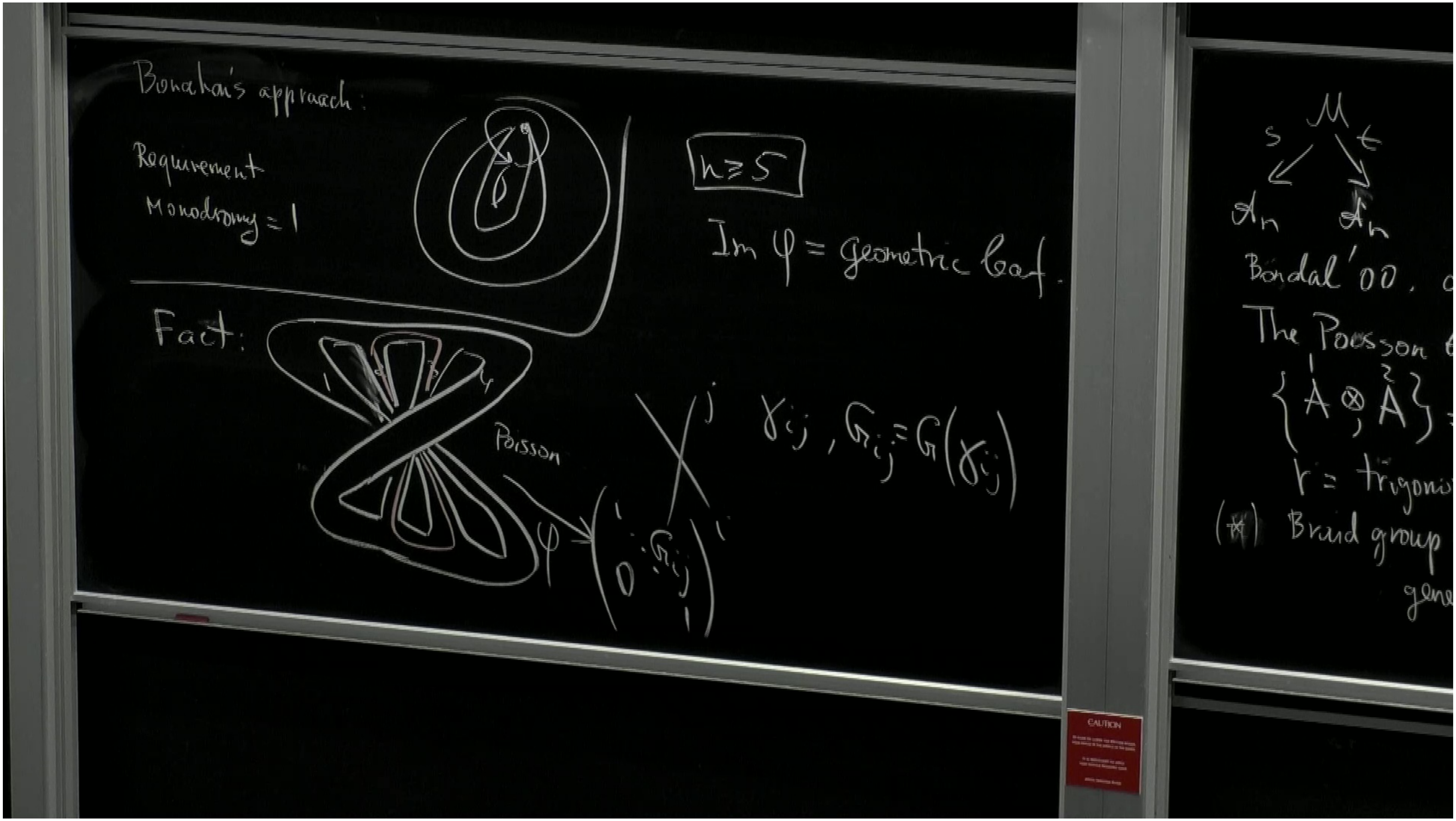

Bonahon's approach:
Requirement
Monodromy = 1
n ≥ 5
Im φ = geometric leaf.
Fact:
Poisson
φ
γij , Gij = G(γij)
Bondal '00.
The Poisson
r = trigono
(*) Braid group

racket.

- Description of nice coordinates for symplectic groupoid.

Key hint: B determines A uniquely.

(B,A) Ex. $n=2$. $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ $A=\begin{pmatrix} 1 & a_{12} \\ 0 & 1 \end{pmatrix}$

$$BAB^t \overset{*}{=} \begin{pmatrix} b_{11}^2+b_{11}a_{12}b_{12}+b_{12}^2 & \\ b_{11}b_{21}+b_{21}a_{12}b_{12}+b_{22}b_{12}b_{21} & b_{21}^2+b_{21}a_{12}b_{22}+b_{22}^2 \end{pmatrix} \Rightarrow \boxed{\begin{array}{l} \det B = 1 \\ \dfrac{b_{12}}{b_{21}} = -1 \end{array}}$$

(entries $=1$, $=1$, $=0$)

$n>2$.

$\det B = 1$

$$\left(\begin{array}{c|c} & B_K \\ \hline \hat{B}_{n-k} & \end{array}\right) \qquad \frac{B_K}{\hat{B}_{n-k}} = (-1)^{n-1}$$

Remark. Poisson-Lie bracket on GL_n,

$\det B, + \dfrac{B_K}{\hat{B}_{n-k}}$ form all casimirs w.r.t. standard P.L. bracket on GL_n.

racket.

$$\mathring{GL}_n = \left\{ B \in GL_n \;\middle|\; \det B = 1 \,,\; \frac{B_k}{\hat{B}_{n-k}} = (-1)^{n-1} \right\}$$ — symplectic leaf of standard P.C. bracket.

$$\Phi : \mathring{GL}_n \longrightarrow \mathcal{A}_n \times \mathcal{A}_n$$

$$B \longmapsto (A, \tilde{A})$$

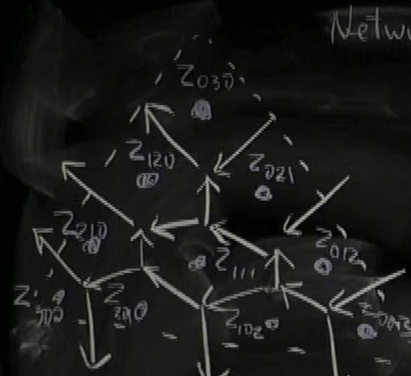
$$\tilde{A} = B A B^T$$

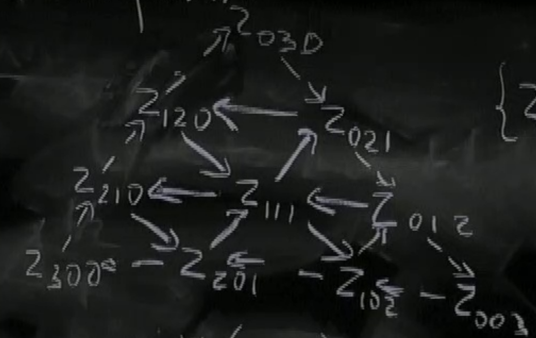
Φ is Poisson.

for $n=3$. $\dim \mathring{GL}_3 = 6$.

$\dim \operatorname{Im} \varphi = 5$.

Networks and quivers



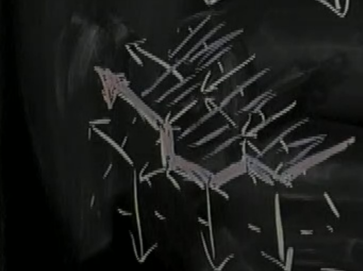


$$\{Z_\alpha, Z_\beta\} = \varepsilon Z_\alpha Z_\beta$$

$$\varepsilon = \left(\text{weight}(\alpha \to \beta) - \text{weight}(\beta \to \alpha)\right)$$

$$\text{weight}(\dashrightarrow) = \tfrac{1}{2}$$

$$\text{weight}(\longrightarrow) = 1$$



weight of path = product of face weights on the right

Ex weight of red path $= Z_{030} Z_{120} Z_{021} Z_{111} Z_{012}$

Transport matrix $T_{ij} = \sum_{\text{path } p:\, j \rightsquigarrow i} \text{weight}(p)$

$\mathcal{M}$, s, t

$\mathcal{A}_n$ $\mathcal{A}_n$

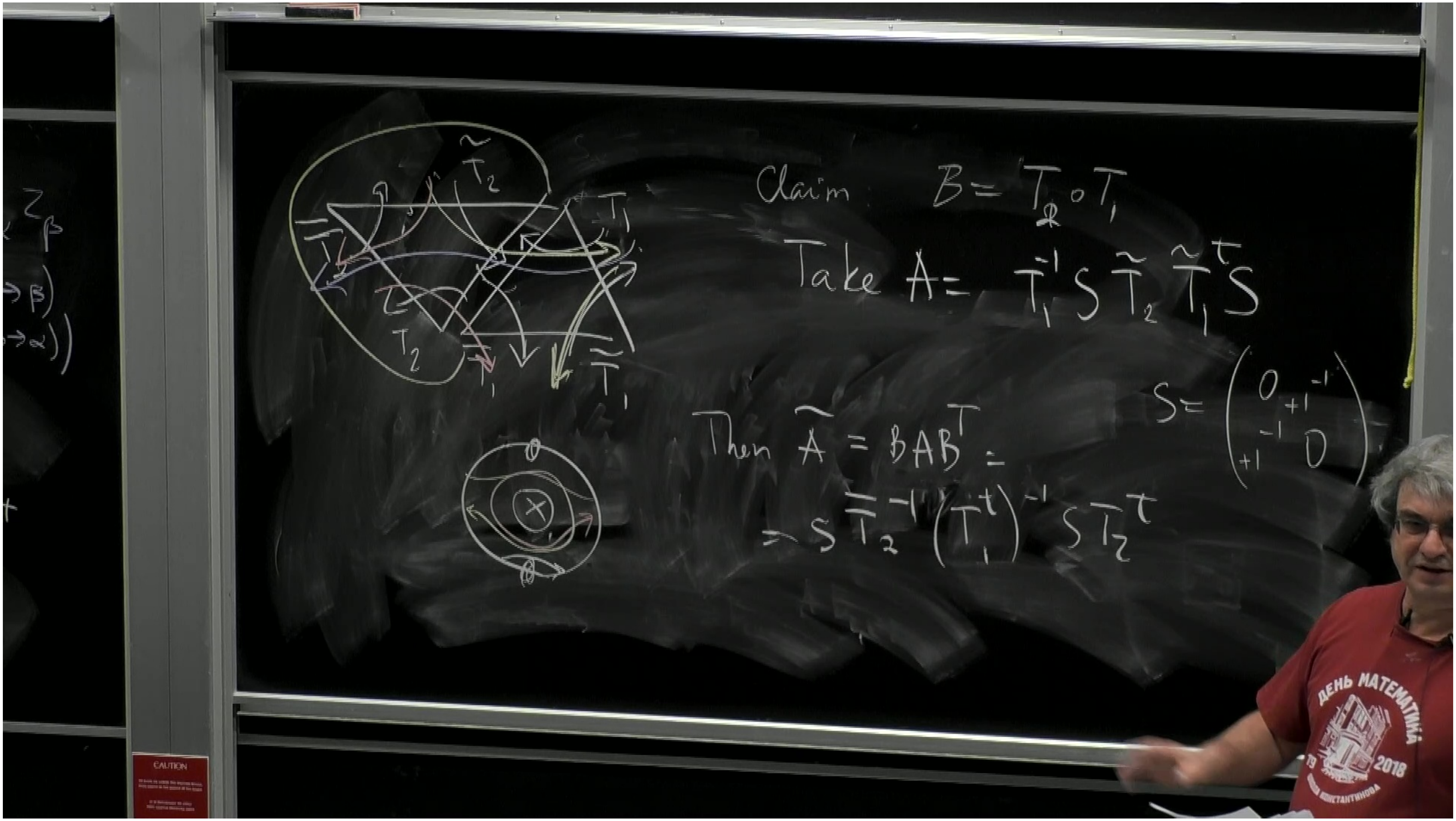
Bondal '00

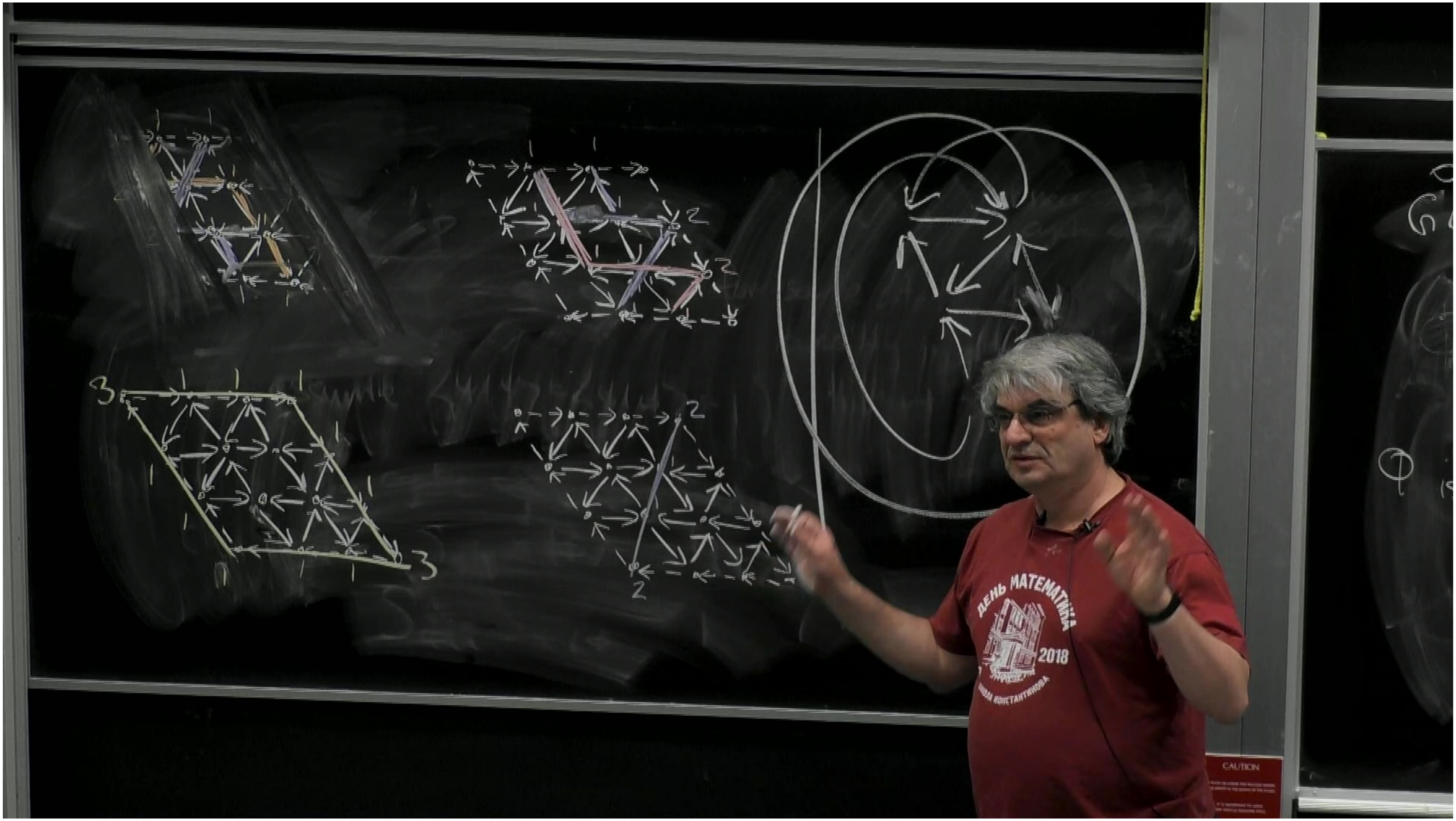
The Poiss

$\{\overset{1}{A} \otimes, \overset{2}{A}$

$r =$ tr

(*) Braid gr


Claim: $B = T_2 \circ T_1$
Take $A = T_1^{-1} S \tilde{T}_2 \tilde{T}_1^{t} S$
Then $\tilde{A} = BAB^{T} =$
$= S \bar{T}_2^{-1} (T_1^{t})^{-1} S T_2^{t}$
CAUTION
ДЕНЬ МАТЕМАТИКИ
2018


ДЕНЬ МАТЕМАТИКА
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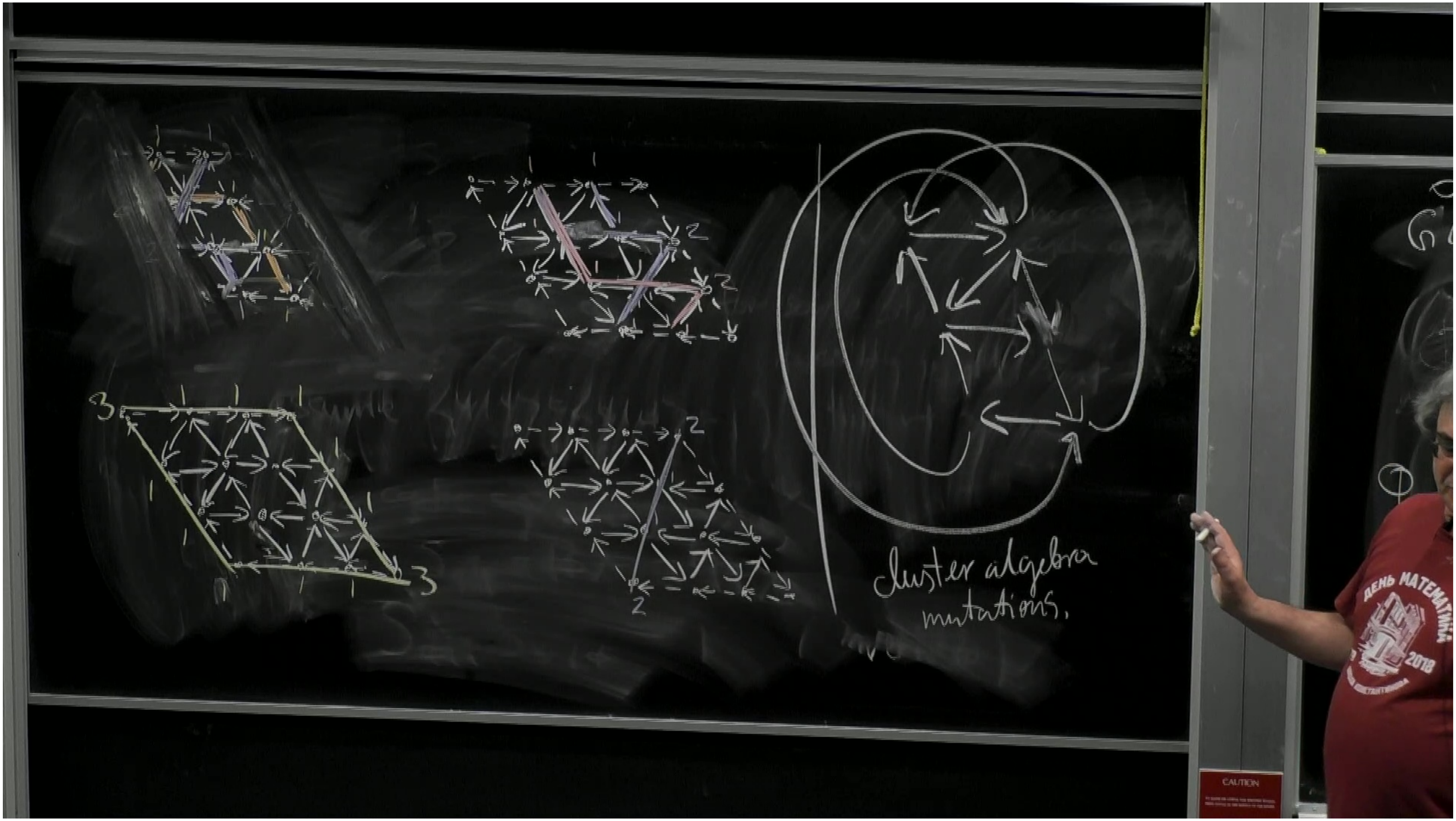
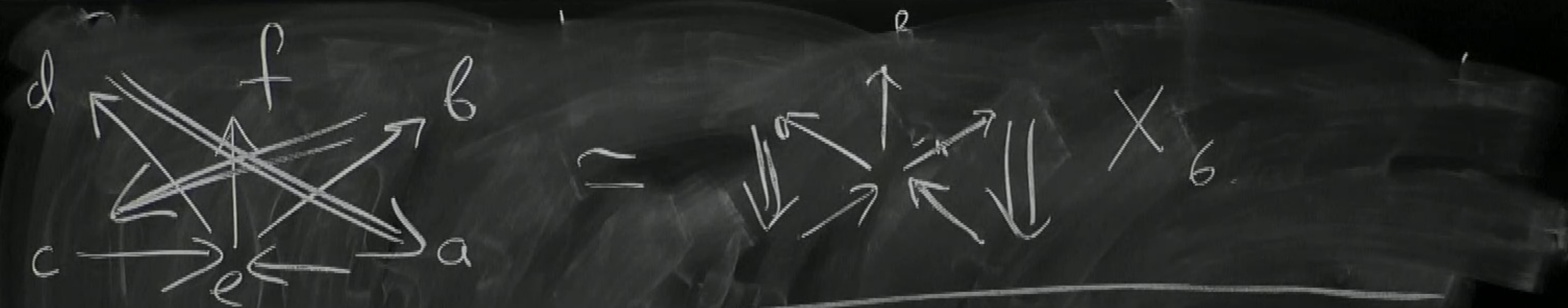
CAUTION


cluster algebra
mutations.



$$\Phi : GL_3 \longrightarrow \mathcal{A}_3 \times \mathcal{A}_3$$

$$B \longrightarrow (A, \tilde{A})$$

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

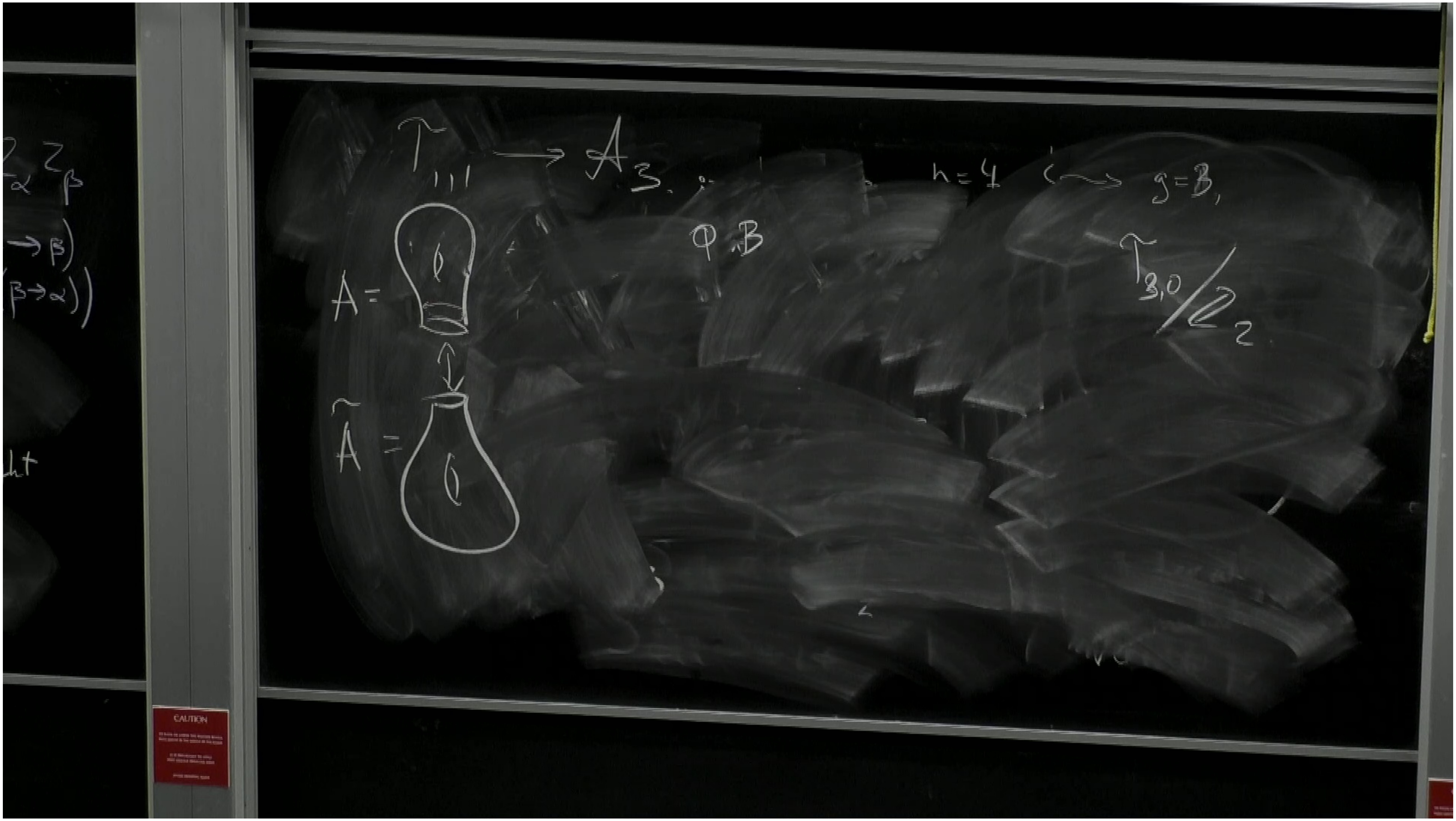
Markov element

$$\mathcal{M} = \underline{a_{12}a_{23}a_{13} - a_{12}^2 - a_{23}^2 - a_{13}^2}$$

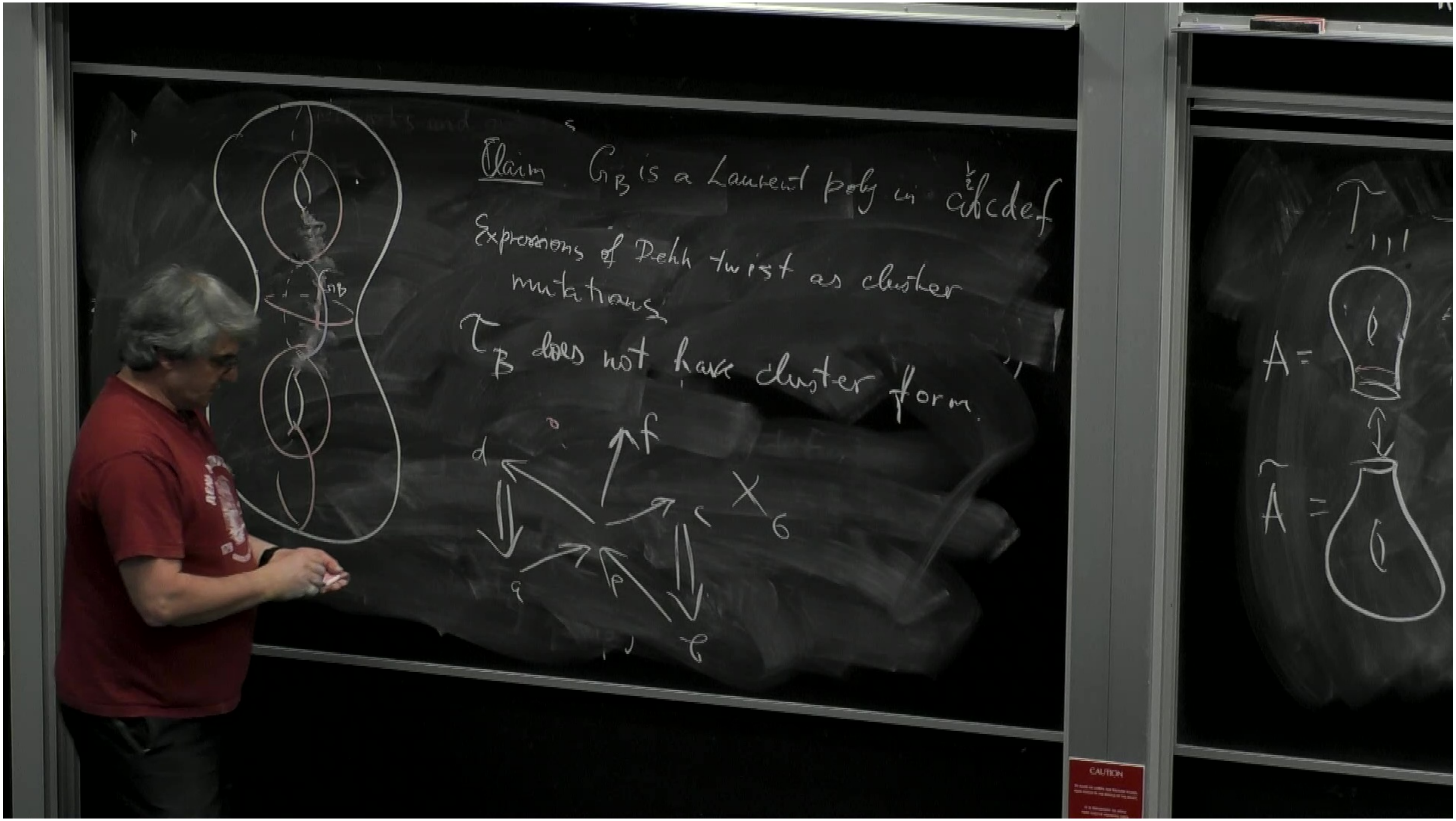
$$= \det(A + A^t)$$

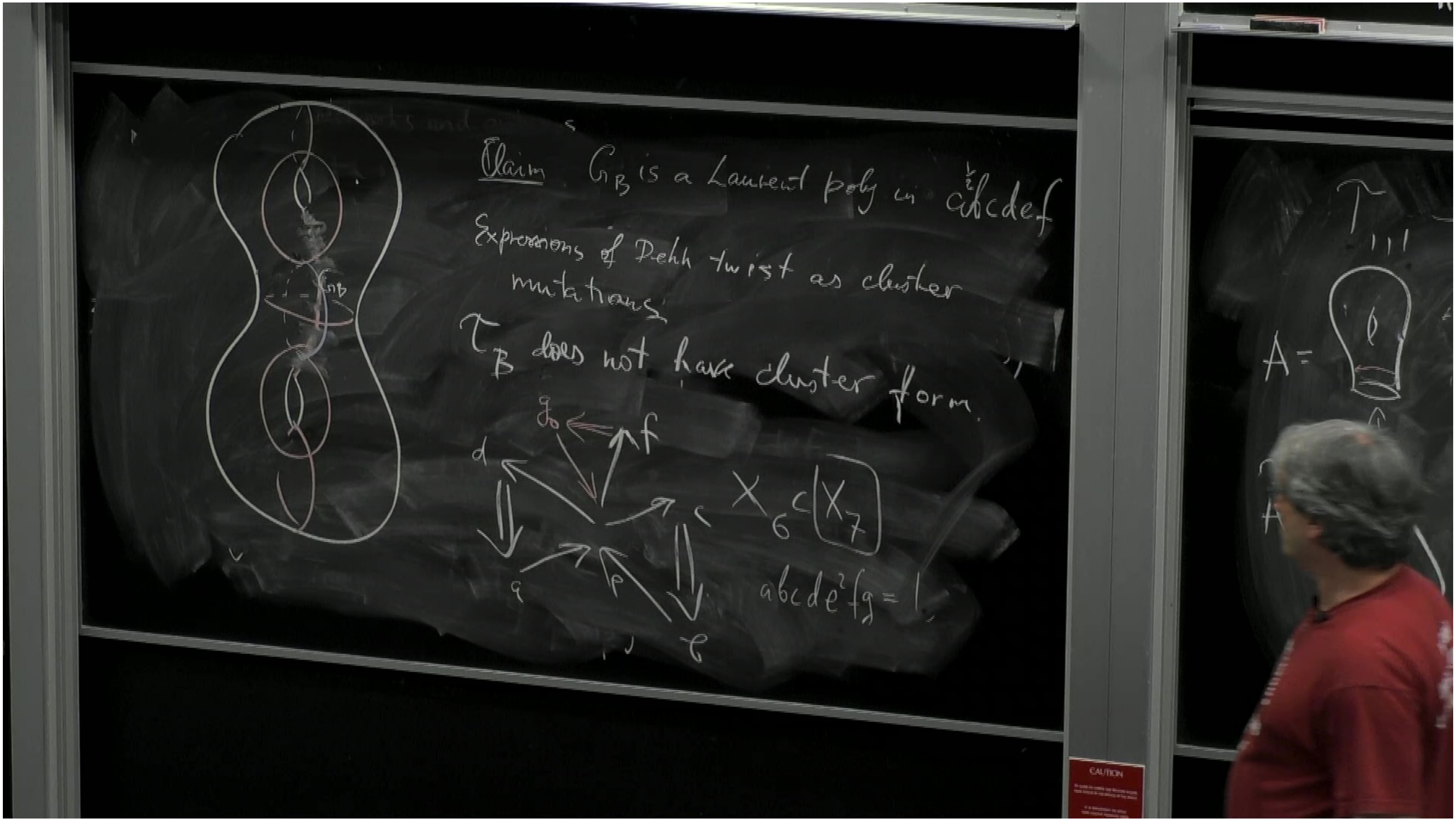
$$\det(\tilde{A} + \tilde{A}^t) = \det(BAB^t + BA^tB^t) =$$

$$= \underline{\det(B)^2}\,\mathcal{M}$$



T̃1,1,1 → A3,
n = 4 ⟷ g = 3,
φ, B
T̃3,0 / Z2
A =
Ã =


Claim G_B is a Laurent poly in $a^{1/2}bcdef$
Expressions of Dehn twist as cluster mutations
τ_B does not have cluster form.
$\mathcal{X}_6$
$A =$
$\tilde{A} =$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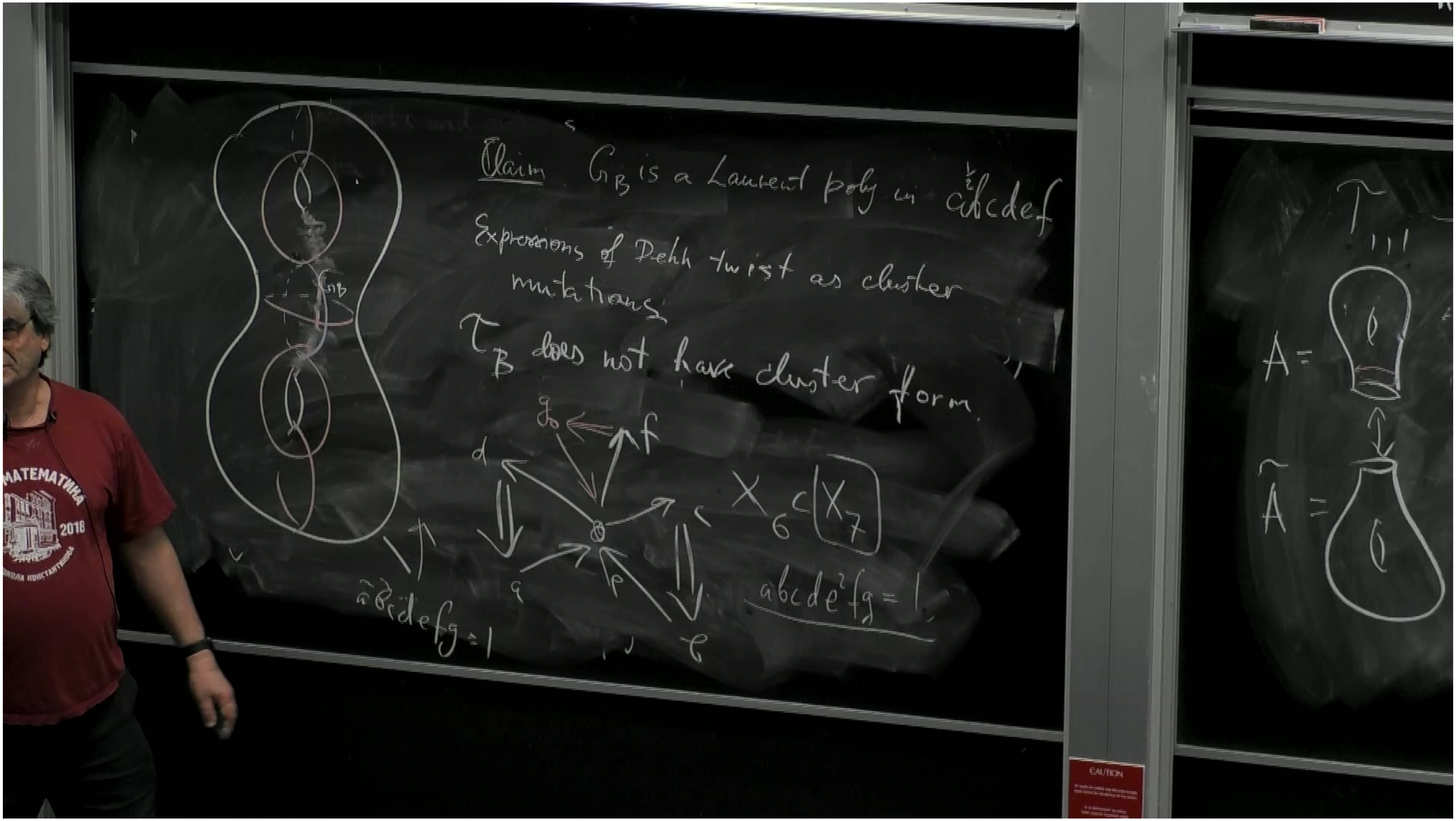
$X_6 \subset X_7$
$abcde^2fg = 1$
A =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